Український математичний вісник
Том 5 (**2008**), № 1, 1 – 15

Wiener criterion for relaxed Dirichlet problem relative to Riemannian p-homogeneous Dirichlet forms

Marco Biroli, Florangela Dal Fabbro, Silvana Marchi

(*Presented by A. E. Shishkov*)

Abstract. We give a Wiener criterion for the relaxed Dirichlet problem relative to a Riemannian p-homogeneous Dirichlet form.

2001 MSC. 31C25, 35B65, 35J70.

Key words and phrases. Dirichlet Forms, Relaxed Dirichlet Problem, Wiener criterion.

Introduction

The Wiener criterion for a relaxed Dirichlet problem has been firstly investigated in an Euclidean framework for linear ellptic coercive operators with bounded measurable coefficients by Dal Maso and Mosco, [18, 19]. The result of Dal Maso and Mosco has been generalized to the case of the subelliptic p-Laplace operator with $p > 1$ (i.e. constructed by means of vector fields satisfying an Hörmander condition), also when we have a source term in a suitably defined Kato class, in [4, 5, 11]. The Kato class of measures relative to a Riemannian bilinear Dirichlet form has been introduced in [10] and the definition has been refined in [6], where Schrödinger problems relative to a Riemannian bilinear Dirichlet form with a potential in the Kato class have been investigated.

A generalization of the definition of strongly local Dirichlet forms to the p-homogeneous case has been given in [7,13,14]. The definition of Riemannian (p-homogeneous) Dirichlet form is given in [15] where the local regularity and the Harnack inequality for the harmonic functions is studied; moreover the Kato class relative to a Riemannian (p-homogeneous)

Received 29.11.2007
The first author has been supported by the MIUR research project n. 2005010173.

ISSN 1810 – 3200.

Dirichlet form has been defined in [5] where the local regularity and Harnack inequality have been proved for the harmonics of a Schrödinger type problem with a potential in Kato class. We recall also that a Wiener criterion at the boundary for harmonic functions relative to a Riemannian (p-homogeneous) Dirichlet form has been proved in [7].

In the present paper we are interested in the Wiener criterion for the solutions of the relaxed Dirichlet problem relative to a Riemannian (p-homogeneous) Dirichlet form. The interest of relaxed Dirichlet problems is twofold:

(1) From the Wiener criterion for relaxed Dirichlet problems a Wiener criterion for regular point of the boundary follows (at least for boundary data derived from functions in the domain of the form relative to the entire space in consideration)

(2) The class of relaxed Dirichlet problems is closed for Γ-convergence and in particular the Γ-limits of Dirichlet problems in open sets with holes and homogeneous Dirichlet condition on the boundary of holes are relaxed Dirichlet problems.

The paper is organized in sections. In the second section we give the definition and the main properties of p-homogeneous Dirichlet forms in the general and in the Riemannian case. In the third section we give the definition of Kato class relative to a Riemannian p-homogeneous Dirichlet form. In the fourth section we introduce the relaxed Dirichlet problem and the relative capacity. In the fifth section we give our main result concerning the Wiener criterion for a relaxed Dirichlet problem and finally in the sixth section we give a sketch of the proof of our result.

1. Dirichlet Functionals and Forms

For the definition and properties of bilinear Dirichlet forms we refer to the book [20] and for the Riemannian case to the paper [9]. We observe that in the nonlinear case we do not have an extension of Beuerling–Deny decomposition formula then we try to define directly a strongly local form.

Firstly we describe the notion of strongly local p-homogeneous Dirichlet form, $p > 1$, as given in [13, 14].

We start with the notion of Dirichlet functional. We consider a locally compact separable Hausdorff space X with a metrizable topology and a positive Radon measure m on X such that $\mathrm{supp}[m] = X$. Let $\Phi : L^p(X, m) \to [0, +\infty]$, $p > 1$, be a l.s.c. strictly convex functional with domain D, i.e. $D = \{v \in L^p(X, m) : \ \Phi(v) < +\infty\}$, such that $\Phi(0) = 0$.

We assume that D is dense in $L^p(X, m)$ and that the following conditions hold:

(H_1) D is a dense linear subspace of $L^p(X, m)$, which can be endowed with a norm $\|\cdot\|_D$; moreover D has a structure of Banach space with respect to the norm $\|\cdot\|_D$ and the following estimate holds

$$c_1\|v\|_D^p \leq \Phi_1(v) = \Phi(v) + \int_X |v|^p \, dm \leq c_2\|v\|_D^p$$

for every $v \in D$, where $c_1,\ c_2$ are positive constants.

(H_2) We denote by D_0 the closure of $D \cap C_0(X)$ in D (with respect to the norm $\|\cdot\|_D$) and we assume that $D \cap C_0(X)$ is dense in $C_0(X)$ for the uniform convergence on X.

