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ON THE UNIQUENESS OF THE VARIATIONAL SOLUTION FOR THE PROBLEM OF EQUILIBRIUM OF THE PENDING DROP

In this paper we study the uniqueness of the equilibrium forms of the axisymmetrical drops pending from the horizontal plane. In our considerations we take into account the intermediate layer separating the liquid phase from that of the vapor. We prove the uniqueness of the variational solution describing the equilibrium forms.

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In memory of A.D. Myshkis

1. Formulation of the Problem.

In this paper we study the uniqueness of the equilibrium form of the drop pending from horizontal plane P. In the classical case the form satisfy the Laplace condition. This condition affirms that for equilibrium form the mean curvature of the pending drop is to be constant [1]. Now it is well known that this condition fails when one of the principal curvatures of the drop is sufficiently large. As far as in the classical equilibrium case we don't take into account the intermediate layer separating the phases then it is quite natural to try to surpass the difficulties connected with applying Laplace condition by introducing the intermediate layer's width into the consideration. It seems that F. Neumann was the first to do it [2], see also [3]. In the contemporary theories on the subject instead of the Laplace condition there appears a sufficiently complicated function depending on mean and Gauss curvatures [4], see also [5, 6]. But in the case when the width of the intermediate layer and potential functions determining the state of equilibrium are constant we get a simple condition affirming that equilibrium surface has linear combination of mean and Gauss curvatures equal to the constant, see, for example, [7],

$$H + l_p K = \lambda^*, \quad \lambda^* = \lambda + \frac{1}{\sigma}\Gamma\rho$$

$$\lambda = \frac{1}{\pi r^2}\left[2\pi r\cos\gamma + \frac{l_p \pi r^2}{2}\sin^2\gamma - \kappa V\right]$$

$$\cos\gamma - \frac{l_p}{2}\left[\gamma - \frac{\sin 2\gamma}{2}\right] = \beta$$

Here H, K are mean and Gauss curvatures respectively, r is the radii of the projection of the surface S over the plane P, Γ – the function describing the gravitational potential, ρ – the fluid's density, β – the coefficient of the relative adhesion, W – drop's interior part and V – the volume of the domain W, l_p the width of the intermediate layer. We repair that in the case of the width l_p being equal to zero we get the classical condition. Besides when $l_p \neq 0$ we get that the wetting angle γ is different from the classical one. This partly solves the problem of the wetting angle γ, see [1], where this problem is thoroughly discussed.

In the paper [8] of the first of the authors the variational problem generalizing that of the classical case was formulated and solved. We do not discuss the solution of it here. But as we are interested in the uniqueness of the variational solution of equilibrium problem we formulate it now. Let us denote through S the drop's surface and through S^* its projection onto the plane P from which the drop is pending. Let Σ be the circle of the contact of the two surfaces S and S^*. We suppose that the line L generating the surface S is a rectifiable curve whose length is equal to l. We introduce Cartesian coordinates (x, y) in the meridian section of the drop orienting the axis x along the line perpendicular to the plane P and we denote through $w = w(s) = (x(s), y(s)), 0 \leqslant s \leqslant l$, the natural parameterization of the curve L. Let $\mathbf{S}$ be the area of the surface S. In the variational study of the equilibrium problem we consider the functional $F = F(S)$ represented as follows

$$F(S) = \sigma \left(\mathbf{S} + l_p \Xi - \beta \int\limits_{S^*} ds + \lambda V + \sigma^{-1} \iiint\limits_{W} \Gamma \rho \, dv \right) \tag{1}$$

Here

$$\Xi(S) = 2\pi \int\limits_{L} f(\dot{y}) \, ds, \quad \dot{y} = dy/ds,$$

with the function f having the following representation

$$f(\dot{y}) = \frac{1}{2} \left\{ -\sqrt{1-\dot{y}^2} \int\limits_0^{|\dot{y}|} \left(\arcsin\sigma + \sigma\sqrt{1-\sigma^2} - \frac{\pi}{2} \right) \left(1-\sigma^2\right)^{-3/2} d\sigma + {} + E_0 \sqrt{1-\dot{y}^2} \right\} \tag{2}$$

The functional (1) differs from the classical one by the term Ξ which is responsible for the appearance of the Gauss curvature in the Euler condition for the extremal surface [9].

Variational Problem. Let M be the class of the surfaces we have described.

