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THE BEHAVIOR OF SOLUTIONS OF THE MIXED BOUNDARY VALUE PROBLEM FOR A LINEAR SECOND-ORDER ELLIPTIC EQUATION IN A NEIGHBOURHOOD OF INTERSECTING EDGES

In this paper we deals with the mixed boundary value problem for second-order elliptic equations in a polyhedral domain. We obtain exact estimates for solutions of the problem in a neighbourhood of an vertex. A special section is dedicated to the examples.

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1. Introduction.

Let $G \subset \mathbb{R}^3$ be a bounded domain. We consider the problem with mixed boundary conditions

$$\begin{cases} Lu := a_{ij}(x)u_{x_ix_j} + a_i(x)u_{x_i} + a(x)u = f(x), & x \in G, \quad (1.1)\\ \dfrac{\partial u}{\partial \vec{n}} = 0, & x \in \Gamma_1, \quad (1.2)\\ u(x) = 0, & x \in \partial G \setminus \Gamma_1. \quad (1.3)\end{cases}$$

Here and throughout summation from 1 to 3 over repeated indices is assumed. The problem with boundary conditions (1.2)–(1.3), also known as the Zaremba problem. The main purpose of this paper is to analyze the behavior of solutions, in the case when the boundary of G contain singular points having the type of the vertex of a polyhedron. The assumptions on the coefficients of the equation are essential for obtaining sharp estimates of the modulus of a solution. Elliptic boundary problems in nonsmooth domains have been studied in many works. In particular, the exact solution estimates for boundary value problems in domains with angular and conical points at the boundary were obtained in [1]. Mixed boundary problem with conormal derivative was studied in [6] and in [7]. There the authors have obtained Schauder and weighted L^p estimates of solutions for equations with constant coefficients in polyhedral domains. The vast bibliography of elliptic boundary problems in nonsmooth domains was compiled by authors of [1].

2. Basic symbols, definitions and assumptions.

Let us introduce the following notations: $x = (x_1, x_2, x_3)$: an element of $\mathbb{R}^3$; $(r, \omega) = (r, \omega_1, \omega_2)$: spherical coordinates in $\mathbb{R}^3$, defined by: $x_1 = r\cos\omega_1 \sin\omega_2$, $x_2 = r\sin\omega_1\sin\omega_2$, $x_3 = r\cos\omega_2$; $\vec{n}$: exterior unit normal vector on ∂G; δ_i^j: Kronecker's delta; the quasi-distance function $r_\varepsilon(x) := \left(\sum\limits_{i=1}^{3}(x_i + \delta_i^3\varepsilon)^2\right)^{1/2}$; $u_x := \left(\sum\limits_{i=1}^{3} u_{x_i}^2\right)^{1/2}$; $u_{xx} := \left(\sum\limits_{i,j=1}^{3} u_{x_i x_j}^2\right)^{1/2}$. On the unit sphere S^2 we consider the domain $\Omega = \{\omega\colon \omega_1 \in [0, \varpi_1],\ \omega_2 \in [0, \varpi_2]\}$, where $\varpi_1 \in (0, 2\pi)$, $\varpi_2 \in (0, \pi)$. Our domain G coincides in some neighbourhood of the boundary point $\mathcal{O}_1$ (point of origin) with the domain $\{(r,\omega)\colon \omega \in \Omega\}$. $\partial G = \bigcup\limits_{i=1}^{3} \overline{\Gamma}_i$, Γ_1 coincides in some neighbourhood of $\mathcal{O}_1$ with the set $\{(r,\omega)\colon \omega \in \partial\Omega,\ \omega_1 = 0\}$, Γ_2 with the set $\{(r,\omega)\colon \omega \in \partial\Omega,\ \omega_1 = \varpi_1\}$ and Γ_3 with the set $\{(r,\omega)\colon\ \in \partial\Omega\ \omega_2 = \varpi_2\}$. ℓ_i are edges of boundary, $\ell_1 := \overline{\Gamma}_1 \cap \overline{\Gamma}_2$, $\ell_2 := \overline{\Gamma}_2 \cap \overline{\Gamma}_3$, $\ell_3 := \overline{\Gamma}_2 \cap \overline{\Gamma}_3$, $\ell := \bigcup\limits_{i=1}^{3} \ell_i$. Edges ℓ_i intersect at the vertices $\mathcal{O} := \bigcap\limits_{i=1}^{3} \ell_i$ of the polyhedron. We suppose that $\mathcal{O} = \{\mathcal{O}_1, \mathcal{O}_2\}$ and analyze the behavior in a neighbourhood of $\mathcal{O}_1$. Γ_i are smooth surfaces everywhere except at $\mathcal{O}$. We introduce the notation $G_a^b := \{x \in G\colon 0 \le a < r < b\}$. We denote the following spaces. $C^k(G)$: the Banach space of functions having all the derivatives of order at most k ($k \in \mathbb{N}$) continuous in G; $C_0^k(G)$: the set of functions in $C^k(G)$ with compact support in G; $L^p(G)$: the space of functions whose absolute value raised to the p-th power ($p \ge 1$) has a finite Lebesgue integral; $W^{k,p}(G)$: Sobolev space, is defined to be the subset of $L^p(G)$ such that function and its weak derivatives up to some order k ($k \in \mathbb{N}$) have a finite L^p norm, for given $p \ge 1$; $W_0^{k,p}(G)$: is the closure of $C_0^\infty(G)$ with respect to the norm $||\cdot||_{W^{k,p}(G)}||$; $W_{\mathrm{loc}}^{k,p}(G)$: the space of functions that belong to $W^{k,p}(G')$, for all $G' \subset G$; $V_{2,\alpha}^2(G)$: weighted Sobolev space, is defined as the closure of $C_0^\infty(\overline{G} \setminus \mathcal{O}_1)$ with respect to the norm

