Український математичний вісник
Том 5 **(2008)**, № 1, 16 – 31

On a classical solvability of a Florin problem

Galina I. Bizhanova

(*Presented by E. Ya. Khruslov*)

Abstract. There is considered the multidimensional two-phase Stefan problem with a small parameter κ at the velocity of a free boundary in a Stefan condition. The unique solvability and coercive uniform with respect to κ estimate of the solution for $t \le T_0$, T_0 — independent on κ, are proved and on the basis of this the existence, uniqueness and estimate of the solution of a Florin problem (Stefan problem with $\kappa = 0$) are obtained in the Hölder spaces.

2000 MSC. 35R35, 35B25, 35K60.

Key words and phrases. Free boundary problem, parabolic equation, small parameter, Hölder space, existence, uniqueness, estimate of solution.

1. Statement of the problems. Main results

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \ge 2$, with a boundary Σ. In Ω there is a closed surface $\gamma(t)$, $t \in [0, t_0]$, which divides Ω into two sub-domains $\Omega_1(t)$ and $\Omega_2(t)$ with the boundaries $\partial\Omega_1(t) = \Sigma \cup \gamma(t)$, $\partial\Omega_2(t) = \gamma(t)$. Denote $\gamma(0) := \Gamma \subset \Omega$ and $\Omega_j(0) := \Omega_j$, $j = 1, 2$. We assume $\operatorname{dist}(\Gamma, \Sigma) \ge d_0 = \mathrm{const} > 0$, $\operatorname{diam}\Omega_2 \ge d_0$ to guarantee that a surface $\gamma(t)$ will not touch Σ and a domain $\Omega_2(t)$ will not degenerate for small time.

Let $\Gamma \in C^{2+\alpha}$, $\alpha \in (0,1)$, then we can represent $\gamma(t)$ for small $t \le t_0$ by an equation [8, 9]

$$x = \xi + \rho(\xi, t)\, N(\xi), \quad \xi = \xi(x) \in \Gamma, \quad t \in [0, t_0], \tag{1.1}$$

where $\rho\big|_{t=0} = 0$, $N(\xi) = (N_1, \dots, N_n) \in C^{2+\alpha}(\Gamma; \mathbb{R}^n)$ is a unit vector field on Γ satisfying condition $\nu_0(\xi)\, N^T(\xi) \ge d_1 = \mathrm{const} > 0$, $\nu_0(\xi)$ is a unit normal to Γ directed into Ω_2

Received 8.02.2008

Here and further by symbol "^{T}" we denote transposed matrix A^T and column-vector N^T; d_k, C_k, $k=1,2,\dots$, are positive constants.

Let $\Omega_T = \Omega\times(0,T)$, $\Sigma_T = \Sigma\times[0,T]$, $\Gamma_T = \Gamma\times[0,T]$, $\Omega_{jT} = \Omega_j\times(0,T)$, $Q_{jT} = \big\{(x,t): x\in\Omega_j(t),\ t\in(0,T)\big\}$, $j=1,2$.

Consider two-phase Stefan problem with the unknown functions $u_j(x,t)$, $j=1,2$, and $\rho(\xi,t)$ satisfying the parabolic equations, initial and boundary conditions

$$\partial_t u_j - a_j\,\Delta u_j = 0 \ \text{ in } Q_{jT}, \quad j=1,2, \tag{1.2}$$

$$\gamma(t)\big|_{t=0} = \Gamma, \quad u_j\big|_{t=0} = u_{0j}(x) \quad \text{in} \ \ \Omega_j, \quad j=1,2, \tag{1.3}$$

$$u_1\big|_{\Sigma} = p(x,t), \quad t\in(0,T), \tag{1.4}$$

and conditions on a free boundary $\gamma(t)$, $t\in(0,T)$,

$$u_1 = u_2 = 0, \tag{1.5}$$

$$\lambda_1\,\partial_\nu u_1 - \lambda_2\,\partial_\nu u_2 = -\kappa\,\nu N^T\,\partial_t\rho, \tag{1.6}$$

where a_j, λ_j, $j=1,2$, are positive constants; $\kappa>0$ — small parameter, $\nu(x,t)$ — a unit normal to $\gamma(t)$ directed into $\Omega_2(t)$, $\nu N^T\partial_t\rho{=}V_\nu$ is a velocity of a free boundary on the direction of ν due to (1.1); $\partial_t = \partial/\partial t$, $\partial_\nu = \partial/\partial\nu = \nu\nabla^T$ is the normal derivative, $\nabla = \partial_{x_1},\dots,\partial_{x_n}$.

Letting κ to zero in the condition (1.6) we shall have degenerate Stefan or Florin [14] problem with unknown functions u_j, $j=1,2$, ρ:

$$\partial_t u_j - a_j\,\Delta u_j = 0 \ \text{ in } \ Q_{jT}, \quad j=1,2, \tag{1.7}$$

$$\gamma(t)\big|_{t=0} = \Gamma,\ u_j\big|_{t=0} = u_{0j}(x) \ \text{ in } \ \Omega_j, \quad j=1,2, \tag{1.8}$$

$$u_1\big|_{\Sigma} = p(x,t), \quad t\in(0,T), \tag{1.9}$$

$$u_1 = u_2 = 0, \quad \lambda_1\,\partial_\nu u_1 - \lambda_2\,\partial_\nu u_2 = 0 \ \text{ on } \ \gamma(t), \quad t\in(0,T). \tag{1.10}$$

Classical solvability of the multidimensional Stefan problem was studied by A. Friedman and D. Kinderlehrer [15], L. A. Caffarelli [11, 12], D. Kinderlehrer and L. Nirenberg [17], A. M. Meirmanov [19], E. I. Hanzawa [16], B. V. Bazaliy [1], E. V. Radkevich [20], B. V. Bazaliy and S. P. Degtyarev [2], M. A. Borodin [10], G. I. Bizhanova [5,6], G. I. Bizhanova and V. A. Solonnikov [9]. In [21] J. F. Rodrigues, V. A. Solonnikov and F. Yi have obtained the existence of the multidimensional one-phase Florin problem locally in time in the Hölder space $C^{2+\beta,1+\beta/2}$, $0<\beta<\alpha$, with the help of the imbedding theorem applied to the solution from $C^{2+\alpha,1+\alpha/2}$, $\alpha\in(0,1)$ of the corresponding Stefan problem with the small parameter.

Solvability in $C^{2+\alpha,1+\alpha/2}$, $\alpha \in (0,1)$, for small time of the multidimensional one-phase Florin problem was established by A. Fasano, M. Primicerio and E. V. Radkevich [13]. In [5, 6] G. I. Bizhanova has proved existence, uniqueness and estimates of the solution of multidimensional two-phase Florin problem in the classical and weighted Hölder spaces with time power weights [3], when free boundary is a graph of function on the plane $x_n = 0$ and on the unit sphere.

We are considering (1.2)–(1.6) as a problem with a small parameter κ at the principle term — velocity of a free boundary in the condition (1.6). Comparing Theorems 1.1 and 1.2 we can see that the smoothness of a free boundary in the Stefan and Florin problems is different and it is higher in the Stefan problem. That is the problem (1.2)–(1.6) with a small parameter is singularly perturbed.

We note that applying of the method of a small parameter permits us to obtain required results for the solutions of the problems, in which one of the unknowns is given in the implicit form, like in the Florin problem a free boundary is set.