For a given values of the layer's width l_p, the coefficient of the relative adhesion β and for a given volume V it is necessary to find a surface $S_e \in M$ and a constant λ such that

$$F(S_e) = \inf \{F(S),\ S \in M\}$$

2. The study of the uniqueness problem.

We are interested in the investigation of the uniqueness problem for variational solutions of the variational problem we have just formulated. In this section we'll give detailed proof of the uniqueness theorem based on the ideas exposed in the paper [10]. Let

$$S_1,\ S_2$$

be two different solutions of the variational problem described by the graphs of the functions $y_1 : [0, r_1] \to R$, $y_2 : [0, r_2] \to R$ which are monotone over their domains of definition. In the sequel we'll suppose that the functions y_1, y_2 are defined over the same interval $[0, r_0]$, $r_0 = \max\{r_1, r_2\}$.

Let us consider the following homotopy of y_1 into y_2

$$y^2(x,\theta) = (1-\theta)\, y_1^2(x) + \theta y_2^2(x), \quad 0 \leqslant \theta \leqslant 1 \tag{3}$$

First of all we prove the following lemma.

Lemma 1. *The functional S satisfies the following condition of the convexity*

$$S(y_\theta) - (1-\theta)\, S(y_1) - \theta S(y_2) \leqslant 0 \tag{4}$$

Proof. Let $z_i = z_i(x,y)$ be nonparametric representation of the surface $S(y_i)$, $i = 1, 2$, which is located in the upper half-space $\left\{(x,y,z) \in R^3 \middle|\, z > 0\right\}$,

$$z_i(x,y) = \sqrt{y_i^2 - y^2} \tag{5}$$

Let us consider the surface $S^*_{1/2}$ with the following non-parametric representation

$$z^* = z^*(x,y) = [z_1(x,y) + z_2(x,y)]\, 2^{-1} \tag{6}$$

We'll suppose that the functions $z_1 = z_1(x,y)$, $z_2 = z_2(x,y)$, $z^* = z^*(x,y)$ are defined over the same domain.

We'll denote as $S_{1/2}$ the union of the surface $S^*_{1/2}$ and its mirror's reflection in the plane (x,y). The surface $S_{1/2}$ is not axisymmetrical one. We consider the domain $B_{1/2}$ bounded by the surface $S_{1/2}$.

We consider Shwartz symmetrization [11] of the body $B_{1/2}$. Due to this symmetrization we substitute each x-section $\left(B_{1/2}\right)_x$ of the body $B_{1/2}$ by the disc lying in the plane orthogonal to the x-axis, centered at the point of intersection of this plane with the axis and having the same measure as this section. Let us calculate the radii $r = r\,(x)$ of this disc.

$$\pi r^2 = 2\int\limits_0^{r^*(x)} z^*\,(x,y)\;dy = \int\limits_0^{y_1(x)} \sqrt{y_1^2\,(x) - y^2}\,dy + \int\limits_0^{y_2(x)} \sqrt{y_2^2\,(x) - y^2}\,dy = \\ = \left[y_1^2\,(x) + y_2^2\,(x)\right]\pi \quad (7)$$

Here $|0, r^*(x)|$ is the projection of x-section of the body $B_{1/2}$ over the y-axis.

We now get from the equality (7) that the radii $r = r\,(x)$ satisfies the following condition

$$r^2\,(x) = \frac{1}{2}\left[y_1^2 + y_2^2\right].$$

These calculations show that the area of the x-section of the body $B\left(y_{1/2}\right)$ coincides with that of the $B^*_{1/2}$.

The surface $S^*_{1/2}$ is the boundary of the domain $B^*_{1/2}$ which is the Shwartz symmetrrization of the domain $B_{1/2}$ bounded by the surface $S_{1/2}$. As under such a symmetrization the area of the surface does not increase we get the following inequality

$$2\pi\int\limits_0^a y\,(x,1/2)\,\sqrt{1 + y'^2\,(x,1/2)}\,dx \leqslant \\ \leqslant 2\iint\limits_{pr_{(x,y)}S^*_{1/2}} \sqrt{1 + z_x^{*2}\,(x,y) + z_y^{*2}\,(x,y)}\,dx\,dy \quad (8)$$

Here the segment $[0,a]$ is the projection of the surface $S_{1/2}$ over the x-axis and $pr_{(x,y)}S^*_{1/2}$ is the projection of the surface $S^*_{1/2}$ onto the plane (x,y).