$$||u||_{V_{2,\alpha}^2(G)} = ||r^{\alpha/2}u_{xx}||_{L^2(G)} + ||r^{\alpha/2-1}u_x||_{L^2(G)} + ||r^{\alpha/2-2}u||_{L^2(G)},$$

where $\alpha \in \mathbb{R}$.

Definition 2.1. A *(strong) solution* of the problem (1.1)–(1.3) in domain G is a function $u \in W^{2,2}_{\mathrm{loc}}(\overline{G} \setminus \mathcal{O}) \cap C^0(\overline{G})$, which satisfies the equations (1.1) for almost $x \in G$, boundary condition (1.2) in the sense of traces and the boundary condition (1.3) for all $x \in \overline{\Gamma}_2 \cup \overline{\Gamma}_3$.

In the following we assume that the coefficients $a_{ij}(x)$, $a_i(x)$ and $a(x)$ satisfy the following conditions.

Assumption 2.2. The uniform ellipticity condition

$$\nu|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \mu|\xi|^2, \qquad \forall \xi \in \mathbb{R}^3,\ x \in \overline{G}.$$

with some ν, $\mu > 0$.

Assumption 2.3. $a_{ij}(0) = \delta_i^j$.

Assumption 2.4. $a_{ij} \in C^0(\overline{G})$, $a_i \in L^p(G)$ and $a \in L^{p/2}(G)$, where $p > 3$.

Assumption 2.5. There exists a monotonically increasing nonnegative function $\mathcal{A}$ such that

$$\left(\sum_{i,j=1}^{n} |a_{ij}(x) - a_{ij}(y)|^2\right)^{1/2} \le \mathcal{A}(|x-y|),$$

$$r\left(\sum_{i=1}^{3} a_i^2(x)\right)^{1/2} + r^2|a(x)| \le \mathcal{A}(r),$$

for x, $y \in \overline{G}$.

3. Some auxiliary assertions.

We denote by $\tilde{G} := \{x\colon \omega_1 \in [0, 2\pi],\ \omega_2 \in [0, \varpi_0]\}\}$, $\varpi_0 \in (0, \pi)$. Obviously, there exists an ϖ_0, such that $G \subset \tilde{G}$. It is easy to see (see lemma 1.11 [1], lemma 1.4.1 [2]) that $r_\varepsilon(x)$ has the following properties.

Lemma 3.1. *There exists an $h > 0$ such that*

$$r_\varepsilon(x) \ge h \cdot r, \qquad r_\varepsilon(x) \ge h \cdot \varepsilon$$

for all $x \in \overline{G}$, where $h = 1$ if $\varpi_0 \in (0, \pi/2]$ and $h = \sin(\varpi_0)$ if $\varpi_0 \in (\pi/2, \pi)$.