(H_3) For every $u, v \in D \cap C_0(X)$ we have $u \vee v \in D \cap C_0(X)$, $u \wedge v \in D \cap C_0(X)$ and

$$\Phi(u \vee v) + \Phi(u \wedge v) \leq \Phi(u) + \Phi(v)$$

The functional Φ satisfying the assumptions (H_1), (H_2), (H_3) is called a *Dirichlet functional.*

We recall that (under the above assumptions) we can define a Choquet capacity cap(E). Moreover we can also define in a natural way the quasi-continuity of a function and prove that every function in D_0 is quasi-continuous and is defined quasi-everywhere (i.e. up to sets of zero capacity), [14].

The assumptions (H_1), (H_2), (H_3) have a global character; now we will recall the definition of *strongly local Dirichlet functional* with a homogeneity degree $p > 1$. Let Φ satisfy (H_1), (H_2), (H_3); we say that Φ is a *strongly local Dirichlet functional* with a homogeneity degree $p > 1$ if the following conditions hold:

(H_4) Φ has the following representation on D_0: $\Phi(u) = \int_X \alpha(u)(dx)$ where α is a non-negative bounded Radon measure depending on $u \in D_0$, which does not charge sets of zero capacity. We say that $\alpha(u)$ is the *energy* (*measure*) of our functional. The energy $\alpha(u)$ (of our functional) is convex with respect to u in D_0 in the space of measures , i.e. if $u, v \in D_0$ and $t \in [0,1]$ then $\alpha(tu+(1-t)v) \leq t\alpha(u)+(1-t)\alpha(v)$, and it is homogeneous of degree $p > 1$, i.e. $\alpha(tu) = |t|^p\alpha(u)$, $\forall\, u \in D_0$, $\forall\, t \in \mathbb{R}$.

Moreover the following closure property holds: if $u_n \to u$ in D_0 and $\alpha(u_n)$ converges to χ in the space of measures then $\chi \geq \alpha(u)$.

(H_5) α is of strongly local type, i.e. if $u, v \in D_0$ and $u - v =$ const on an open set A we have $\alpha(u) = \alpha(v)$ on A.

(H_6) $\alpha(u)$ is of Markov type, if $\beta \in C^1(\mathbf{R})$ is such that $\beta'(t) \leq 1$ and $\beta(0) = 0$ and $u \in D \cap C_0(X)$, then $\beta(u) \in D \cap C_0(X)$ and $\alpha(\beta(u)) \leq \alpha(u)$ in the space of measures.

Let $\Phi(u) = \int_X \alpha(u)(dx)$ be a strongly local Dirichlet functional with domain D_0. Assume that for every $u, v \in D_0$ we have

$$\lim_{t \to 0} \frac{\alpha(u + tv) - \alpha(u)}{t} = \mu(u, v)$$

in the weak* topology of $\mathcal{M}$ (where $\mathcal{M}$ is the space of Radon measures on X) uniformly for u, v in a compact set of D_0, where $\mu(u, v)$ is defined on $D_0 \times D_0$ and is linear in v. We say that $a(u, v) = \int_X \mu(u, v)(dx)$ is a *strongly local p-homogeneous Dirichlet form.*

We observe that (H_3) is a consequence of (H_1), (H_2), (H_4)–(H_6). The strong locality property allow us to define the domain of the form with respect to an open set O, denoted by $D_0[O]$ and the local domain of the form with respect to an open set O, denoted by $D_{loc}[O]$. We recall that, given an open set O in X we can define a Choquet capacity $\mathrm{cap}(E; O)$ for a set $E \subset \overline{E} \subset O$ with respect to the open set O. Moreover the sets of zero capacity are the same with respect to O and to X.

We summarize in the following Proposition the main properties of a strongly local p-homogeneous Dirichlet form

Proposition 1.1. *Let $a(u, v) = \int_X \mu(u, v)(dx)$ be a (p-homogeneous, strongly local) Dirichlet form. For any u, v belonging to $D_{loc}[X] \cap L^\infty(X, m)$ we have*

(i) *$\mu(u, v)$ is homogeneous of degree $p - 1$ in u and linear in v*

(ii) *for any $a \in \mathbb{R}^+$*

$$|\mu(u, v)| \leq \alpha(u + v) \leq 2^{p-1} a^{-p} \alpha(u) + 2^{p-1} a^{p(p-1)} \alpha(v)$$