The functional

$$\iint \sqrt{1 + z_x^2 + z_y^2}\,dx\,dy$$

is convex relatively to the linear homotopy $(1-\theta)z_1 + \theta x_2$, $0 \leqslant \theta \leqslant 1$. Using inequality (8) we arrive at the following condition

$$2\pi \int\limits_0^a y\,(x,1/2)\,\sqrt{1+y'^2\,(x,1/2)}\,dx \leqslant$$

$$\leqslant \iint\limits_{pr_{(x,y)}S_1} \sqrt{1+z_{1x}^2\,(x,y)+z_{1y}^2\,(x,y)}\,dx\,dy+$$

$$+\iint\limits_{pr_{(x,y)}S_2} \sqrt{1+z_{2x}^2\,(x,y)+z_{2y}^2\,(x,y)}\,dx\,dy=$$

$$=\pi \int\limits_0^a y_1\,(x)\,\sqrt{1+{y_1'}^2\,(x)}\,dx+\pi \int\limits_0^a y_2\,(x)\,\sqrt{1+{y_2'}^2\,(x)}\,dx. \quad (9)$$

Here $pr_{(x,y)}S_i$ is the projection of the surface S_i, $i=1,2$, onto the plane (x,y).

>From the definition of homotopy y_θ it now follows that

$$y^2_{\frac{\theta'+\theta''}{2}}=\frac{1}{2}\left(y^2_{\theta'}+y^2_{\theta''}\right) \quad (10)$$

We now get from the conditions (9) and (10) the convexity of the functional S.

The lemma is proved.

Let us now compare the following functionals

$$S\,(y)=\iint \sqrt{1+z_\xi^2+z_\eta^2}\,d\xi\,d\eta, \quad S_0\,(y)=\int \sqrt{1+y'^2}\,dx$$

First of all we'll prove the following lemma

Lemma 2. *Let*

$$\delta^*:=\sqrt{1+z_\xi^2\,(1/2;\xi,\eta)+z_\eta^2\,(1/2;\xi,\eta)}-$$

$$-\,(1/2)\,\sqrt{1+z_{1\xi}^2+z_{1\eta}^2}-(1/2)\,\sqrt{1+z_{2\xi}^2+z_{2\eta}^2}$$

$$z_k\,(\xi,\eta)=\sqrt{y_k^2-\eta^2}, \quad k=1,2, \quad z\,(1/2)=\left[z_1\,(\xi,\eta)+z_2\,(\xi,\eta)\right](1/2)$$

and

$$\delta:=\sqrt{1+\left[(y_1'+y_2')\,(1/2)\right]^2}-(1/2)\,\sqrt{1+{y_1'}^2}-(1/2)\,\sqrt{1+{y_2'}^2}$$

Then we have the following inequality

$$(-\delta^*)\geqslant \frac{1}{8}\,(-\delta)\,. \quad (11)$$

Proof. Let us denote as e_1^*, e_2^* the following quantities

$$e_1^* := \sqrt{1 + (\nabla z)^2 (1/2; \xi, \eta)} + \\ + (1/2) \sqrt{1 + (\nabla z_1)^2 (\xi, \eta)} + (1/2) \sqrt{1 + (\nabla z_2)^2 (\xi, \eta)}$$

$$e_2^* = (1/2) \left[1 + \nabla z_1 \nabla z_2 + \sqrt{\left[1 + (\nabla z_1)^2\right] \left[1 + (\nabla z_1)^2\right]} \right].$$

Then we have

$$\delta^* = \Bigg\{ -\frac{1}{4} (z_{1\xi} z_{2\eta} - z_{2\xi} z_{1\eta})^2 - \\ - \frac{1}{4} \left[(z_{1\xi} - z_{2\xi})^2 + (z_{1\eta} - z_{2\eta})^2 \right] \Bigg\} (e_1^* e_2^*)^{-1} \quad (12)$$

In conformity with the definition of z_1, z_2 we obtain now the following representations

$$z_{1\xi} - z_{2\xi} = \frac{y_1 y_1'}{\sqrt{y_1^2 - \eta^2}} - \frac{y_2 y_2'}{\sqrt{y_2^2 - \eta^2}}$$

$$z_{1\eta} - z_{2\eta} = -\frac{\eta}{\sqrt{y_1^2 - \eta^2}} + \frac{\eta}{\sqrt{y_2^2 - \eta^2}}$$

$$z_{1\xi} z_{2\eta} - z_{1\eta} z_{2\xi} = -\frac{\eta (y_1 y_1' - y_2 y_2')}{\sqrt{y_1^2 - \eta^2}\sqrt{y_2^2 - \eta^2}} \quad (13)$$