Using the solution of the Stefan problem (1.2)–(1.6) and letting κ to zero we shall prove existence, uniqueness and estimate of the solution of the Florin problem (1.7)–(1.10) without loss of a smoothness of this solution. We can not apply for that available results on the solvability of Stefan problem, because the time T_0 of an existence of the solution and a constant in the estimate for it depend on a small parameter κ.

In Chapter 2 we prove Theorem 1.1 for the solution of Stefan problem with T_0 and a constant in the estimate of a solution independent on κ and in Chapter 3 on the basis of Theorem 1.1 we obtain Theorem 1.2 on the solvability of a Florin problem.

The problems are considering in the classical Hölder spaces $C_{x\ t}^{l,l/2}(\bar{\Omega}_T)$, l is positive non–integer, of the functions $u(x,t)$ with the norm [18]

$$|u|_{\Omega_T}^{(l)} := \sum_{2k+|m|<l} |\partial_t^k \partial_x^m u|_{\Omega_T} + \sum_{2k+|m|=[l]} [\partial_t^k \partial_x^m u]_{\Omega_T}^{(l-[l])} + \sum_{2k+|m|=[l]-1} [\partial_t^k \partial_x^m u]_{t,\Omega_T}^{\left(\frac{1+l-[l]}{2}\right)},$$

where the last term is omitted, if $[l]=0$, $|\,v|_{\Omega_T} = \max_{(x,t)\in\overline{\Omega}_T} |v|$,

$$[v]_{\Omega_T}^{(\alpha)} = [v]_{x,\Omega_T}^{(\alpha)} + [v]_{t,\Omega_T}^{(\alpha/2)},$$
$$[v]_{x,\Omega_T}^{(\alpha)} = \max_{(x,t),(z,t)\in\overline{\Omega}_T} \left|v(x,t) - v(z,t)\right| |x-z|^{-\alpha},$$

$$[v]^{(\alpha)}_{t,\Omega_T} = \max_{(x,t),\,(x,t_1)\in\overline{\Omega}_T} \big|v(x,t) - v(x,t_1)\big|\,|t-t_1|^{-\alpha}, \quad \alpha\in(0,1).$$

$\overset{\circ}{C}{}^{\,l,l/2}_{x\ \ t}(\overline{\Omega}_T)$ is a sub-space of the functions $u(x,t)\in C^{\,l,l/2}_{x\ \ t}(\overline{\Omega}_T)$ satisfying the conditions $\partial_t^k u\big|_{t=0} = 0,\ k\le [l/2]$.

We formulate the main results of the paper.

Theorem 1.1. *Let* $\Sigma,\ \Gamma\in C^{2+\alpha}$, $\alpha\in(0,1)$.

For any functions $u_{0j}\in C^{2+\alpha}(\overline{\Omega}_j)$, $j=1,2$, $p\in C^{2+\alpha,1+\alpha/2}_{x\qquad t}(\Sigma_T)$ *satisfying the compatibility conditions of zero and the first order on* Σ *and* Γ *and the conditions*

$$0<\kappa\le\kappa_0, \qquad \partial_{\nu_0} u_{0j}\big|_\Gamma \le -d_2 < 0, \quad j=1,2, \tag{1.11}$$

there exists $T_0>0$ *such that the Stefan problem* (1.2)–(1.6) *has a unique solution* $u_j\in C^{2+\alpha,1+\alpha/2}_{x\qquad t}(\overline{Q}_{jT_0})$, $j=1,2$, $\rho\in C^{2+\alpha,1+\alpha/2}_{x\qquad t}(\Gamma_{T_0})$, $\kappa\partial_t\rho\in C^{1+\alpha,1+\alpha/2}_{x\qquad t}(\Gamma_{T_0})$ *and the following estimate holds for* $t\in(0,T_0]$:

$$\sum_{j=1}^2 |u_j|^{(2+\alpha)}_{Q_{jt}} + |\rho|^{(2+\alpha)}_{\Gamma_t} + |\kappa\partial_t\rho|^{(1+\alpha)}_{\Gamma_t} \le C_1\Big(\sum_{j=1}^2 |u_{0j}|^{(2+\alpha)}_{\Omega_j} + |p|^{(2+\alpha)}_{\Sigma_t}\Big), \tag{1.12}$$

where T_0 *and a constant* C_1 *do not depend on* κ.

Theorem 1.2. *Let* $\Sigma,\ \Gamma\in C^{2+\alpha}$, $\alpha\in(0,1)$. *For any functions* $u_{0j}\in C^{2+\alpha}(\overline{\Omega}_j)$, $j=1,2$, $p\in C^{2+\alpha,1+\alpha/2}_{x\qquad t}(\Sigma_T)$ *satisfying the compatibility conditions of zero and the first order on* Σ *and* Γ *and the condition* $\partial_{\nu_0} u_{0j}\big|_\Gamma\le -d_2$, $j=1,2$, *there exists* $T_0>0$ *such that the Florin problem* (1.7)–(1.10) *has a unique solution* $u_j\in C^{2+\alpha,1+\alpha/2}_{x\qquad t}(\overline{Q}_{jT_0})$, $j=1,2$, $\rho\in C^{2+\alpha,1+\alpha/2}_{x\qquad t}(\Gamma_{T_0})$ *and the following estimate holds for* $t\in(0,T_0]$:

$$\sum_{j=1}^2 |u_j|^{(2+\alpha)}_{Q_{jt}} + |\rho|^{(2+\alpha)}_{\Gamma_t} \le C_2\Big(\sum_{j=1}^2 |u_{0j}|^{(2+\alpha)}_{\Omega_j} + |p|^{(2+\alpha)}_{\Sigma_t}\Big). \tag{1.13}$$

We note that the compatibility conditions for a Florin problem are the compatibility conditions for a Stefan problem with $\kappa=0$.

2. Proof of Theorem 1.1

We apply coordinate transformation [8, 9, 16] to the problem (1.2)–(1.6) to reduce it to the problem in given domains $\Omega_1\cup\Omega_2$

$$\begin{gathered} x = y + \chi(\lambda(y))\,\rho(\xi,\tau)\,N(\xi), \qquad y\in\mathcal{O},\ \xi=\xi(y)\in\Gamma, \\ x=y,\ y\in\overline{\Omega}\backslash\mathcal{O},\ t=\tau, \end{gathered} \tag{2.1}$$

where $\mathcal{O}$ is a $2\lambda_0$-neighborhood of Γ, $\lambda_0 > 0$ is sufficiently small value depending on Γ and such that $\gamma(t) \subset \mathcal{O}$ for $\forall t \in [0, t_0]$, $\lambda(y)$ is the distance between a point $\xi = \xi(y) \in \Gamma$ and a point $y \in \mathcal{O}$ lying on a vector $N(\xi)$ or it's continuation (see [9]), $\chi(\lambda)$ is a smooth cut-off function: $\chi = 1$, $|\lambda| < \lambda_0$, $\chi = 0$, $|\lambda| \geq 2\lambda_0$.

The mapping (2.1) transforms Γ into $\gamma(t)$ and the domains Ω_j into the unknown ones $\Omega_j(t)$, $j = 1, 2$. We keep the variable t instead of a new one τ.