(iii) *$\mu(u, u) = p\alpha(u)$*

(iv) *(Leibnitz rule on the second argument) for any $v, w \in D_{loc}[X] \cap L^\infty(X, m)$ we have $vw \in D_{loc}[X] \cap L^\infty(X, m)$ and*

$$\mu(u, vw) = v\mu(u, w) + w\mu(u, v)$$

(v) *(Schwarz inequality) For any $f \in L^{p'}(X, \alpha(u))$ and $g \in L^p(X, \alpha(v))$, $u, v \in D_{loc}[X]$ and $1/p + 1/p' = 1$, fg is integrable with respect to*

the absolute variation of $\mu(u,v)$ *and* $\forall\, a \in \mathbb{R}^+$

$$\int_X |fg|\,|\mu(u,v)|(dx) \\ \le 2^{p-1}a^{-p}\int_X |f|^{p'}\alpha(u)(dx) + 2^{p-1}a^{p(p-1)}\int_X |g|^p\alpha(v)(dx)$$

(*vi*) (*Chain rule*) *If* $u, v \in D_{loc}[X] \cap L^\infty(X,m)$ *and* $g \in C^1(\mathbb{R})$ *with* g' *bounded on* $\mathbb{R}$, *then* $g(u)$, $g(v)$ *belong to* $D_{loc}[X] \cap L^\infty(\Omega,m)$ *and*

$$\mu(g(u),v) = |g'(u)|^{p-2}g'(u)\mu(u,v),$$

$$\mu(u,g(v)) = g'(v)\mu(u,v)$$

(*vii*) (*Truncation rule*) *For every* u *and* v *in* $D_{loc}[X]$ *we have* $u^+, v^+ \in D_{loc}[X]$

$$\mu(u^+,v) = 1_{\{u>0\}}\mu(u,v),$$
$$\mu(u,v^+) = 1_{\{v>0\}}\mu(u,v)$$

(*where we denote again by* u *and* v *the quasi-continuous representative of* u).

Assume now that a quasi-distance d is given on X. We denote by $B(x,r)$ the (open) ball for the distance d with center x and radius r. We assume that

(H_8) the topology induced by d is equivalent to the original topology of X. Moreover, given a compact subset K of X, there exist constants $c_1 > 0$, $\nu \ge 1$ such that for every $x \in K$ and every $0 < r \le r_0$ we have

$$m(B(x,r)) \le c_1 m(B(x,s))\left(\frac{r}{s}\right)^\nu \tag{1.1}$$

(Duplication property)

Remark 1.1. (a) If we assume that for every x and y in X with $x \ne y$ there exists a function φ in $D_0 \cap C_0(X)$ with $L^\infty(X,m)$ energy density such that $\varphi(x) \ne \varphi(y)$, then

$$d(x,y) = \sup\{\varphi(x) - \varphi(y)\},$$

where the sup is on the set

$$\{\varphi \in D_0 \cap C_0(X),\ \mu(\varphi) \le m \text{ on } X\},$$

if finite, is a distance on X such that $\mu(d) \le m$.

(b) Under the assumption (H_8) X is a space of homogeneous type, [17]. We also observe that the following property

$$0 < m(B(x,2r)) \leq c_0 m(B(x,r))$$

(where x belongs to a compact set K, $0 < r \leq 2r_0$) implies the duplication property in (H_8) .

The following assumption (H_9) gives a relation between the metric, the measure on X and the measure valued map α.

(H_9) We assume that, given a compact subset K of X, there exist constants $c_2 > 0$ and $k \geq 1$ such that for every $x \in K$ and every $0 < r \leq r_0$ the following Poincaré inequality of order p holds

$$\int_{B(x,r)} |u - \overline{u}_r|^p m(dx) \leq c_2 r^p \int_{B(x,kr)} \alpha(u)(dx) \tag{1.2}$$

for every $u \in D_{loc}[B(x,kr)]$, where

$$\overline{u}_r = [m(B(x,r))]^{-1} \int_{B(x,kr)} u\, m(dx).$$

Let us assume $p < \nu$. Under the above assumptions the following *Sobolev inequality* holds

$$\begin{aligned}\left(\frac{1}{m(B(x,r))} \int_{B(x,r)} |u|^{p^*} m(dx)\right)^{\frac{p}{p^*}} &\leq c\left(\frac{r^p}{m(B(x,r))} \int_{B(x,kr)} \alpha(u)(dx)\right) \\ &\quad + \left(\frac{1}{m(B(x,r))} \int_{B(x,r)} |u|^p m(dx)\right)\end{aligned} \tag{1.3}$$

where $x \in K$, $0 < r < r_0$ and $p^* = \frac{p\nu}{\nu - p}$ and c depending only on c_0 and c_2. If $\nu \geq p$, then (1.3) holds again where $p^\star$ is any finite positive number greater than p. Moreover from (1.3) we have the compact embedding of the space $D_0(B(x,r))$ into $L^p(B(x,r),m)$, see [8,12] for the bilinear case and [21] for the general case. A Dirichlet functional on a quasi-metric space X with a quasi-distance d, for which (H_1)–(H_6), (H_7), (H_8) hold, is called a Dirichlet–Poincaré functional. A Dirichlet–Poincaré functional