Using formulas (13) we estimate now the quantity $-\delta^*$ from below. Let δz be the following expression

$$\delta z := (z_{1\xi} z_{2\eta} - z_{2\xi} z_{1\eta})^2 + (z_{1\xi} - z_{2\eta})^2$$

The direct calculations give us the equation

$$\delta z = \Big[y_1^2 y_2^2 {y_1'}^2 + y_1^2 y_2^2 y_2^2 - 2 y_1 y_2 y_1' y_2' \sqrt{y_1^2 - \eta^2} \sqrt{y_2^2 - \eta^2} - \\ - 2 y_1 y_2 \eta^2 y_1' y_2' \Big] \left[\left(y_1^2 - \eta^2\right) \left(y_2^2 - \eta^2\right) \right]^{-1} \quad (14)$$

Let δy be the following expression

$$\delta y := \left[\sqrt{y_1^2 - \eta^2} \sqrt{y_2^2 - \eta^2} + \eta^2 - y_1 y_2 \right] \quad (15)$$

then we get

$$\eta^{-2}\delta y = -\left(y_1^2 - y_2^2\right)^2 \delta y_1^* \delta y_2^* \tag{16}$$

>From the representations (15), (16) we arrive at the inequality

$$y_1 y_2 \geqslant \sqrt{y_1^2 - \eta^2}\sqrt{y_2^2 - \eta^2} + \eta^2 \tag{17}$$

The inequality (17) implies that

$$\delta z \geqslant y_1^2 y_2^2 \left(y_1' - y_2'\right)^2 \left[\left(y_1^2 - \eta^2\right)\left(y_2^2 - \eta^2\right)\right]^{-1} \tag{18}$$

Using now the definition (12) of δ^* and inequality (18) we get

$$-\delta^* \geqslant \left(y_1' - y_2'\right) \frac{1}{4} y_1^2 y_2^2 \left[\left(y_1^2 - \eta^2\right)\left(y_1^2 - \eta^2\right)\right]^{-1} \left(e_1^* e_2^*\right)^{-1} \tag{19}$$

As the quantity $-\delta$ is equal to the following expression

$$-\delta = \frac{1}{4}\left(y_1' - y_2'\right)^2 \left(e_1 e_2\right)^{-1}$$

where

$$e_1 = \sqrt{1 + \left(\frac{y_1' + y_2'}{2}\right)^2} + \frac{1}{2}\sqrt{1 + {y_1'}^2} + \frac{1}{2}\sqrt{1 + {y_2'}^2}$$

$$e_2 = \frac{1}{2} + \frac{1}{2} y_1' y_2' + \frac{1}{2}\sqrt{1 + y_1^2}\sqrt{1 + y_2^2}$$

then in order to compare $-\delta^*$ and $-\delta$ it is sufficient now to compare the quantities

$$y_1^2 y_2^2 \left[\left(y_1^2 - \eta^2\right)\left(y_2^2 - \eta^2\right)\right]^{-1} \left(e_1^* e_2^*\right)^{-1}$$

and

$$\left(e_1 e_2\right)^{-1}$$

In these calculations we suppose that neither y_1 nor y_2 are equal to zero. The case when one of these functions could be equal to zero will be considered later. Now using the definitions of the quantities e_1^*, e_1^* we easily arrive at the following estimations

$$\begin{aligned}
&\left(y_1^2 - \eta^2\right)\left(y_2^2 - \eta^2\right) e_1^* e_2^* \leqslant \\
&\quad \leqslant \left\{ \left[y_1^2 y_2^2 + y_1^2 y_2^2 \left(\frac{|y_1'| + |y_2'|}{2}\right) + \frac{y_1^2 y_2^2}{4} \right]^{1/2} + \frac{1}{2} y_1 y_2 \left(1 + {y_1'}^2\right)^{1/2} + \right. \\
&\qquad \left. + \frac{1}{2} y_1 y_2 \left(1 + {y_2'}^2\right)^{1/2} \right\} \sqrt{\left(y_1^2 - \eta^2\right)\left(y_2^2 - \eta^2\right)} e_2^* \leqslant \\
&\qquad\quad \leqslant \frac{\sqrt{5}}{2} y_1 y_2 e_1 \sqrt{\left(y_1^2 - \eta^2\right)\left(y_2^2 - \eta^2\right)} e_2^* \leqslant \frac{\sqrt{5}}{2} y_1^2 y_2^2 e_1 e_2
\end{aligned} \tag{20}$$

Using the inequality (20) and definitions of δ^* and δ we arrive at the following inequality

$$-\delta^*(x) \geqslant -\frac{2\sqrt{5}}{5}\delta(x) \tag{21}$$

The inequality (21) takes place for values of the variable x for which neither y_1 nor y_2 are equal to zero.