We consider the problem of the eigenvalues for the Laplace-Beltrami operator Δ_ω on the unit sphere.

$$\begin{cases} \Delta_\omega v + \vartheta v = 0, & \omega \in \Omega, \quad (3.1) \\ \dfrac{\partial v}{\partial \omega_1} = 0, & \omega_1 = 0, \quad (3.2) \\ v(\varpi_1, \omega_2) = v(\omega_1, 0) = v(\omega_1, \varpi_2) = 0. & \quad (3.3) \end{cases}$$

According to the variational principle of eigenvalues we have the Wirtinger inequality (see 2.3.1, 2.4.6 [1], 2.2.1 [2]).

Theorem 3.2. (the Wirtinger inequality). *The following inequality is valid for all $v \in W^{2,2}(\Omega)$, that satisfies (3.2)–(3.3) in the sense of traces*

$$\int\limits_\Omega v^2(\omega) d\Omega \le \frac{1}{\vartheta_0} \int\limits_\Omega \left(\left(\frac{\partial v}{\partial \omega_1} \right)^2 + \frac{1}{\sin^2 \omega_1} \left(\frac{\partial v}{\partial \omega_2} \right)^2 \right) d\Omega,$$

where ϑ_0 is the smallest positive eigenvalue of the problem (3.1)–(3.3).

Let us define the value

$$\lambda = \frac{-1 + \sqrt{1 + 4\vartheta_0}}{2}, \tag{3.4}$$

where ϑ_0 is the smallest positive eigenvalue of the problem (3.1)–(3.3). Next theorem follows from the Wirtinger inequality (see 2.5.2–2.5.9, corollary 2.29 [1], 2.3.2–2.3.9, corollary 2.3.6 [2]).

Theorem 3.3. *Let $v \in W^{2,2}(G_0^d)$ satisfy the boundary value condition (1.2)–(1.3) in the sense of traces. Then following estimates are held*

$$\int\limits_{G_\varepsilon^d} r^{\alpha-4} v^2 dx \le \frac{1}{\lambda(\lambda+1)} \int\limits_{G_\varepsilon^d} r^{\alpha-2} v_x^2 dx, \qquad \alpha \in \mathbb{R} \tag{3.5}$$

$$\int\limits_{G_\varepsilon^d} r^{\alpha-4} v^2 dx \le H(\lambda, \alpha) \int\limits_{G_\varepsilon^d} r^{\alpha-2} v_x^2 dx, \qquad \alpha \le 1, \tag{3.6}$$

where $H(\lambda, \alpha) = ((1-\alpha)^2/4 + \lambda(\lambda+1))^{-1}$, $\varepsilon \in [0, d]$ i

$$\int\limits_{G_0^d} r_\varepsilon^{\alpha-2} r^{-2} v^2 dx \le \left(\frac{3}{h} \right)^{2-\alpha} \frac{1}{\lambda(\lambda+1)} \int\limits_{G_0^d} r_\varepsilon^{\alpha-2} v_x^2 dx, \qquad \alpha \in \mathbb{R}, \tag{3.7}$$

where h is a number from the lemma 0.0, also, if $V(\rho) := \int\limits_{G_0^\rho} r^{-1} v_x^2 dx < \infty$, *then*

$$\int\limits_{\Omega} \left(\rho v \frac{\partial v}{\partial r} + \frac{v^2}{2} \right) \Bigg|_{r=\rho} d\Omega \leq \frac{\rho}{2\lambda} V'(\rho), \quad \rho \in (0, d). \tag{3.8}$$

4. Integral estimates.

At first, we will obtain a local integral estimate in the neighbourhood of an edge.

Lemma 4.1. *Let $u(x)$ be a solution of (1.1)–(1.3). Suppose that* $\lim\limits_{r\to+0} \mathcal{A}(r) = 0$ *and that* $f \in L^2(G)$. *Then there are* $d > 0$ *and constant* $c > 0$ *depends only on* ν, μ, α, λ, $\max\limits_{x\in G} \mathcal{A}(|x|)$ *and* G, *such that*

$$||u_{xx}||_{L^2(G_d^{2d})} \leq c \left(||u_x||_{L^2(G_{d/2}^{3d})} + ||u||_{L^2(G_{d/2}^{3d})} + ||f||_{L^2(G_{d/2}^{3d})} \right). \tag{4.1}$$