$\Phi(u) = = \int_X \alpha(u)(dx)$ on the space X endowed with a distance d, such that $d \in D_{loc}[X]$ and $\alpha(d) \leq m$ in the measures, is called a *Riemannian–Dirichlet functional* (for an example of distance satisfying the above assumptions see Remark 2.1). The corresponding Dirichlet forms (if they exist i.e. if (H_7) also holds) are called respectively a *Dirichlet–Poincaré form* or a *Riemannian–Dirichlet form.*

Remark 1.2. If $u \in D_0[B(x,r)]$ the Poincaré and Sobolev inequalities on $B(x,r)$ holds without the presence of the term

$$\frac{1}{m(B(x,r))} \int\limits_{B(x,r)} |u|^p m(dx)$$

Consider a Riemannian–Dirichlet functional $\int \alpha(u)(dx)$, we denote by d the distance on X. Let ψ be a C^1-function decreasing and such that $\psi = 1$ on $(0,s)$, $\psi = 0$ on $[t, r_0]$, $0 < s < t < r_0$, $0 \leq \psi \leq 1$, $\psi' \leq \frac{c}{(t-s)}$; taking into account that $\alpha(d) \leq m$ and using the chain rule we can prove that $\psi(d(x,\cdot))$ is a "*cut off*" function between the balls $B(x,s)$ and $B(x,t)$ with the same properties as in the classical Euclidean frame.

Proposition 1.2. *Given any two concentric balls $B(x,s)$ and $B(x,t)$ with $0 < s < t < r_0$ there exists a function $\varphi \in D_0 \cap C_0(X)$ such that $0 \leq \varphi \leq 1$, $\varphi(y) = 1$ for $y \in B(x,s)$, $\operatorname{supp} \varphi \subset B(x,t)$ and $\alpha(\varphi) \leq \frac{c}{(t-s)^p} m$, where c is any fixed constant with $c > 1$.*

Remark 1.3. As a consequence of the assumptions on X and d and of the Poincaré inequality we have the following estimate on the capacity of a ball [7]: for every fixed compact set K there exists positive constants c_4 and c_5 such that

$$c_4 \frac{m(B(x,r))}{r^p} \leq p - \operatorname{cap}(B(x,r), B(x,2r)) \leq c_5 \frac{m(B(x,r))}{r^p}$$

where $x \in K$ and $0 < 2r < r_0$.

Examples of (p-homogeneous) Riemannian–Dirichlet forms are:

(a) The forms relative to a subelliptic p-Laplacian also in the weighted case

(b) The form (if it exists) relative to the p-energy on a measure metric space, whose corresponding Sobolev space satisfies on the assumptions on D_0 (such a form exists if the corresponding Sobolev space is uniformly convex), and in particular the form relative the p-energy on a Cheeger type metric structure [16].

2. The Kato class and the relaxed Dirichlet problem

We give now the notion of *Kato class* of measures relative to a Riemannian p-homogeneous Dirichlet form. In [5] the Kato class was defined in the case of the subelliptic p-Laplacian and in [6] the following definition of Kato class relative to a Riemannian p-homogeneous Dirichlet form has been given:

Definition 2.1. *Let λ be a Radon measure. We say that λ is in the Kato space $K(X)$ if*

$$\lim_{r\to 0} \Lambda(r) = 0 \tag{2.1}$$

where

$$\Lambda(r) = \sup_{x\in X} \int_0^r \left(\frac{|\lambda|(B(x,\rho))}{m(B(x,\rho))} \rho^p \right)^{1/(p-1)} \frac{d\rho}{\rho}$$

Let $\Omega \subset X$ be an open set; $K(\Omega)$ is defined as the space of Radon measures λ on Ω such that the extension of λ by 0 out of Ω is in $K(X)$.

In [6] the properties of the space $K(\Omega)$ are investigated. In particular it is proved that if Ω is a relatively compact open set of diameter $\frac{\bar{R}}{2}$, then

$$\|\lambda\|_{K(\Omega)} := \Lambda(\bar{R})^{p-1}$$

is a norm on $K(\Omega)$ and we can prove, as in [4] for the bilinear case, that $K(\Omega)$ endowed with this norm is a Banach space. Moreover in the above paper it is proved that $K(\Omega)$ is contained in $D^{'}[\Omega]$, where $D^{'}[\Omega]$ denotes the dual of $D_0[\Omega]$.