We construct auxiliary functions [18] $\rho_0(\xi, t) \in C_y^{3+\alpha, \frac{3+\alpha}{2}}{}_t(\Gamma_T)$ under the conditions

$$\rho_0\big|_{t=0} = 0, \quad \partial_t\rho_0\big|_{t=0} \equiv \partial_t\rho\big|_{t=0} = -\frac{a_j \Delta u_{0j}\big|_\Gamma}{\nu_0 N^T \partial_{\nu_0} u_{0j}\big|_\Gamma}, \quad j = 1, 2,$$

and $V_j(y, t) \in C_y^{2+\alpha, 1+\alpha/2}{}_t(\mathbb{R}^n_T)$, $j = 1, 2$, as the solutions of the Cauchy problems

$$\partial_t V_j - a_j\, \Delta V_j - \chi\, \partial_t \rho_0 N\, \nabla^T V_j = 0 \ \text{ in } \mathbb{R}^n_T, \tag{2.2}$$

$$V_j\big|_{t=0} = \widetilde{u}_{0j}(y) \text{ in } \mathbb{R}^n. \tag{2.3}$$

These functions satisfy the estimates

$$|\rho_0|^{(3+\alpha)}_{\Gamma_T} \leq C_3 |u_{0j}|^{(2+\alpha)}_{\Omega_j}, \quad |V_j|^{(2+\alpha)}_{\mathbb{R}^n_T} \leq C_4 \sum_{j=1}^{2} |u_{0j}|^{(2+\alpha)}_{\Omega_j}, \quad j = 1, 2. \tag{2.4}$$

Here symbol "$\widetilde{\ }$" denotes the smooth extension of a function into $\mathbb{R}^n$, $\mathbb{R}^n_T = \mathbb{R}^n \times (0, T)$; $\rho\big|_{t=0}$ is found, when we reduce the compatibility conditions. We note also that the functions ρ_0, V_1, V_2 are one and the same for the Stefan and Florin problems.

In the problem (1.2)–(1.6) we make the following substitutions

$$\rho(\xi, t) = \rho_0(\xi, t) + \psi(\xi, t), \quad u_j(y + \chi\, \rho N, t) = v_j(y, t) + V_j(y, t), \quad j = 1, 2, \tag{2.5}$$

where ψ, v_j are the new unknown functions satisfying zero initial conditions $\partial_t^k v_j\big|_{t=0} = 0$, $\partial_t^k \psi\big|_{t=0} = 0$, $k = 0, 1$; $j = 1, 2$.

Jacobian matrix of the transformation (2.1) $J = \{\partial x_i / \partial y_j\}_{1 \leq i, j \leq n}$ may be represented in the form [8]

$$\begin{aligned} J &= \big\{\delta_{ij} + \partial_{y_j}\big(N_i\, \chi(\rho_0 + \psi)\big)\big\}_{1 \leq i,\, j \leq n} \\ &= I + \big(\nabla^T N \chi\, (\rho_0 + \psi)\big)^T := I + J_{01} + J_1 = J_0 + J_1, \\ J_0 &= I + J_{01}, \qquad J_{01} = \big(\nabla^T N \chi\, \rho_0\big)^T, \\ J_1 &= \big(\nabla^T N \chi\, \psi\big)^T = N^T \chi \nabla \psi + \psi \big(\nabla^T (N\chi)\big)^T := J_{11} + J_{12}, \end{aligned} \tag{2.6}$$

where δ_{ij} is a Kronecker delta, I is identity matrix, $\nabla = (\partial_{y_1}, \dots, \partial_{y_n})$.

With the help of the expansion formulae of the inverse Jacobian matrix J^{-1} and J_0^{-1}: $J^{-1} \equiv (I+B)^{-1} = I - BJ^{-1}$, $B = J_{01} + J_1$, $J_0^{-1} \equiv (I + J_{01})^{-1} = I - J_{01}J_0^{-1}$, we extract linear principal terms with respect to unknown functions, known functions and remainder terms containing the rests after separating linear terms and known functions. Then we obtain the problem in a given domain $\Omega_1 \cap \Omega_2$ for the unknown functions v_j, $j = 1, 2$, ψ satisfying zero initial data

$$\partial_t v_j - a_j\,\Delta v_j - (\partial_t\psi - a_j\,\Delta\psi)\,\chi\,N J_0^{-T}\,\nabla^T V_j = f_j(y,t) + F_j(v_j,\psi) \quad \text{in } \Omega_{jT}, \quad j = 1,2, \tag{2.7}$$

$$v_1\big|_\Sigma = p_1(y,t), \qquad t \in (0,T), \tag{2.8}$$

$$v_j\big|_\Gamma = \eta_j(y,t), \qquad t \in (0,T),\ j = 1,2, \tag{2.9}$$

$$\begin{aligned}\big(\lambda_1\,\partial_{\nu_0}v_1 - \lambda_2\,\partial_{\nu_0}v_2 + \kappa\,\nu_0\,N^T\,\partial_t\psi \\ - \nu_0\,N^T\big[(\lambda_1\,\nabla V_1 - \lambda_2\,\nabla V_2)\,J_0^{-1}J_0^{-T} + \kappa\,N J_0^{-T}\,\partial_t\rho_0\big]\,\nabla^T\psi\big)\Big|_\Gamma \\ = \varphi(y,t;\kappa) + \Phi(v_1,v_2,\psi;\kappa)\Big|_\Gamma, \quad t \in (0,T),\end{aligned} \tag{2.10}$$

where the symbol "T" means transposed matrix and column-vector, $\nu_0\,N^T \geq d_1 > 0$,

$$f_j = \chi\,\partial_t\rho_0 N J_0^{-T}\,\nabla^T V_j - \partial_t V_j + a_j(J_0^{-T}\,\nabla^T)^T J_0^{-T}\,\nabla^T V_j, \quad j = 1,2, \tag{2.11}$$

$$\begin{aligned}F_j = \chi\,\partial_t(\rho_0+\psi)\,N J^{-T}(\nabla^T v_j - J_1^T J_0^{-T}\,\nabla^T V_j) \\ + a_j\big[\nabla\,B^T + \big(B^T\,J^{-T}\,\nabla^T\big)^T\,J^{-T} J_{11}^T \\ - (J_0^{-T} J_1^T J^{-T}\,\nabla^T)^T + (J^{-T}\,\nabla^T)^T J^{-T} J_{12}^T\big]\,J_0^{-T}\,\nabla^T V_j \\ - a_j\big[\nabla\,B^T + \big(B^T\,J^{-T}\nabla^T\big)^T\big]\,J^{-T}\nabla^T v_j \\ - a_j(\nabla\psi)\nabla^T(\chi N J_0^{-T}\nabla^T V_j), \quad j = 1,2,\end{aligned} \tag{2.12}$$

$$p_1 = \big(p(y,t) - V_1(y,t)\big)\Big|_\Sigma, \quad \eta_j = -V_j(y,t)\Big|_\Gamma, \quad j = 1,2, \tag{2.13}$$

$$\varphi = -\nu_0\,J_0^{-1}\big[J_0^{-T}\nabla^T(\lambda_1 V_1 - \lambda_2 V_2)\big|_\Gamma + \kappa N^T\partial_t\rho_0\big], \tag{2.14}$$

$$\begin{aligned}\Phi = \nu_0\big(B^T + J^{-1}B\big)\,J^{-T}\nabla^T(\lambda_1\,v_1 - \lambda_2\,v_2) \\ - \nu_0\,\mathcal{M}\nabla^T(\lambda_1\,V_1 - \lambda_2\,V_2) \\ + \kappa\,\nu_0 J^{-1}\,\big(B\,N^T\partial_t\psi + \big(J_{12} - B\,J_{11}\big)\,J_0^{-1}N^T\partial_t\rho_0\big),\end{aligned} \tag{2.15}$$