Proof. Let us introduce the function $v(x) = u(x)\eta(x)$, where $\eta(x) \in C^2(G_0^{2d})$ is a cutoff function such that: $\eta(x) \equiv 1$ if $r(x) \in [d, 2d]$, $0 \leq \eta(x) \leq 1$ if $r(x) \in (d/2, d) \cup (2d, 3d)$ and $\eta(x) \equiv 0$ when $r(x) \in [0, d/2] \cup [3d, \infty)$. Then the function v satisfies the equation

$$a_{ij}(x) v_{x_i x_j} + a_i(x) v_{x_i} + a(x) v = f_1(x), \tag{4.2}$$

where $f_1 = f\eta + a_{ij}(2u_{x_i}\eta_{x_j} + u\eta_{x_i x_j}) + a_i u \eta_{x_i}$. Since $a_{ij}(0) = \delta_i^j$, we have

$$\Delta v = f_1(x) - (a_{ij}(x) - a_{ij}(0)) v_{x_i x_j} - a_i(x) v_{x_i} - a(x) v := f_2(x). \tag{4.3}$$

For the equation (4.3) we use (7.19) [5] ($f_2 \in V_{2,0}^0(G_{d/2}^{3d})$), applying it for the domains $G_{d/2}^{3d}$ with edges on the boundary

$$||v_{xx}||_{L^2(G_{d/2}^{3d})} \leq c_1 ||\Delta v||_{L^2(G_{d/2}^{3d})}.$$

Using the assumption (0.0) we obtain

$$||v_{xx}||_{L^2(G_{d/2}^{3d})} \leq c_2 \cdot \mathcal{A}^2(3d) ||v_{xx}||_{L^2(G_{d/2}^{3d})} +$$

$$+ c_3 \left(||v_x||_{L^2(G_{d/2}^{3d})} + ||v||_{L^2(G_{d/2}^{3d})} + ||f_1||_{L^2(G_{d/2}^{3d})} \right).$$

Now, let $d > 0$ chosen according to the inequality $c_2 \cdot \mathcal{A}^2(3d) < 1$, then from properties of the cutoff function we obtain (4.1).

Theorem 4.2. *Let $u(x)$ be a solution of (1.1)–(1.3) and λ be defined by (3.4). Suppose that $\lim\limits_{r\to+0} \mathcal{A}(r) = 0$, $f \in V^0_{2,\alpha}(G)$, where $\alpha \in (1 - 2\lambda, 2]$. Then $u \in V^2_{2,\alpha}(G^d_0)$ and*

$$||u||_{V^2_{2,\alpha}(G^d_0)} \le c\left(||u||_{V^0_{2,\alpha}(G^d_0)} + ||u_x||_{V^0_{2,\alpha}(G^d_0)} + ||f||_{V^0_{2,\alpha}(G^d_0)}\right), \qquad (4.4)$$

where $c > 0$ depends only on ν, μ, α, λ, $\max\limits_{x\in G} \mathcal{A}(|x|)$ i G.

Proof. Let us introduce the function $v(x) = u(x)\eta(x)$, where $\eta(x) \in C^2(G^{2d}_0)$ is a cutoff function such that: $\eta(x) \equiv 1$ if $r(x) \in [0, d]$, $0 \le \eta(x) \le 1$ if $r(x) \in (d, 2d)$ and $\eta(x) \equiv 0$ when $r(x) \in [2d, \infty)$.
Case I: $1 \le \alpha \le 2$. We multiply both parts of the (4.3) by $r^{\alpha-2}v(x)$ and integrate over the domain G^{2d}_ε. Twice integrating by parts we obtain the analog of (4.3.6) [1] (see also (4.2.6) [2])

$$\begin{gathered}\varepsilon^{\alpha-2}\int\limits_{\Omega_\varepsilon} v\frac{\partial v}{\partial r}d\Omega_\varepsilon + \int\limits_{G^{2d}_\varepsilon} r^{\alpha-2}v_x^2dx + \frac{(2-\alpha)\varepsilon^{\alpha-3}}{2}\int\limits_{\Omega_\varepsilon} v^2 d\Omega_\varepsilon + \\ + \frac{(2-\alpha)(\alpha-1)}{2}\int\limits_{G^{2d}_\varepsilon} r^{\alpha-4}v^2dx = \\ = \frac{2-\alpha}{2}\int\limits_{\Gamma_1\cap\partial G^{2d}_\varepsilon} r^{\alpha-4}v^2x_i\cos(\vec{n}, x_i)d\sigma + \\ + \int\limits_{G^{2d}_\varepsilon} r^{\alpha-2}v(-f_1(x) + (a_{ij}(x) - a_{ij}(0))v_{x_ix_j} + a_i(x)v_{x_i} + a(x)v)dx.\end{gathered} \qquad (4.5)$$

where $d\sigma$ area element of Γ_1. Since $x_i\cos(\vec{n}, x_i) = x_2\cos(\vec{n}, x_2) = -x_2 = 0$, $\forall x \in \Gamma_1 \cap \partial G^{2d}_\varepsilon$, therefore

$$\int\limits_{\Gamma_1\cap\partial G^{2d}_\varepsilon} r^{\alpha-4}v^2x_i\cos(\vec{n}, x_i)d\sigma = 0. \qquad (4.6)$$