Let $a(u,v) = \int_X \mu(u,v)(dx)$ be a Riemannian p-homogeneous Dirichlet form of domain D_0 and let Ω be a r.c. open set in X. We denote by σ a Borel (positive) measure on Ω, that does not charge sets of zero capacity. Let g be a continuous function on the closure of Ω, which belongs also to $D_0[\Omega]$ and λ a measure in the Kato class (relative to Ω).

Definition 2.2. *The function $u \in D_{loc}[\Omega] \cap L^p_{loc}(\Omega,\sigma)$ is a local solution of the relaxed Dirichlet problem relative to μ, Ω, σ, g and λ if $u - g \in L^p_{loc}(\Omega,\sigma)$ and*

$$\int_\Omega \mu(u,v)(dx) + \int_\Omega |u-g|^{p-2}(u-g)\, v\, \sigma(dx) = \int_\Omega v\, \lambda(dx) \tag{2.2}$$

for any $v \in D_0[\Omega] \cap L^p(\Omega,\sigma)$ with compact support in Ω (we observe that the condition $(u-g)$ in $L^p_{loc}(\Omega,\sigma)$ can be imposed due to the fact that u is q.e defined on every compact subset of Ω).

We introduce now a notion of σ*-capacity* related to our relaxed Dirichlet problem:

Definition 2.3. *We say that a Borel subset E of an open subset $B \subset \Omega$ is σ-admissible (with respect to B) if there exists a function $v \in D_0[B]$ such that $(w-1) \in L^p(B, \sigma|_E)$, where $\sigma|_E$ is the restriction of σ to E. If E is not σ-admissible, then we define* $\mathrm{cap}_\sigma(E, B) = +\infty$. *If E is σ-admissible, then we define*

$$p - \mathrm{cap}_\sigma(E, B) = \min \left\{ \int_B \alpha(v)(dx) + \int_B |v-1|^p \sigma|_E(dx) \right\}$$

where the minimum is taken on the set

$$\{v \in D_0(B); (v-1) \in L^p(B; \sigma|_E)\}$$

The function v_E which realizes the minimum is called the σ-potential of E relative to Ω.

Remark 2.1. Let ω be an open set with closure contained in Ω and define σ_ω as the measure defined by

$$\sigma_\omega(E) = m(E) \quad \text{if} \quad \mathrm{cap}(E \cap \omega^c) = 0$$

$$\sigma_\omega(E) = +\infty \quad \text{otherwise}$$

Let $x_0 \in \partial\omega$ such that $B(x_0, 2r) \subset \Omega$. Then $B(x_0, r)$ is admissible in $B(x_0, 2r)$ with respect to σ_ω and we have

$$\mathrm{cap}_{\sigma_\omega}(B(x_0, r); B(x_0, 2r)) = \mathrm{cap}(\omega^c \cap B(x_0, r); B(x_0, 2r))$$

3. The Wiener criterion for the relaxed Dirichlet problem

At first we give the definition of regular point for the relaxed Dirichlet problem

Definition 3.1. *A point $x_0 \in \Omega$ is a regular point if every local solution u of the relaxed Dirichlet problem relative to a neighbourhood of x_0 in Ω and an arbitrary g, λ satisfying the condition required in Definition* 2.2, *is continuous at x_0 and $u(x_0) = g(x_0)$.*

Remark 3.1. The regularity of a point x_0 for (2.2) does not depend on Ω, g, λ.

We now give the definition of Wiener point:

Definition 3.2. *A point $x_0 \in \Omega$ is a regular point if*

$$\int_0^R \delta(\rho)^{\frac{1}{p-1}} \frac{d\rho}{\rho} = +\infty$$

where

$$\delta(\rho) = \delta(\rho; x_0) = \frac{\mathrm{cap}_\sigma(B(x_0, \rho), B(x_0, 2\rho))}{\mathrm{cap}(B(x_0, \rho), B(x_0, 2\rho))}$$

We are now in position to state the main result of this paper:

Theorem 3.1. *The point x_0 is regular if and only if x_0 is a Wiener point.*

Remark 3.2. Using the same notations of Remark 2.1 we have that in the case $\sigma = \sigma_\omega$ it is equivalent for a point $x_0 \in \partial\omega$ to be a regular (Wiener) point for the relaxed Dirichlet problem relative to σ_ω or to be a regular (Wiener) point of the boundary for the Dirichlet problem in ω. The proof is easy if $g \in D_0[\Omega]$, but it is enough to prove the equivalence for $g = 0$.