$\mathcal{M} = J^{-1}\big[B\,J_{11}^T + J_{01}^T J_0^{-T} J_{11}^T - J_0^{-T} J_{12}^T\big]\,J^{-T} + J^{-1}\big(B\,J_{11} - J_{12}\big)\,J_0^{-1} J_0^{-T}.$

In the same manner we reduce Florin problem (1.7)–(1.10) to the problem with unknown functions v_j, $j = 1,2$, ψ satisfying zero initial conditions

$$\partial_t v_j - a_j\,\Delta v_j - (\partial_t\psi - a_j\,\Delta\psi)\,\chi\,N J_0^{-T}\,\nabla^T V_j = f_j(y,t) + F_j(v_j,\psi) \quad \text{in } \Omega_{jT}, \quad j = 1,2, \tag{2.16}$$

$$v_1\big|_\Sigma = p_1(y,t), \quad t\in(0,T), \qquad v_j\big|_\Gamma = \eta_j(y,t), \quad j = 1,2, \tag{2.17}$$

$$\big(\lambda_1\,\partial_{\nu_0} v_1 - \lambda_2\,\partial_{\nu_0} v_2 - \nu_0\,N^T(\lambda_1\,\nabla V_1 - \lambda_2\,\nabla V_2)\,J_0^{-1} J_0^{-T}\,\nabla^T\,\psi\big)\Big|_\Gamma = \varphi(y,t;0) + \Phi(v_1,v_2,\psi;0)\Big|_\Gamma, \quad t\in(0,T), \tag{2.18}$$

where functions f_j, F_j, p_1, η_j, φ, Φ are determined by formulae (2.11)–(2.15).

Theorem 2.1. *Let the assumptions of Theorem* 1.1 *be fulfilled. Then there exists $T_0 > 0$, such that the Stefan problem* (2.7)–(2.10) *has a unique solution $v_j \in \overset{\circ}{C}{}_y^{2+\alpha,}{}_t^{1+\alpha/2}(\overline{\Omega}_{jT_0})$, $j = 1,2$, $\psi \in \overset{\circ}{C}{}_y^{2+\alpha,}{}_t^{1+\alpha/2}(\Gamma_{T_0})$, $\kappa\partial_t\psi \in \overset{\circ}{C}{}_y^{1+\alpha,}{}_t^{\frac{1+\alpha}{2}}(\Gamma_{T_0})$ and this solution satisfies an estimate for $t\le T_0$*

$$\sum_{j=1}^{2}|v_j|_{\Omega_{jt}}^{(2+\alpha)} + |\psi|_{\Gamma_t}^{(2+\alpha)} + |\kappa\partial_t\psi|_{\Gamma_t}^{(1+\alpha)} \le C_5\bigg(\sum_{j=1}^{2}|u_{0j}|_{\Omega_j}^{(2+\alpha)} + |p\,|_{\Sigma_t}^{(2+\alpha)}\bigg), \tag{2.19}$$

where T_0 and a constant C_5 do not depend on κ.

Consider the functions f_j, p_1, η_j, $j = 1,2$, φ determined by (2.11), (2.13), (2.14).

Lemma 2.1. *Let Σ, $\Gamma \in C^{2+\alpha}$, $\alpha\in(0,1)$. For any functions $u_{0j} \in C^{2+\alpha}(\overline{\Omega}_j)$, $j = 1,2$, $p \in C_y^{2+\alpha,}{}_t^{1+\alpha/2}(\Sigma_T)$ satisfying the compatibility conditions of zero and the first order on Σ and Γ there exists $t_1 > 0$, such that $f_j \in \overset{\circ}{C}{}_y^{\alpha,}{}_t^{\alpha/2}(\overline{\Omega}_{jt_1})$, $\eta_j \in \overset{\circ}{C}{}_y^{2+\alpha,}{}_t^{1+\alpha/2}(\Gamma_{t_1})$, $j = 1,2$, $p_1 \in \overset{\circ}{C}{}_y^{2+\alpha,}{}_t^{1+\alpha/2}(\Sigma_{t_1})$, $\varphi \in \overset{\circ}{C}{}_y^{1+\alpha,}{}_t^{\frac{1+\alpha}{2}}(\Gamma_{t_1})$ and an estimate holds*

$$\sum_{j=1}^{2}\big(|f_j|_{\Omega_{jt}}^{(\alpha)} + |\eta_j|_{\Gamma_t}^{(2+\alpha)}\big) + |p_1|_{\Sigma_t}^{(2+\alpha)} + |\varphi|_{\Gamma_t}^{(1+\alpha)} \le C_6\big(\sum_{j=1}^{2}|u_{0j}|_{\Omega_j}^{(2+\alpha)} + |p|_{\Sigma_t}^{(2+\alpha)}\big), \tag{2.20}$$

for $t\le t_1$, $\kappa\in(0,\kappa_0]$, where constant C_6 does not depend on κ.

Proof. This estimate is derived with the help of the estimates (2.4) for the functions ρ_0, V_1, V_2 and an estimate $\|J_0^{-1}\|_{\Gamma_t}^{(\alpha+\nu)} \le 1/(1-q)$, $\nu = 0,1$, $q \in (0,1)$, of the inverse matrix J_0^{-1} existing for $t \le t_1$ under the conditions $\rho_0(\xi(y),t) \in C_y^{3+\alpha,\frac{3+\alpha}{2}}{}_t(\Gamma_T)$, $\rho_0\big|_{t=0} = 0$ (see [8]) (here $\|\{a_{ij}\}_{1\le i,j\le n}\|_{\Gamma_T}^{(l)} := n \max_{i,j} |a_{ij}|_{\Gamma_T}^{(l)}$). The functions f_j satisfy zero initial data by (2.2), (2.3), the functions p_1, η_j, $j = 1,2$, φ — due to the compatibility conditions. □

Consider a linear problem with the unknowns satisfying zero initial data

$$\partial_t Z_j - a_j\,\Delta Z_j - \alpha_j(x,t)\big(\partial_t\Psi - a_j\,\Delta\Psi\big) = f_j(x,t) \quad \text{in } \Omega_{jT},\ j = 1,2, \tag{2.21}$$

$$Z_1\big|_{\Sigma} = p_1(x,t), \quad t \in (0,T), \tag{2.22}$$

$$Z_j\big|_{\Gamma} = \eta_j(x,t), \quad t \in (0,T),\ j = 1,2, \tag{2.23}$$

$$\big(\lambda_1\,\partial_{\nu_0} Z_1 - \lambda_2\,\partial_{\nu_0} Z_2\big)\big|_{\Gamma} + \kappa\,\partial_t\Psi + d(x,t)\,\nabla^T\Psi = \varphi(x,t), \quad t \in (0,T), \tag{2.24}$$

where λ_j, a_j are positive constants, $j = 1,2$, $d = (d_1,\dots,d_n)$.