Let us suppose now that, for example, the equation $y_1(x) = 0$ takes place. In this case we easily get the following inequalities

$$-\delta^* \geqslant \frac{1}{8}\frac{{y_2'}^2}{1+{y_2'}^2}, \quad -\delta \leqslant \frac{{y_2'}^2}{1+{y_2'}^2}$$

The last inequalities mean that the following estimation takes place

$$-\delta^*(x) \geqslant \frac{1}{8}(-\delta(x)) \tag{22}$$

$>$From the estimations (21), (22) we conclude that the inequality (11) is valid for all the values of x,

$$x \in [0, a].$$

The lemma is proved.

Let us now consider linear combination of the functionals of the following type

$$Q(y) := \int_0^a y\sqrt{1+y'^2}\,dx - \kappa^* \int_0^a \sqrt{1+y'^2}dx, \quad \kappa^* > 0. \tag{23}$$

We'll prove now the following lemma

Lemma 3. *Let $y = y(x)$, $x \in [0, a]$, be a function whose graph represents the line L generating the surface S delivering the minimum value to the functional F. Let us suppose that the following inequality takes place*

$$\kappa^* < r_0/8 \tag{24}$$

Then the functional Q is convex under the transformations (3).

Proof. Let us consider difference

$$\Delta := \frac{1}{2}Q(y_1) + \frac{1}{2}Q(y_2) - Q(y(x, 1/2))$$

Using inequality (9) we see that

$$\Delta > \frac{1}{2}\int_0^a\int_0^{r_0}\sqrt{1+z_{1x}^2(x,y)+z_{1y}^2(x,y)}\,dx\,dy+$$

$$+\frac{1}{2}\int_0^a\int_0^{r_0}\sqrt{1+z_{2x}^2(x,y)+z_{2y}^2(x,y)}\,dx\,dy-$$

$$-\int_0^a\int_0^{r_0}\sqrt{1+z_x^{*2}(x,y)+z_y^{*2}(x,y)}\,dx\,dy-$$

$$-\kappa^*\left[-\int_0^a\sqrt{1+y'^2(x,1/2)}\,dx+\right.$$

$$\left.+\frac{1}{2}\int_0^a\sqrt{1+{y_1'}^2(x)}\,dx+\frac{1}{2}\int_0^a\sqrt{1+{y_2'}^2(x)}\,dx\right]\geqslant$$

$$\geqslant\int_0^a\int_0^{r_0}\left\{[-\delta_*(x)]-\kappa^* r_0^{-1}[-\delta(x)]\right\}\,dx\,dy\geqslant$$

$$\geqslant\int_0^a\int_0^{r_0}\left[\frac{1}{8}-\kappa^* r_0^{-1}\right][-\delta(x)]\,dx\,dy\geqslant 0$$

The lemma is proved.

We proceed now with the investigation of the principal parts of the functionals constituting the body of the functional F. To this end we study the behavior of the functional Ξ under homotopies defined by the expression (3). In order to formulate a proposition describing this behavior we introduce some notations. First of all we express this functional in terms of the functions written in the rotated coordinate system. Let us consider (σ,τ) – coordinate system rotated by the angle -45° relatively the (x,y) – coordinate system. The coordinate transformation has the following form

$$x=\frac{\sqrt{2}}{2}(\sigma+h(\sigma)),\quad y=\frac{\sqrt{2}}{2}(-\sigma+h(\sigma)) \tag{25}$$

Let $h=h(\sigma)$ be the representation of the function $y=y(x)$ in this system. The functional Ξ acquires now the following representation

$$\Xi(h)=\frac{1}{2}\int_\Delta\left[\frac{\sqrt{2}}{2}\left(1+h'\right)(\sigma)\int_0^{N(h')}f_0(u)\,du\right]d\sigma+\frac{\sqrt{2}}{4}\int_\Delta E_0\left(1+h'\right)\,d\sigma$$