4. Sketch of the proof of Theorem 3.1

We begin by the proof of the sufficient part of our criterion

First we prove that a suitable truncation of a solution u of the relaxed Dirichlet problem is a subsolution of the Dirichlet problem:

Proposition 4.1. *Let λ be a Radon measure in Ω such that $\lambda \in D'[\Omega]$, and let u be a local solution of* (2.2)*. Then*

$$\int_\Omega \mu((u \mp k)^\pm, v)(dx) \le \int_\Omega v|\lambda|(dx)$$

$\forall\, v \in D_0[\Omega]$, $v \ge 0$ a.e. in Ω, where $g^\pm \le k$ in Ω.

We think that the result of above Proposition has an interest in itself.

We observe now that we may assume without loss of generality $g(x_0) = 0$.

The second step is to prove that the result follows from an inequality for the energy of a suitable function.

Let $x_0 \in \Omega$. Assume that $u \in D_0[\Omega] \cap L^p_{loc}(\Omega, \sigma)$ is a local weak solution of (2.2). Let $r \le \frac{3R}{4}$, $\overline{B(x_0, 2R)} \subseteq \Omega$. From Proposition 4.1

$u_k := (u-k)^+$, where $k \geq \sup_{B(x_0,2r)} g$, is a local weak subsolution of the relaxed Dirichlet problem with $\sigma = 0$, that is it satisfies

$$\int_{B(x_0,2r)} \mu(u_k,\varphi)(dx) \leq \int_{B(x_0,2r)} \varphi|\lambda|(dx)$$

$\forall \varphi \in D_0[B(x_0,2r)]$, $\varphi \geq 0$ a.e. in $B(x_0,2r)$. Then it is locally bounded in $B(x_0,2r)$ (see [6]). Let us define

$$M(r) = \sup_{B(x_0,r)} u_k$$

Let $\xi(r)$ be a positive increasing function such that $\xi(r) \to 0$ when $r \to 0$ and suppose

(κ_1) $\xi(r)^{-p}\Lambda(r) \to 0$ when $r \to 0$ if $p \geq 2$

(κ_2) $\xi(r)^{-2}\Lambda(r) \to 0$ when $r \to 0$ if $1 < p < 2$

For example we can choose $\xi(r) = \Lambda(r)^{\frac{1-\epsilon}{p}}$ if $p \geq 2$ and $\xi(r) = \Lambda(r)^{\frac{1-\epsilon}{2}}$ if $1 < p < 2$, when $1-\epsilon > 0$. Let us observe that we will suppose r small enough to have $\xi(r) \leq 1$.

Let

$$v = \frac{1}{M(r) - u_k + \xi(r)}$$

Proposition 4.2. *Let $p \in (1,\nu]$, $r \leq \frac{r_0}{8k^2}$ and $\eta \in D_0[B(x_0,\frac{r}{2})] \cap L^\infty(B(x_0,\frac{r}{2}))$ with $\alpha(\eta) \leq \frac{c}{r^p}$ a.e. in Ω, for a positive constant c. Then there exists a constant $C > 0$ dependent only on Ω, p but independent on x_0, r such that*

$$\frac{r^p}{m(B(x_0,r))} \int_\Omega \alpha(\eta v^{-1})(dx) + \frac{r^p}{m(B(x_0,r))} \int_\Omega |v^{-1} - M(r) - \xi(r)|^p \eta^p \sigma(dx) \leq C\,[M(r)+\xi(r)] \left\{ \left[M(r) - M\Big(\frac{r}{2}\Big) + \xi(r)\right]^{p-1} + \Sigma(r)^{(p-1)} \right\}$$

where

$$\Sigma(r)^{p-1} := C\big(|\lambda|(B(x_0,r))^{\frac{1\vee(p-1)}{p}} + \Lambda(r)^{p-1}\big) \leq C\big(\Lambda(r)^{\frac{[1\vee(p-1)](p-1)}{p}} + \Lambda(r)^{p-1}\big)$$

We prove now that the result follows from Proposition 4.2. From Proposition 4.2 we obtain the following inequality

$$M(\frac{r}{2}) \leq \left[1 - C^{-\frac{1}{p-1}}\delta\left(\frac{r}{2}\right)^{\frac{1}{p-1}}\right] M(r) + \xi(r) + \Sigma(r)$$

From the above inequality and a well known iteration result [22] the sufficient part of our Wiener criterion follows. We can also obtain an estimate on the rate of convergence of $u(x)$ to $g(x_0)$ as x converges to x_0. In particular if g is Hölder continuous, $\Lambda(r) \leq Cr^\gamma$ then the rate of convergence is of the type $|x - x_0|^\tau$, with $0 < \tau < \gamma$ suitable.

Then to prove the sufficient part of our Wiener criterion is enough to prove the inequality in Proposition 4.2.