Let us estimate in the above equation the integrals over Ω_ε. We consider the set $G^{2\varepsilon}_\varepsilon$ and we have $\Omega_\varepsilon \subset \partial G^{2\varepsilon}_\varepsilon$. Now we use the inequality (6.23) [4]

$$\int\limits_{\Omega_\varepsilon} |w|d\Omega_\varepsilon \le c_1\int\limits_{G^{2\varepsilon}_\varepsilon} (|w| + |w_x|)dx. \qquad (4.7)$$

Setting $w = v\dfrac{\partial v}{\partial r}$ we find (see (4.3.8) [1], (4.2.8) [2])

$$\int\limits_{\Omega_\varepsilon} \left| v\frac{\partial v}{\partial r} \right| d\Omega_\varepsilon \le c_2 \int\limits_{G_\varepsilon^{2\varepsilon}} \left(r^2 v_{xx}^2 + v_x^2 + r^{-2} v^2 \right) dx. \tag{4.8}$$

Twice using (3.5) we obtain

$$\int\limits_{G_\varepsilon^{2\varepsilon}} (v_x^2 + r^{-2} v^2 dx)\, dx \le c_3 \int\limits_{G_\varepsilon^{2\varepsilon}} v_x^2 dx \le$$
$$\le 4c_3\varepsilon^2 \int\limits_{G_\varepsilon^{2\varepsilon}} r^{-2} v_x^2 dx \le c_4\varepsilon^2 \int\limits_{G_\varepsilon^{2\varepsilon}} v_{xx}^2 dx \le c_5 \int\limits_{G_\varepsilon^{2\varepsilon}} r^2 v_{xx}^2 dx,$$

therefore from (4.8) we get

$$\int\limits_{\Omega_\varepsilon} \left| v\frac{\partial v}{\partial r} \right| d\Omega_\varepsilon \le c_6 \int\limits_{G_\varepsilon^{2\varepsilon}} r^2 v_{xx}^2 dx.$$

Applying the local integral estimate (4.1) we obtain

$$\begin{aligned} &\int\limits_{\Omega_\varepsilon} \left| v\frac{\partial v}{\partial r} \right| d\Omega_\varepsilon \le \\ &\quad \le c_6 \int\limits_{G_\varepsilon^{2\varepsilon}} r^2 v_{xx} dx \le c_7 \int\limits_{G_{\varepsilon/2}^{3\varepsilon}} r^2 (v_x^2 + v^2 + f_1^2) dx \le \\ &\qquad \le c_8 \varepsilon^{2-\alpha} \int\limits_{G_{\varepsilon/2}^{3\varepsilon}} r^\alpha (v_x^2 + v^2 + f_1^2) dx \le \\ &\qquad \le c_8 \varepsilon^{2-\alpha} \int\limits_{G_{\varepsilon/2}^{2d}} r^\alpha (v_x^2 + v^2 + f_1^2) dx. \end{aligned} \tag{4.9}$$

Let us apply again (4.7), in analogy to (4.9) we have

$$\begin{aligned}
&\int\limits_{\Omega_\varepsilon} v^2 d\Omega_\varepsilon \leq \\
&\quad \leq c_9 \int\limits_{G_\varepsilon^{2\varepsilon}} (v^2 + |v||v_x|)dx \leq c_{10} \int\limits_{G_\varepsilon^{2\varepsilon}} (rv_x^2 + r^{-1}v^2)dx \leq \\
&\leq c_{11} \int\limits_{G_\varepsilon^{2\varepsilon}} r^3 v_{xx}^2 dx \leq c_{12}\varepsilon^{3-\alpha} \int\limits_{G_{\varepsilon/2}^{3\varepsilon}} r^\alpha (v_x^2 + v^2 + f_1^2)dx \leq \\
&\qquad\qquad \leq c_{12}\varepsilon^{3-\alpha} \int\limits_{G_{\varepsilon/2}^{2d}} r^\alpha (v_x^2 + v^2 + f_1^2)dx.
\end{aligned} \tag{4.10}$$