Here Δ is the projection of the graph of the function of the function h (we assume that all the functions participating in the calculations to follow are defined over the same domain),

$$N = N\left(t = \frac{\sqrt{2}}{2}\frac{1-t}{\sqrt{1+t^2}}\right)$$

$$f_0\left(u\right) = \left(\arcsin u + u\sqrt{1-u^2-\frac{\pi}{2}}\right)\left(1-u^2\right)^{-\frac{3}{2}}, \quad u \in (0,1).$$

Alongside with the functional Ξ we also consider the modification Ξ^* of it,

$$\Xi^* := \int\limits_{\Delta} \Lambda_3\left(h'\right)(\sigma)\, d\sigma + \frac{1}{2}\int\limits_{\Delta} E_0\left(1 + h'(\sigma)\right)\, d\sigma, \tag{26}$$

$$\Lambda_3 = \Lambda_3\left(t\right) = \kappa^*\sqrt{1+t^2} + \frac{\sqrt{2}}{2} l_p\left(1+t\right)\int\limits_0^{N(t)} f_0\left(u\right)\, du.$$

Now the following functions will be used in the future

$$\Lambda_3' = \Lambda_3'\left(t\right) = \kappa^*\frac{t}{\sqrt{1+t^2}} + \frac{\sqrt{2}}{2} l_p \int\limits_0^{N(t)} f_0\left(u\right)\, du + \\ + \frac{\sqrt{2}}{2} l_p\left(1+t\right) f_0\left(N\left(t\right)\right) N'\left(t\right)$$

$$q\left(t\right) = 3\left(1-t^2\right)\frac{f_0\left(N\left(t\right)\right)}{2\left(1+t^2\right)^{\frac{5}{2}}} - \sqrt{2}\left(1+t\right)^3\frac{f_0'\left(N\left(t\right)\right)}{4\left(1+t^2\right)^3}$$

$$\Lambda_3''\left(t\right) = \kappa^* p\left(t\right) - l_p q\left(t\right), \quad p\left(t\right) = \left(1+t^2\right)^{-\frac{3}{2}}$$

Let

$$c_0 = \sup\left\{q\left(t\right), t \in [-1,1]\right\}$$

Lemma 4. *Let κ^*, l_p be the numbers connected by the inequality*

$$\kappa^* - 2c_0 l_p > 0 \tag{27}$$

Let $h = h\left(1/2, \sigma\right)$ be the representation of the function $y = y\left(1/2, x\right)$ defined by the functions $y_1 = y_1\left(x\right)$, $y_2 = y_2\left(x\right)$ in conformity with the condition

$$y^2\left(1/2, x\right) = \frac{1}{2}\left(y_1^1 + y_2^2\right)$$

and $h_k = h_k\left(\sigma\right)$ – representation of the function $y_k = y_k\left(x\right)$, $k = 1, 2$, in the rotated coordinate system.

Then the following inequality takes place

$$\Xi^{*}\left(h\left(1/2,\sigma\right)\right) \leqslant \frac{1}{2}\Xi^{*}\left(h_1\left(\sigma\right)\right)+\frac{1}{2}\Xi^{*}\left(h_2\left(\sigma\right)\right) \tag{28}$$

Proof. First of all let us note that the functional

$$\int\limits_{\Delta} E_0 l_p\left(1+h'\left(\sigma\right)\right)\,d\sigma \tag{29}$$

behaves itself like the linear functional. Really as the homotopy in question leaves the endpoints of curves fixed the following equation takes place

$$\int\limits_{\Delta} E_0\left(1+h'\left(1/2,\sigma\right)\right)\,d\sigma-\frac{1}{2}\int\limits_{\Delta} E_0\left(1+h_1'\left(\sigma\right)\right)\,d\sigma- \\ -\frac{1}{2}\int\limits_{\Delta} E_0\left(1+h_2'\left(\sigma\right)\right)\,d\sigma=0.$$

This property of the functional (29) implies that we can select the sign of the first derivative of the function Λ_3 at our will. In the classical case this means that we can add an arbitrary linear function to the function under consideration and this would not affect its convexity. At the same time it permits us to change the signal of its first derivative.

In the study of the function $\delta\Lambda_3$,

$$\delta\Lambda_3=\Lambda_3\left(h'\left(1/2,\sigma\right)\right)-\frac{1}{2}\Lambda_3\left(h_1'\left(\sigma\right)\right)-\frac{1}{2}\Lambda_3\left(h_2'\left(\sigma\right)\right),$$

it is necessary to take into account the different relations between the functions y_1, y_2 and their derivatives. As we suppose that the functions y_1, y_2 are monotone ones there exists not more than countable set of the arcs where these functions are comparable. The union of this arcs covers the union of the curves in question.