Theorem 2.2. *Let* Σ, $\Gamma \in C^{2+\alpha}$, $\alpha \in (0,1)$, $\alpha_j(x,t) \in C_x^{\alpha,\alpha/2}{}_t(\overline{\Omega}_{jT})$, $d_i(x,t) \in C_x^{1+\alpha,1+\alpha/2}{}_t(\Gamma_T)$, $j = 1,2$, $i = 1,\dots,n$, *and*

$$0 < \kappa \le \kappa_0, \quad \alpha_j(x,0)\big|_{\Gamma} \le -d_3 < 0, \quad j = 1,2. \tag{2.25}$$

Then for every functions $f_j \in \overset{\circ}{C}{}_x^{\alpha,\alpha/2}{}_t(\overline{\Omega}_{jT})$, $p_1 \in \overset{\circ}{C}{}_x^{2+\alpha,1+\alpha/2}{}_t(\Sigma_T)$, $\eta_j \in \overset{\circ}{C}{}_x^{2+\alpha,1+\alpha/2}{}_t(\Gamma_T)$, $j = 1,2$, $\varphi \in \overset{\circ}{C}{}_x^{1+\alpha,\frac{1+\alpha}{2}}{}_t(\Gamma_T)$ *the problem* (2.21)–(2.24) *has a unique solution* $Z_j \in \overset{\circ}{C}{}_x^{2+\alpha,1+\alpha/2}{}_t(\overline{\Omega}_{jT})$, $j = 1,2$, $\Psi \in \overset{\circ}{C}{}_x^{2+\alpha,1+\alpha/2}{}_t(\Gamma_T)$, $\kappa\partial_t\Psi \in \overset{\circ}{C}{}_x^{1+\alpha,\frac{1+\alpha}{2}}{}_t(\Gamma_T)$ *and it satisfies an estimate*

$$\sum_{j=1}^{2} |Z_j|_{\Omega_{jt}}^{(2+\alpha)} + |\Psi|_{\Gamma_t}^{(2+\alpha)} + |\kappa\partial_t\Psi|_{\Gamma_t}^{(1+\alpha)} \le C_7\bigg(\sum_{j=1}^{2}\big(|f_j|_{\Omega_{jt}}^{(\alpha)} + |\eta_j|_{\Gamma_t}^{(2+\alpha)}\big) + |p_1|_{\Sigma_t}^{(2+\alpha)} + |\varphi|_{\Gamma_t}^{(1+\alpha)}\bigg), \quad t \le T, \tag{2.26}$$

where T and constant C_7 do not depend on κ.

This theorem is proved by standard technique. The proof is based on the following model problem with unknown functions $\psi(x',t)$, $u_j(x,t)$, $j=1,2$,

$$\begin{gathered}\partial_t u_j - a_j\,\Delta u_j = 0 \ \text{ in } \ D_{jT}, \quad j=1,2,\\ u_j\big|_{t=0} = 0 \text{ in } \ D_j, \quad j=1,2;\\ \psi\big|_{t=0} = 0 \text{ on } R;\\ u_j + \alpha_j\psi = 0 \text{ on } R_T, \quad j=1,2,\\ b\nabla^T u_1 - c\nabla^T u_2 + h'\nabla'^T\psi + \kappa\,\partial_t\psi = g(x',t) \text{ on } R_T,\end{gathered} \tag{2.27}$$

where all coefficients are constant; $D_1 := \mathbb{R}^n_-$, $D_2 := \mathbb{R}^n_+$, $D_{jT} := D_j \times (0,T)$; R is a plane $x_n = 0$ in $\mathbb{R}^n$, $R_T := R \times [0,T]$; $b = (b_1,\dots,b_n)$, $c = (c_1,\dots,c_n)$, $h' = (h_1,\dots,h_{n-1})$; α_j, $j=1,2$, are coefficients $\alpha_j(\xi_0,0)$, $\xi_0 \in \Gamma$ in the equations (2.21).

In the Hölder spaces this problem with arbitrary κ was studied by B. V. Bazaliy [1], E. V. Radkevich [20], G. I. Bizhanova [4]. J. F. Rodrigues, V. A. Solonnikov, F. Yi [21] have established the uniform on κ estimates of the solution of a one-phase problem.

In [7] the following theorem was proved.

Theorem 2.3. *Let $\alpha_j < 0$, $j=1,2$, $b_n > 0$, $c_n > 0$, $0 < \kappa \le \kappa_0$. For every function $g \in \overset{\circ}{C}{}^{1+\alpha,\frac{1+\alpha}{2}}_{x'\ \ \ t}(R_T)$, $\alpha \in (0,1)$, the problem* (2.27) *has a unique solution $u_j \in \overset{\circ}{C}{}^{2+\alpha,1+\alpha/2}_{x\ \ \ \ \ t}(D_{jT})$, $j=1,2$, $\psi \in \overset{\circ}{C}{}^{2+\alpha,1+\alpha/2}_{x'\ \ \ \ \ t}(R_T)$, $\kappa\,\partial_t\psi \in \overset{\circ}{C}{}^{1+\alpha,\frac{1+\alpha}{2}}_{x'\ \ \ t}(R_T)$, and it satisfies the estimate*

$$\sum_{j=1}^{2} |u_j|^{(2+\alpha)}_{D_{jT}} + |\psi|^{(2+\alpha)}_{R_T} + |\kappa\partial_t\psi|^{(1+\alpha)}_{R_T} \le C_8 |g|^{(1+\alpha)}_{R_T},$$

where T and a constant C_8 do not depend on κ.

Proof of Theorem 2.1. We introduce the Hölder spaces. Let $\overset{\circ}{\mathcal{D}}{}^{2+\alpha}(\Gamma_T)$ be the space of functions $\psi(\xi,t)$ such that $\psi(\xi,t) \in \overset{\circ}{C}{}^{2+\alpha,1+\alpha/2}_{y\ \ \ \ \ t}(\Gamma_T)$, $\kappa\partial_t\psi \in \overset{\circ}{C}{}^{1+\alpha,\frac{1+\alpha}{2}}_{y\ \ \ t}(\Gamma_T)$. Let

$$\mathcal{B}(\Omega_T) := \overset{\circ}{C}{}^{2+\alpha,1+\alpha/2}_{y\ \ \ \ \ t}(\overline{\Omega}_{1T}) \times \overset{\circ}{C}{}^{2+\alpha,1+\alpha/2}_{y\ \ \ \ \ t}(\overline{\Omega}_{2T}) \times \overset{\circ}{\mathcal{D}}{}^{2+\alpha}(\Gamma_T),$$

$$\begin{aligned}\mathcal{H}(\Omega_T) := {}& \overset{\circ}{C}{}^{\alpha,\alpha/2}_{y\ \ t}(\overline{\Omega}_{1T}) \times \overset{\circ}{C}{}^{\alpha,\alpha/2}_{y\ \ t}(\overline{\Omega}_{2T}) \times \overset{\circ}{C}{}^{2+\alpha,1+\alpha/2}_{y\ \ \ \ \ t}(\Sigma_T)\\ &\times \overset{\circ}{C}{}^{2+\alpha,1+\alpha/2}_{y\ \ \ \ \ t}(\Gamma_T) \times \overset{\circ}{C}{}^{2+\alpha,1+\alpha/2}_{y\ \ \ \ \ t}(\Gamma_T) \times \overset{\circ}{C}{}^{1+\alpha,\frac{1+\alpha}{2}}_{y\ \ \ t}(\Gamma_T)\end{aligned}$$

be the spaces of the functions $w=(v_1,v_2,\psi)$ and $h=(f_1,f_2,p_1,\eta_1,\eta_2,\varphi)$ respectively with the norms

$$\|w\|_{\mathcal{B}(\Omega_T)}:=\sum_{j=1}^{2}|v_j|^{(2+\alpha)}_{\Omega_{jT}}+|\psi|^{(2+\alpha)}_{\Gamma_T}+|\kappa\partial_t\psi|^{(1+\alpha)}_{\Gamma_T},$$

$$\|h\|_{\mathcal{H}(\Omega_T)}:=\sum_{j=1}^{2}|f_j|^{(\alpha)}_{\Omega_{jT}}+|p_1|^{(2+\alpha)}_{\Sigma_T}+\sum_{j=1}^{2}|\eta_j|^{(2+\alpha)}_{\Gamma_T}+|\varphi|^{(1+\alpha)}_{\Gamma_T}.$$