Writing the inequality (4.1) for the $\rho \in (0, d)$ and taking into account that $\rho \sim r$ in $G_\rho^{2\rho}$, we obtain

$$\int\limits_{G_\rho^{2\rho}} r^\alpha v_{xx}^2 dx \leq c_{13} \int\limits_{G_{\rho/2}^{3\rho}} r^\alpha (v_x^2 + v^2 + f_1^2)dx.$$

We replace ρ by $2^{-k}d$. Summing up this inequalities for $k = 0, 1, \ldots,$ $[\log_2(d/\varepsilon)] + 1$, we get

$$\int\limits_{G_\varepsilon^{2d}} r^\alpha v_{xx}^2 dx \leq c_{14} \int\limits_{G_{\varepsilon/4}^{2d}} r^\alpha (v_x^2 + v^2 + f_1^2)dx. \tag{4.11}$$

Applying assumption (0.0) together with the Hölder and the Cauchy inequality

$$\begin{aligned}
&r_\epsilon^{\alpha-2} v \left((a_{ij}(x) - \delta_i^j)v_{x_i x_j} + a_i(x)v_{x_i} + a(x)v\right) \leq \\
&\quad \leq c_{15}\mathcal{A}(r) \left(r_\epsilon^{\alpha-2} r^2 v_{xx}^2 + r_\epsilon^{\alpha-2} v_x^2 + r_\epsilon^{\alpha-2} r^{-2} v^2\right) \\
&\qquad\qquad r_\epsilon^{\alpha-2} v f_1 \leq \frac{\delta}{2} r_\epsilon^{\alpha-2} r^{-2} v^2 + c_{16} r^\alpha f_1^2,
\end{aligned} \tag{4.12}$$

for all $\delta > 0$, $\epsilon \geq 0$. Let $\epsilon = 0$. From (4.5), (4.6) and (4.9)–(4.12) follows that

$$\int\limits_{G_\varepsilon^{2d}} r^\alpha v_{xx}^2 dx + \int\limits_{G_\varepsilon^{2d}} r^{\alpha-2} v_x^2 dx + \frac{(2-\alpha)(\alpha-1)}{2} \int\limits_{G_\varepsilon^{2d}} r^{\alpha-4} v^2 dx \le$$

$$\le c_{17} \int\limits_{G_\varepsilon^{2d}} \left(\mathcal{A}(2d)(r^{\alpha-2}v_x^2 + r^{\alpha-4}v^2) + \delta r^{\alpha-4} v^2\right) dx +$$

$$+ c_{18} \int\limits_{G_{\varepsilon/4}^{2d}} r^\alpha (v_x^2 + v^2 + f_1^2) dx,$$

for all $\delta > 0$ and $0 < \varepsilon < d$. Furthermore, if $(2-\alpha)(\alpha-1) = 0$, then we apply the inequality(3.5). Now, let $\delta > 0$, $d > 0$ are small enough. Then we obtain

$$\int\limits_{G_\varepsilon^{2d}} (r^\alpha v_{xx}^2 + r^{\alpha-2} v_x^2 + r^{\alpha-4} v^2) dx \le c_{19} \int\limits_{G_{\varepsilon/4}^{2d}} r^\alpha (v_x^2 + v^2 + f_1^2) dx,$$

where the constants c_{19} do not depend on ε. Letting $\varepsilon \to +0$ we obtain the assertion of our theorem in the case I.

Case II: $1 - 2\lambda < \alpha < 1$, $\alpha \ge 0$. From the inequality (4.1) we have

$$\int\limits_{G_\rho^{2\rho}} \rho^2 (\rho+\varepsilon)^{\alpha-2} v_{xx}^2 dx \le c_{20} \int\limits_{G_{\rho/2}^{3\rho}} \rho^2 (\rho+\varepsilon)^{\alpha-2} (v_x^2 + v^2 + f_1^2) dx.$$

Since $r_\varepsilon \le r + \varepsilon \le 2r_\varepsilon/h$ in $\overline{G}$, we obtain

$$\int\limits_{G_\rho^{2\rho}} r^2 r_\varepsilon^{\alpha-2} v_{xx}^2 dx \le c_{21} \int\limits_{G_{\rho/2}^{3\rho}} r^2 r_\varepsilon^{\alpha-2} (v_x^2 + v^2 + f_1^2) dx.$$