The proof of Proposition 4.2 is divided in different steps. In the first step choosing as test function $\eta^p(\frac{1}{w})^{(p-1)}$ where $w = v^{-1}$ and $\eta \in D_0[B(x_0, r)] \cap L^\infty(B(x_0, r))$ with $\eta = 1$ in $B(x_0, \frac{3}{4}r)$ and $\alpha(\eta) \leq cr^{-p}m$ for a positive constant c we prove that

$$\int_{B(x_0,r)} \alpha(lg(w))(dx) = \int_{B(x_0,r)} \alpha(lg(v))(dx) \leq C\frac{m(B(x_0, r))}{r^p}$$

From the above inequality we obtain that there are constants C and σ_0 such that for $|\tilde{\sigma}| \leq \sigma_0$, and $0 < r < \frac{3}{4k}r_0$

$$\left(\frac{1}{m(B(x_0, r))}\int_{B(x_0,r)} v^{\tilde{\sigma}} m(dx)\right)\left(\frac{1}{m(B(x_0, r))}\int_{B(x_0,r)} v^{-\tilde{\sigma}} m(dx)\right) \leq C$$

The second step is the proof of a weak Harnack inequality for v. We have that v is a subsolution of the problem with $\sigma = 0$. Then, using the estimate in [6] we obtain

$$\left[M(r) - M\left(\frac{r}{2}\right) + \xi(r)\right]^{-q} \leq \frac{C}{m(B(x_0, 3r/4))}\int_{B(x_0,3r/4)} v^q m(dx) + C\left[\xi(r)^{-(p\vee 2)}\Lambda(r)\right]^q$$

for any $q > 0$. If we take in particular $0 < q \leq \sigma_0$, from the above inequalities we have

$$\frac{1}{m(B(x_0, 3r/4))}\int_{B(x_0,3r/4)} v^{-q} m(dx) \leq C\left[M(r) - M\left(\frac{r}{2}\right) + \xi(r)\right]^q$$

whenever $0 < q \le \sigma_0$. We have taken into account that

$$\Big[M(r) - M\Big(\frac{r}{2}\Big) + \xi(r)\Big]^q \Big[\xi(r)^{-(p\vee 2)}\Lambda(r)\Big]^q \le \frac{1}{2}$$

The third step is the proof that the above inequality holds also for $q > \sigma_0$. Let $\tau < 0$ such that $p(\tau+1) > 1$. Let $\beta = \tau p + p - 1$. Let us observe that β is positive. We use as test function

$$\varphi = \eta^p \psi \ge 0$$

where $\eta \in D_0(B(x_0,r)) \cap L^\infty(B(x_0,r))$, $\eta \ge 0$ and

$$\psi = \Big(v^\beta - \Big(\frac{1}{M(r)+\xi(r)}\Big)^\beta\Big)$$

and we obtain

$$\int_{B(x_0,r)} \alpha(\eta v^\tau)(dx) \le K(\tau)\Bigg[\int_{B(x_0,r)} v^{p\tau}\alpha(\eta)(dx) + \overline{\Sigma}(r)\Bigg]$$

where $K(\tau) \simeq |\tau|^p + \beta^{-p}$ and $\xi(r)^{-(p\vee 2)}\Lambda(r)|\lambda|(B(x_0,r)) =: \overline{\Sigma}(r)$. The Sobolev inequality and a finite iteration of Moser type gives the result.

In the last step we conclude the proof choosing as test function $\varphi = \eta^p u_k$ where $\eta \in D_0[B(x_0,\frac{r}{2})] \cap L^\infty(B(x_0,\frac{r}{2})$ with $\alpha(\eta) \le \frac{c}{r^p} m$ for a positive constant c and using for a suitable choice of the exponents in the Hölder inequality.

We have now to sketch the proof of the necessary part. The proof is by contradiction. For a given σ we consider the σ-potential $B(x_0,R)$ in $B(x_0,2R)$ denoted by v_R and we denote $w_R = v_R + 1$. Let x_0 be a regular point such that

$$\int_0^{2R} \Big(\frac{p-\mathrm{cap}_\sigma(B(x_0,\rho),B(x_0,2\rho))}{p-\mathrm{cap}(B(x_0,\rho),B(x_0,2\rho))}\Big)^{\frac{1}{p-1}} \frac{d\rho}{\rho} < +\infty$$

Using this relation we obtain

$$w_R(x_0) \ge \frac{3}{4}$$

The above relation gives a contradiction with the assumption of the regularity of x_0, which implies that $w_R(x_0) = 0$ with continuity. We remark that the proof follows the lines of the proof of the necessity part of Wiener criterion at the boundary in [7] and of the one given in the subelliptic case in [11]

References

[1] M. Biroli, *Nonlinear Kato measures and nonlinear subelliptic Schrödinger problems* // Rend. Acc. Naz. Scie. detta dei XL, Memorie di Matematica e Applicazioni, **21** (1997), 235–252.