We write the problem (2.7)–(2.10) in the operator form

$$\mathcal{A}[w]=h+\mathcal{N}[w], \tag{2.28}$$

where $w=(v_1,v_2,\psi)$ is unknown vector, $h=(f_1,f_2,p_1,\eta_1,\eta_2,\varphi)$ — given one, $\mathcal{A}$ is a linear operator determined by all the terms in the left-hand sides of the equations and conditions of the problem (2.7)–(2.10), $\mathcal{N}=(F_1,\,F_2,\,0,\,0,\,0,\,\Phi)$ — nonlinear operator, and $\mathcal{A}\colon\mathcal{B}(\Omega_T)\to\mathcal{H}(\Omega_T)$, $\mathcal{N}\colon\mathcal{B}(\Omega_T)\to\mathcal{H}(\Omega_T)$.

In the left-hand sides of the equations and conditions of the problem (2.7)–(2.10) there are the same linear terms as in the problem (2.21)–(2.24). The condition (2.25): $\alpha_j(x,0)\big|_\Gamma\le -d_3<0$ with $\alpha_j(x,0)\big|_\Gamma=\chi N J_0^{-T}\nabla^T V_j\big|_{\Gamma,\,t=0}=\partial_N u_{0j}\big|_\Gamma=\nu_0 N^T\partial_{\nu_0}u_{0j}\big|_\Gamma$ is fulfilled by $\nu_0N^T\ge d_1>0$ and (1.11). So due to Theorem 2.2 and an estimate (2.26) we can represent the problem (2.28) in the form

$$w=\mathcal{A}^{-1}[h+\mathcal{N}[w]] \tag{2.29}$$

and obtain an estimate

$$\begin{aligned}\|w\|_{\mathcal{B}(\Omega_T)}&\equiv\|\mathcal{A}^{-1}\left[h+\mathcal{N}[w]\right]\|_{\mathcal{B}(\Omega_T)}\\&\le C_9\biggl(\|h\|_{\mathcal{H}(\Omega_T)}+\sum_{j=1}^{2}|F_j(v_j,\psi)|^{(\alpha)}_{\Omega_{jT}}+|\Phi(v_1,v_2,\psi;\kappa)|^{(1+\alpha)}_{\Gamma_T}\biggr).\end{aligned} \tag{2.30}$$

Let $B(M)\subset\mathcal{B}(\Omega_{T_0})$ be a closed ball with the center at zero: $B(M):=\{w\,|\,v_j\in\overset{\circ}{C}{}^{2+\alpha,1+\alpha/2}_{y\quad\ t}(\overline{\Omega}_{jT_0}),\ j=1,2,\ \psi\in\overset{\circ}{C}{}^{2+\alpha,1+\alpha/2}_{y\quad\ t}(\Gamma_{T_0}),\ \kappa\partial_t\psi\in\overset{\circ}{C}{}^{1+\alpha,\frac{1+\alpha}{2}}_{y\quad\ t}(\Gamma_{T_0}),\ \|w\|_{\mathcal{B}(\Omega_{T_0})}\le M,\ t\le T_0\}$, $M=C_9\|h\|_{\mathcal{H}(\Omega_{T_0})}(1-q)^{-1}$, $q\in(0,1)$.

To prove that an operator $\mathcal{A}^{-1}[h+\mathcal{N}[w]]$ acts from the closed ball $B(M)$ into itself and is a contractive one we estimate the norm (2.30) and the following one

$$\|\mathcal{A}^{-1}[h+\mathcal{N}[w]]-\mathcal{A}^{-1}[h+\mathcal{N}[\widetilde{w}]]\|_{\mathcal{B}(\Omega_t)} \equiv \|\mathcal{A}^{-1}\big[\mathcal{N}[w]-\mathcal{N}[\widetilde{w}]\big]\|_{\mathcal{B}(\Omega_t)} \le C_9\bigg(\sum_{j=1}^{2}|F_j(v_j,\psi)-F_j(\widetilde{v}_j,\widetilde{\psi})|^{(\alpha)}_{\Omega_{jt}} + |\Phi(v_1,v_2,\psi;\kappa)-\Phi(\widetilde{v}_1,\widetilde{v}_2,\widetilde{\psi};\kappa)|^{(1+\alpha)}_{\Gamma_t}\bigg) \quad (2.31)$$

for $\forall\, w,\widetilde{w}\in B(M)$.

With the help of the estimates $\|J^{-1}\|^{(\alpha+\nu)}_{\Gamma_t}\le C_{10}\big(1+t^{\frac{1-\nu}{2}}\,|\psi|^{(2+\alpha)}_{\Gamma_t}\big)$, $t\le t_2$; $\|J_0^{-1}\|^{(\alpha+\nu)}_{\Gamma_t}\le 1/(1-q)$, $q\in(0,1)$, $t\le t_1$; $\|J_{11}\|^{(\alpha+\nu)}_{\Gamma_t}\le C_{11}\,t^{\frac{1-\nu}{2}}|\psi|^{(2+\alpha)}_{\Gamma_t}$, $\|J_{12}\|^{(\alpha+\nu)}_{\Gamma_t}\le C_{12}\,t^{\frac{2-\nu}{2}}|\psi|^{(2+\alpha)}_{\Gamma_t}$, $\|J_{01}\|^{(\alpha+\nu)}_{\Gamma_t}\le C_{13}\,t^{\frac{2+\alpha-\nu}{2}}\times |\rho_0|^{(3+\alpha)}_{\Gamma_t}$, $\nu=0,1$, of the inverse Jacobian matrix J^{-1} and J_0^{-1} and the matrices $J_1=J_{11}+J_{12}$, J_{01} determined by (2.6) we evaluate the norms (2.30), (2.31) containing the functions (2.12) F_j, $j=1,2$, and (2.15) Φ, then we derive

$$\|\mathcal{A}^{-1}[h+\mathcal{N}[w]]\|_{\mathcal{B}(\Omega_t)}\le C_9\,\|h\|_{\mathcal{H}(\Omega_t)}+r_1(t,|\psi|^{(2+\alpha)}_{\Gamma_t})\,\|w\|_{\mathcal{B}(\Omega_t)}, \quad (2.32)$$

$$\|\mathcal{A}^{-1}\big[\mathcal{N}[w]-\mathcal{N}[\widetilde{w}]\big]\|_{\mathcal{B}(\Omega_t)} \le r_2(t,|v_1|^{(2+\alpha)}_{\Omega_{1t}},\,|v_2|^{(2+\alpha)}_{\Omega_{2t}},\,|\psi|^{(2+\alpha)}_{\Gamma_t})\,\|w-\widetilde{w}\|_{\mathcal{B}(\Omega_t)}, \quad (2.33)$$

where $r_1(0,M)=0$, $r_2(0,M,M,M)=0$.