Let us consider an arc where $y_1<y_2$ and $y_1'<y_2'$. On this arc the functions $h'=h'\left(1/2,\sigma\right)$, $h_1'=h_1'\left(\sigma\right)$, $h_2'=h_2'\left(\sigma\right)$ are connected by the following inequalities

$$h_1'\left(\sigma\right)\leqslant h'\left(1/2,\sigma\right)\leqslant h_2'\left(\sigma\right) \tag{30}$$

Really

$$\frac{h_1'\left(\sigma\right)-h'\left(1/2,\sigma\right)}{2}= \\ =\left(\frac{1+y_1'\left(x\right)}{1-y_1'\left(x\right)}-\frac{1+y'\left(1/2,x\right)}{1-y'\left(1/2,x\right)}\right)\frac{1}{2}=\frac{y_1'\left(x\right)-y'\left(1/2,x\right)}{\left(1-y_1'\left(x\right)\right)\left(1-y'\left(1/2,x\right)\right)}< \\ <\frac{y_1y_1'-2y_1y'\left(1/2,x\right)}{y_1\left(1-y_1'\left(x\right)\right)\left(1-y'\left(1/2,x\right)\right)}\leqslant\frac{\left(y_1^2\right)'-2\left(y^2\left(1/2,x\right)\right)'}{y_1\left(1-y_1'\left(x\right)\right)\left(1-y'\left(1/2,x\right)\right)}<0 \tag{31}$$

>From the inequality (31) we get the left part of the inequality (30). In the same manner we arrive at the following evaluation

$$\frac{h_2'(\sigma)-h'(1/2,\sigma)}{2}=\frac{y_2'(x)-y'(1/2,x)}{(1-y_2'(x))(1-y'(1/2,x))}=$$
$$=\frac{y_2y_2'-y_2y'(1/2,x)}{y_2(1-y_2'(x))(1-y'(1/2,x))}>\frac{2\left(y_2^2\right)'-\left(y^2(1/2,x)\right)'}{y_2(1-y_2'(x))(1-y'(1/2,x))}>0 \quad (32)$$

>From the inequality (32) we get the right part of the expression (30). Using the same arguments we arrive at the following equation

$$\frac{h_1'(\sigma)-h'(1/2,\sigma)}{2}+\frac{h_2'(\sigma)-h'(1/2,\sigma)}{2}=\frac{y_1'-y_2'}{(1-y_1')((1-y_2'))} \quad (33)$$

>From the condition (33) we deduce the following possibilities for the relations between $h'=h'(1/2,\sigma), h_1'=h_1'(\sigma), h_2'=h_2'(\sigma)$ and y_1', y_2':

a) $y_1'\leqslant y_2' \Rightarrow \dfrac{h_1'(\sigma)+h_2'(\sigma)}{2}-h'(1/2,\sigma)\leqslant 0$

b) $y_1'\geqslant y_2' \Rightarrow \dfrac{h_1'(\sigma)+h_2'(\sigma)}{2}-h'(1/2,\sigma)\geqslant 0$

When investigating the functional Ξ^* we take into account each of these cases separately. As it was already said the signal of the constant E_0 can be selected at our will. In the case a) we select this signal being negative so that the derivative of the function

$$\Lambda_3^*(t)=\Lambda_3(t)+\frac{E_0}{2\sqrt{2}}(1+t)$$

is also negative. Taking this property into account we arrive at the following inequality

$$\delta\Lambda_3^*=(\Lambda_3^*)'(\tau_1)\frac{h'(1/2,\sigma)-h_1'(\sigma)}{2}-(\Lambda_3^*)'(\tau_2)\frac{-h'(1/2,\sigma)+h_2'(\sigma)}{2}\leqslant$$
$$\leqslant\left[(\Lambda_3^*)'(\tau_1)-(\Lambda_3^*)'(\tau_2)\right]\frac{h_2'(\sigma)-h'(1/2,\sigma)}{2}=$$
$$=(\Lambda_3^*)''(\tau_3)(\tau_1-\tau_2)\frac{h_2'(\sigma)-h'(1/2,\sigma)}{2} \quad (34)$$

Here

$$h_1'(\sigma)\leqslant\tau_1\leqslant h'(1/2,\sigma)\leqslant\tau_2\leqslant h_2'(\sigma), \tau_1\leqslant\tau_3\leqslant\tau_2 \quad (35)$$

From the formulas (34), (35) we get the inequality

$$\delta\Lambda_3^*(\sigma)\leqslant 0,\quad \sigma\in\Delta_1 \quad (36)$$

at the points of the subset Δ_1 of the set Δ where the condition a) takes place.