[2] M. Biroli, *Weak Kato measures and Schrödinger problems for a Dirichlet form* // Rend. Acc. Naz. Scie. detta dei XL, Memorie di Matematica e Applicazioni, **24** (2000), 235–252.

[3] M. Biroli, *Schrödinger type and relaxed Dirichlet problems for the subelliptic p-Laplacian* // Potential Analysis, **15** (2001), 1–16.

[4] M. Biroli, *Nonlinear p-homogeneous Dirichlet forms on nonreflexive Banach spaces* // Rend. Acc. Naz. Scie. detta dei XL, Memorie di Matematica e Applicazioni, **29** (2005), 235–252.

[5] M. Biroli, S. Marchi, *Harnack inequality for Schrödinger problem relative to strongly local Riemannian p-homogeneous forms with a potential in the Kato class*, Boundary Value Problems, Special Issue "Harnack Estimates and Local Behavior of degenerate and singular Partial Differential Equations", 2007, Art. ID 24806, pp. 19.

[6] M. Biroli, S. Marchi, *Oscillation estimates relative to p-homogeneuous forms and Kato measures data* // Le Matematiche, **51** (II) (2006), 335–361.

[7] M. Biroli, S. Marchi, *Wiener criterion at the boundary related to p-homogeneous strongly local Dirichlet forms* // Le Matematiche, **52** (2007), in print.

[8] M. Biroli and U. Mosco, *Sobolev inequalities on homogeneous spaces* // Potential Analysis, **4** (1995), 311–324.

[9] M. Biroli, U. Mosco, *A Saint Venant type principle for Dirichlet forms on discontinuous media* // Ann. Mat. Pura e Appl., **169** (IV) (1995), 125–181.

[10] M. Biroli, U. Mosco, *Kato spaces for Dirichlet forms* // Potential Analysis, **10** (1999), 1–16.

[11] M. Biroli, N. Tchou, *Relaxed Dirichlet problem for the subelliptic p-Laplacian* // Ann. Mat. Pura e Appl., **129** (IV), 39–64.

[12] M. Biroli, S. Tersian, *On the existence of nontrivial solutions to a semilinear equation relative to a Dirichlet form*, Ist. Lombardo, Accad. Sci. Lett. Rend. A, **31** (1997), 151–168.

[13] M. Biroli, G. P. Vernole, *Strongly Local Nonlinear Dirichlet Functionals*, Ukrainian Math. Bullettin, **1**, 2004, pp. 485-500.

[14] M. Biroli, G.P. Vernole, *Strongly Local Nonlinear Dirichlet Functionals and Forms* // Advances in Mathematical Sciences and Applications, **15** (2005), 655–682.

[15] M. Biroli, G. P. Vernole, *Harnack inequality for harmonic functions relative to a nonlinear p-homogeneous Riemannian Dirichlet form* // Nonlinear Analysis TMA, **64** (2005), 51–68.

[16] J. Cheeger, *Differentiability of Lipschitz functions on metric measure spaces* // Geom. Func. Anal., **9** (1999), 428–517.

[17] R. R. Coifman and G. Weiss, *Analyse harmonique non commutative sur certaines espaces homogénes*, Lectures Notes in Math., 242, Springer-Verlag, 1971.

[18] G. Dal Maso, U. Mosco, *Wiener criteria and energy decay for relaxed Dirichlet problems* // Arch. Rat. Mech. An., **95** (1986), 345–387.

[19] G. Dal Maso, U. Mosco, *Wiener's criterion and* Γ*-convergence* // J. Appl. Math. Opt.,**15** (1987), 15–63.

[20] M. Fukushima, Y. Oshima, M. Takeda, *Dirichlet forms and Markov processes*, W. De Gruyter and Co., 1994.

[21] J. Maly, U. Mosco, *Remarks on measure-valued Lagrangians on homogeneous spaces* // Ricerche Mat., **48** (1999), 217–231.

[22] U. Mosco, *Wiener criterion and potential estimates for the obstacle problem* // Indiana Un. Math. J., **36** (1987), 455–494.

Contact information

Marco Biroli
Department of Mathematics "F. Brioschi",
Politecnico di Milano,
Piazza Leonardo da Vinci 32,
20133 Milano,
Italy
E-Mail: marco.biroli@polimi.it

Florangela Dal Fabbro
Department of Mathematics "F. Brioschi",
Politecnico di Milano,
Piazza Leonardo da Vinci 32,
20133 Milano,
Italy
E-Mail: forangela.dal_fabbro@polimi.it

Silvana Marchi
Department of Mathematics,
Università di Parma,
Parco delle Scienze 53/A,
Parma
Italy
E-Mail: silvana.marchi@unipr.it