We find T_1 from the inequalities $r_1(t,M)\le q$, $r_2(t,M,M,M)\le q$, $q\in(0,1)$, then from (2.32) and (2.33) we shall have the estimates

$$\|\mathcal{A}^{-1}[h+\mathcal{N}[w]]\|_{\mathcal{B}(\Omega_t)}\le C_9\,\|h\|_{\mathcal{H}(\Omega_t)}+q\|w\|_{\mathcal{B}(\Omega_t)} \le C_9\,\|h\|_{\mathcal{H}(\Omega_t)}+q\,M\le M\equiv\;C_9\,\|h\|_{\mathcal{H}(\Omega_{T_0})}(1-q)^{-1}, \quad (2.34)$$

$$\|\mathcal{A}^{-1}[h+\mathcal{N}[w]]-\mathcal{A}^{-1}[h+\mathcal{N}[\widetilde{w}]]\|_{\mathcal{B}(\Omega_t)}\le q\,\|w-\widetilde{w}\|_{\mathcal{B}(\Omega_t)} \quad (2.35)$$

for all w, $\widetilde{w}\in B(M)$, $\forall\,t\le T_0=\min(t_0,\,t_1,\,t_2,\,T_1)$ (the parametrization of a free boundary (1.1) is valid for $t\le t_0$; for $t\le t_1$ and $t\le t_2$ the inverse matrices J_0^{-1} and J^{-1} exist).

From (2.34) and (2.35) by contraction mapping principle it follows that the problem (2.28) or (2.7)–(2.10) has a unique solution $w=(v_1,v_2,\psi)\in\mathcal{B}(\Omega_{T_0})$. We can see that T_0 and a constant $C_9(1-q)^{-1}$ do not depend on κ.

From (2.29) by (2.34) it follows $\|w\|_{\mathcal{B}(\Omega_t)} \le C_9\,(1-q)^{-1}\,\|h\|_{\mathcal{H}(\Omega_t)}$. Applying an estimate (2.20) for the vector h we find an estimate (2.19)

$$\|w\|_{\mathcal{B}(\Omega_t)} \le C_9\,(1-q)^{-1}\,\|h\|_{\mathcal{H}(\Omega_t)} \le C_5\Big(\sum_{j=1}^{2}|u_{0j}|^{(2+\alpha)}_{\Omega_j}+|p|^{(2+\alpha)}_{\Sigma_t}\Big),\quad t\le T_0, \quad (2.36)$$

with a constant $C_5 = C_6\,C_9(1-q)^{-1}$ independent on κ. □

From the formulae (2.5) with $x = y + \chi\,(\rho_0+\psi)\,N$ and estimates (2.4) for V_j, $j=1,2$, and ρ_0 we shall have Theorem 1.1 and estimate (1.12).

3. Proof of Theorem 1.2

We write down an index κ at the functions v_j, $j=1,2$, ψ of the Stefan problem (2.7)–(2.10). Due to Theorem 2.1 this problem has a unique solution $v_{j\kappa}\in \overset{\circ}{C}{}^{2+\alpha,1+\alpha/2}_{y\ \ t}(\overline{\Omega}_{jT_0})$, $j=1,2$, $\psi_\kappa\in \overset{\circ}{C}{}^{2+\alpha,1+\alpha/2}_{y\ \ t}(\Gamma_{T_0})$, $\kappa\partial_t\psi_\kappa \in \overset{\circ}{C}{}^{1+\alpha,\frac{1+\alpha}{2}}_{y\ \ t}(\Gamma_{T_0})$ and it satisfies a uniform with respect to $\kappa\in(0,\kappa_0]$ estimate (2.36) ((2.19)) for $t\le T_0$:

$$\sum_{j=1}^{2}|v_{j\kappa}|^{(2+\alpha)}_{\Omega_{jt}}+|\psi_\kappa|^{(2+\alpha)}_{\Gamma_t}+|\kappa\partial_t\psi_\kappa|^{(1+\alpha)}_{\Gamma_t} \le C_5\Big(\sum_{j=1}^{2}|u_{0j}|^{(2+\alpha)}_{\Omega_j}+|p|^{(2+\alpha)}_{\Sigma_t}\Big). \quad (3.1)$$

From here it follows that the sequences $\{v_{j\kappa}\}$, $j=1,2$, $\{\psi_\kappa\}$, as $\kappa\to 0$, are compact in $\overset{\circ}{C}{}^{2,1}_{y\,t}(\overline{\Omega}_{jT_0})$, $\overset{\circ}{C}{}^{2,1}_{y\,t}(\Gamma_{T_0})$ respectively. We choose the converging subsequences

$$\{v_{j\kappa_n}\},\quad j=1,2,\quad \{\psi_{\kappa_n}\},\ \kappa_n\to 0, \quad (3.2)$$

and denote

$$\lim_{\kappa_n\to 0} v_{j\kappa_n}=v_j,\qquad \lim_{\kappa_n\to 0}\psi_{\kappa_n}=\psi, \quad (3.3)$$

where $v_j\in \overset{\circ}{C}{}^{2,1}_{y\,t}(\overline{\Omega}_{jT_0})$, $\psi\in \overset{\circ}{C}{}^{2,1}_{y\,t}(\Gamma_{T_0})$. These functions satisfy an estimate

$$\sum_{j=1}^{2}|v_j|_{C^{2,1}_{y\,t}(\overline{\Omega}_{jt})}+|\psi|_{C^{2,1}_{y\,t}(\Gamma_t)} \le C_5\Big(\sum_{j=1}^{2}|u_{0j}|^{(2+\alpha)}_{\Omega_j}+|p|^{(2+\alpha)}_{\Sigma_t}\Big),\quad t\le T_0, \quad (3.4)$$

which is derived from an estimate (3.1) due to (3.3). To show that the functions v_j, $j = 1, 2$, ψ possess higher smoothness we estimate the Hölder constants

$$[\partial_y^2 v_j]^{(\alpha)}_{\Omega_{jT_0}}, \quad [\partial_t v_j]^{(\alpha)}_{\Omega_{jT_0}}, \quad [\partial_y v_j]^{(\frac{1+\alpha}{2})}_{t,\Omega_{jT_0}}, \quad [\partial_y^2 \psi]^{(\alpha)}_{\Gamma_{T_0}}, \quad [\partial_t \psi]^{(\alpha)}_{\Gamma_{T_0}}, \quad [\partial_y \psi]^{(\frac{1+\alpha}{2})}_{t,\Gamma_{T_0}}. \tag{3.5}$$

We evaluate, for instance, the difference $\partial_t\psi(y,t) - \partial_t\psi(z,t)$

$$|\partial_t\psi(y,t) - \partial_t\psi(z,t)| \le |\partial_t\psi(y,t) - \partial_t\psi_{\kappa_n}(y,t)| \\ + |\partial_t\psi(z,t) - \partial_t\psi_{\kappa_n}(z,t)| + |\partial_t\psi_{\kappa_n}(y,t) - \partial_t\psi_{\kappa_n}(z,t)|. \tag{3.6}$$

In (3.6) we apply an estimate (3.1) for the function ψ_{κ_n}

$$|\partial_t\psi_{\kappa_n}(y,t) - \partial_t\psi_{\kappa_n}(z,t)| \le [\partial_t\psi_\kappa]^{(\alpha)}_{y,\,\Gamma_{T_0}} |y-z|^\alpha \\ \le C_5\left(\sum_{j=1}^{2} |u_{0j}|^{(2+\alpha)}_{\Omega_j} + |p|^{(2+\alpha)}_{\Sigma_{T_0}}\right)|y-z|^\alpha$$

and let $\kappa_n \to 0$, then due to (3.3) we obtain an inequality

$$|\partial_t\psi(y,t) - \partial_t\psi(z,t)| \le C_5\left(\sum_{j=1}^{2} |u_{0j}|^{(2+\alpha)}_{\Omega_j} + |p|^{(2+\alpha)}_{\Sigma_{T_0}}\right)|y-z|^\alpha, \quad t \le T_0,$$

which leads to the estimate of the Hölder constant

$$[\partial_t\psi]^{(\alpha)}_{y,\,\Gamma_{T_0}} \le C_5\left(\sum_{j=1}^{2} |u_{0j}|^{(2+\alpha)}_{\Omega_j} + |p|^{(2+\alpha)}_{\Sigma_{T_0}}\right). \tag{3.7}$$