Let us consider now the set $\Delta_2 = \Delta \backslash \Delta_1$ consisting of the points where the condition b) takes place. Using Taylor formula we get

$$-\,\delta\Lambda_3^* = \Lambda_3^{*'}\left(h\left(1/2,\sigma\right)\right)\frac{h_1' - h'\left(1/2,\sigma\right)}{2} + \Lambda_3^{*'}\left(h\left(1/2,\sigma\right)\right)\frac{h_2' - h'\left(1/2,\sigma\right)}{2} + \\ + \Lambda_3^{*'}\left(\tau_1\right)\frac{\left[h_1' - h'\left(1/2,\sigma\right)\right]^2}{2} + \Lambda_3^{*'}\left(\tau_2\right)\frac{\left[h_2' - h'\left(1/2,\sigma\right)\right]^2}{2} \quad (37)$$

In the expression for Λ_3^* the constant E_0 is selected in such a manner that the derivative $\Lambda_3^{*\prime}$ is positive. Of course we construct another function but we preserve the same denotation for it.

Now from the condition (37) we get the inequality

$$\delta\Lambda_3^*\left(\sigma\right) \leqslant 0, \quad \sigma \in \Delta_2 \quad (38)$$

Let us now note that

$$\delta\Xi^* = \Xi^*\left(h\left(1/2,\sigma\right)\right) - \frac{1}{2}\Xi^*\left(h_1\right) - \frac{1}{2}\Xi^*\left(h_2\right) = \\ = \int\limits_{\Delta}\delta\Lambda_3\left(\sigma\right)d\sigma = \int\limits_{\Delta_1\cup\Delta_2}\delta\Lambda_3^*\left(\sigma\right)\,d\sigma \quad (39)$$

Using now the equations (36) and (38) we arrive at the inequality (28).

The lemma is proved.

Now we can prove the main theorem

Theorem 1. *Let the coefficients*

$$\sigma,\ \lambda,\ \rho,\ \beta,\ l_p$$

are such that the following inequality takes place

$$\frac{r_0}{8} - 2c_0 l_p > 0, \quad r_0 < 1$$

Then the solution of the variational problem is unique in the class of the surfaces represented by the graph of the monotone functions.

Proof. Let y_1, y_2 two monotone functions defined over $[0,a]$ with the homotopy (3) between them. First of all let us note that for the functionals

$$I_1 = \int\limits_{S^*} ds, \quad I_2 = \lambda V, \quad I_3 = \iiint\limits_{W}\Gamma\rho\,dv$$

the following equation takes place

$$I_k\left(h\left(1/2,\sigma\right)\right) - \frac{1}{2}I_k\left(h_1\left(\sigma\right)\right) - \frac{1}{2}I_k\left(h_2\left(\sigma\right)\right) = 0, \quad k = 1,\ 2,\ 3. \quad (40)$$

We note also that the functionals

$$\lambda V, \quad \int\limits_{S^*} ds$$

are invariant under homotopical transformation (3). Besides the condition $r_0 < 1$ permits us to consider the variation of Ξ of the linear homotopy of the derivatives y_1', y_2' instead of considering it on the nonlinear (3). Now let us represent the functional F in the following form

$$F(S) = \sigma \left(Q + l_p \Xi^* - \beta \int\limits_{S^*} ds + \lambda V + \sigma^{-1} \iiint\limits_{W} \Gamma \rho \, dv \right)$$

From lemma 3 it follows that the functional

$$Q(y) := \int\limits_0^a y\sqrt{1+y'^2}\, dx - \kappa^* \int\limits_0^a \sqrt{1+y'^2}\, dx,$$

is strictly convex for any value of

$$\kappa^* < \frac{r_0}{8}$$

Now using lemma 4 and the condition (40) we get that

$$F(S(1/2)) < \inf \{F(S), S \in M\}$$

The contradiction we get means that

$$y_1 = y_2$$

The theorem is proved.

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Hidrostroitelei, 23, flat 55,
Krasnodar 350065
Russian Federation
echt@math.kubsu.ru

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