We obtain such estimates for the all other Hölder constants in (3.5). On the basis of (3.4) and estimates of the Hölder constants, as (3.7) we shall have for the limit functions (3.3) that $v_j \in \overset{\circ}{C}{}^{2+\alpha,1+\alpha/2}_{\;y\quad\;\; t}(\overline{\Omega}_{jT_0})$, $j = 1, 2$, $\psi \in \overset{\circ}{C}{}^{2+\alpha,1+\alpha/2}_{\;y\quad\;\; t}(\Gamma_{T_0})$ and

$$\sum_{j=1}^{2} |v_j|^{(2+\alpha)}_{\Omega_{jt}} + |\psi|^{(2+\alpha)}_{\Gamma_t} \le C_5\left(\sum_{j=1}^{2} |u_{0j}|^{(2+\alpha)}_{\Omega_j} + |p|^{(2+\alpha)}_{\Sigma_t}\right), \quad t \le T_0. \tag{3.8}$$

To show that the limit functions v_j, $j = 1, 2$, ψ satisfy the Florin problem (2.16)–(2.18) we rewrite the problem (2.7)–(2.10) for the functions of the subsequences (3.2) and with κ_n instead of κ in a Stefan condition (2.10), in this problem we let κ_n to 0 taking into account (3.3), then we obtain that the functions v_j, $j = 1, 2$, ψ are the solution of the problem (2.16)–(2.18).

We prove a uniqueness of the solution of a Florin problem (2.16)–(2.18). For that we assume there are two solutions of this problem $w = (v_1, v_2, \psi)$ and $\widetilde{w} = (\widetilde{v}_1, \widetilde{v}_2, \widetilde{\psi})$ and let $\{w_{\kappa_n}\}$ and $\{\widetilde{w}_{\kappa_n}\}$ be subsequences converging to w and $\widetilde{w}$ as $\kappa_n \to 0$ respectively. We consider Stefan problem (2.29) written for the functions of the subsequences w_{κ_n} and $\widetilde{w}_{\kappa_n}$ and estimate the difference $w_{\kappa_n} - \widetilde{w}_{\kappa_n} = \mathcal{A}^{-1}[h + \mathcal{N}[w_{\kappa_n}]] - \mathcal{A}^{-1}[h + \mathcal{N}[\widetilde{w}_{\kappa_n}]] = \mathcal{A}^{-1}\big[\mathcal{N}[w] - \mathcal{N}[\widetilde{w}]\big]$ using (2.31)

$$\sum_{j=1}^{2} |v_{j\kappa_n} - \widetilde{v}_{j\kappa_n}|_{\Omega_{jt}}^{(2+\alpha)} + |\psi_{\kappa_n} - \widetilde{\psi}_{\kappa_n}|_{\Gamma_t}^{(2+\alpha)}$$
$$\leq C_9 \Bigg(\sum_{j=1}^{2} |F_j(v_{j\kappa_n}, \psi_{\kappa_n}) - F_j(\widetilde{v}_{j\kappa_n}, \widetilde{\psi}_{\kappa_n})|_{\Omega_{jt}}^{(\alpha)}$$
$$+ |\Phi(v_1, v_2, \psi_{\kappa_n}; \kappa_n) - \Phi(\widetilde{v}_{1\kappa_n}, \widetilde{v}_{2\kappa_n}, \widetilde{\psi}_{\kappa_n}; \kappa_n)|_{\Gamma_t}^{(1+\alpha)} \Bigg).$$

We let κ_n to zero and apply the estimates (2.33), (2.35)

$$\sum_{j=1}^{2} |v_j - \widetilde{v}_j|_{\Omega_{jt}}^{(2+\alpha)} + |\psi - \widetilde{\psi}|_{\Gamma_t}^{(2+\alpha)} \leq C_9 \Bigg(\sum_{j=1}^{2} |F_j(v_j, \psi) - F_j(\widetilde{v}_j, \widetilde{\psi})|_{\Omega_{jt}}^{(\alpha)}$$
$$+ |\Phi(v_1, v_2, \psi; 0) - \Phi(\widetilde{v}_1, \widetilde{v}_2, \widetilde{\psi}; 0)|_{\Gamma_t}^{(1+\alpha)} \Bigg)$$
$$\leq r_2(t, M, M, M) \Bigg(\sum_{j=1}^{2} |v_j - \widetilde{v}_j|_{\Omega_{jt}}^{(2+\alpha)} + |\psi - \widetilde{\psi}|_{\Gamma_t}^{(2+\alpha)} \Bigg),$$

$$\sum_{j=1}^{2} |v_j - \widetilde{v}_j|_{\Omega_{jt}}^{(2+\alpha)} + |\psi - \widetilde{\psi}|_{\Gamma_t}^{(2+\alpha)}$$
$$\leq q \Bigg(\sum_{j=1}^{2} |v_j - \widetilde{v}_j|_{\Omega_{jt}}^{(2+\alpha)} + |\psi - \widetilde{\psi}|_{\Gamma_t}^{(2+\alpha)} \Bigg), \quad t \in (0, T_0],$$

where $q \in (0, 1)$. This inequality leads to the identity $w \equiv \widetilde{w}$ and to the uniqueness of the solution of Florin problem (2.16)–(2.18).

From the formulae (2.5) with $x = y + \chi N\rho$

$$\rho := \rho_0 + \psi, \qquad u_j(x, t) := v_j(x - \chi N\rho, t) + V_j(x - \chi N\rho, t), \quad j = 1, 2, \tag{3.9}$$

we obtain that $\rho \in C_{x\ t}^{2+\alpha,1+\alpha/2}(\Gamma_{T_0})$, $u_j \in C_{x\ t}^{2+\alpha,1+\alpha/2}(\overline{Q}_{jT_0})$, $j = 1,2$, and with the help of the estimates (2.4) for the functions ρ_0, V_j; (3.8) for v_j, ψ , we have got an estimate (1.13) for the functions $u_j(x,t)$, $j = 1,2$, and ρ.

Obtained functions u_j, $j = 1,2$, and ρ (3.9) are the solution of the Florin problem (1.7)–(1.10). Really, we substitute them into equations and conditions (1.7)–(1.10), make coordinate transformation (2.1) and substitutions (2.5) with ρ and u_j, determined by (3.9), then we obtain for the functions v_j, $j = 1,2$, and ψ the Florin problem (2.16)–(2.18). As it was proved, these functions are the unique solution of the problem (2.16)–(2.18), that is the functions $u_j(x,t)$, $j = 1,2$, and ρ determined by (3.9) are the unique solution of the Florin problem (1.7)–(1.10). □

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Contact information

Galina I. Bizhanova
Institute of Mathematics,
Pushkin Str. 125,
Almaty, 050010,
Kazakhstan
E-Mail: galya@math.kz,
galina_math@mail.ru