Let $\rho = 2^{-k} d$. Summing up this inequalities for $k = 0, 1, \dots,$ we finally obtain

$$\int\limits_{G_0^{2d}} r^2 r_\varepsilon^{\alpha-2} v_{xx}^2 dx \le c_{22} \int\limits_{G_0^{2d}} r^2 r_\varepsilon^{\alpha-2} (v_x^2 + v^2 + f_1^2) dx. \tag{4.13}$$

Multiplying both sides of (4.3) by $r_\varepsilon^{\alpha-2} v(x)$ and integrating by parts twice we obtain (compare with case I)

$$\int\limits_{G_0^{2d}} r_\varepsilon^{\alpha-2} v_x^2 dx = \frac{(2-\alpha)(1-\alpha)}{2} \int\limits_{G_0^{2d}} r_\varepsilon^{\alpha-4} v^2 dx +$$
$$+\frac{2-\alpha}{2} \int\limits_{\Gamma_1 \cap \partial G_\varepsilon^{2d}} r_\varepsilon^{\alpha-4} v^2 (x_i + \varepsilon \delta_i^3) \cos(\vec{n}, x_i) d\sigma +$$
$$+ \int\limits_{G_0^{2d}} r_\varepsilon^{\alpha-2} v(-f_1(x) + (a_{ij}(x) - a_{ij}(0)) v_{x_i x_j} + a_i(x) v_{x_i} + a(x) v) dx,$$

where $d\sigma$ area element of Γ_1. The second integral on the right is equal to zero (see (4.6)). Therefore from (4.12) for $\epsilon = \varepsilon$ we get

$$\int\limits_{G_0^{2d}} r_\varepsilon^{\alpha-2} v_x^2 dx \leq \frac{(2-\alpha)(1-\alpha)}{2} \int\limits_{G_0^{2d}} r_\varepsilon^{\alpha-4} v^2 dx +$$
$$+ c_{23} \int\limits_{G_0^{2d}} \left(r_\varepsilon^{\alpha-2} \left(\delta r^{-2} v^2 + \mathcal{A}(2d)(r^2 v_{xx}^2 + v_x^2 + r^{-2} v^2) \right) + r^\alpha f_1^2 \right) dx.$$

Since by case I $u \in V_{2,2}^2(G_0^d)$ and $f_1 \in V_{2,\alpha}^0(G_0^d)$ $(\alpha \geq 0)$ the integral from the right side is finite. Therefore from (3.6), (3.7) and (4.13) we have

$$C(\lambda,\alpha) \int\limits_{G_0^{2d}} r_\varepsilon^{\alpha-2} v_x^2 dx \leq$$
$$\leq c_{24} \int\limits_{G_0^{2d}} \left(r_\varepsilon^{\alpha-2} \left((\mathcal{A}(2d) + \delta) v_x^2 + r^2 (v_x^2 + v^2 + f_1) \right) + r^\alpha f_1^2 \right) dx,$$

where $C(\lambda,\alpha) = 1 - \frac{1}{2}(2-\alpha)(1-\alpha)H(\lambda,\alpha) > 0$. Choosing $\delta > 0$ and $d > 0$ small enough and passing to the limits as $\varepsilon \to 0$, by the Fatou Theorem we obtain the assertion, if we recall (3.6) and (4.13).

Case III: $1 - 2\lambda < \alpha < 1$, $\alpha < 0$. We take any $\alpha_0 \in [\max(-2,\alpha), 0]$. Then we have u, u_x, $f_1 \in V_{2,\alpha_0+2}^0(G_0^d)$. Now, we can repeat verbatim the proof of case II. We get $u \in V_{2,\alpha_0}^2(G_0^d)$ and (4.4). Repeating the stated process k times we obtain $u \in V_{2,\alpha_k}^2(G_0^d)$, where $\alpha_k = \alpha_{k-1} - 2$. Obviously, we can find such an integer k that $\alpha_{k+1} \leq \alpha \leq \alpha_k$. Finally, repeating the proof of case II once again, we obtain the assertion.

Corollary 4.3. Let $u(x)$ be a solution of (1.1)–(1.3). Suppose that

$\lim\limits_{r\to+0}\mathcal{A}(r)=0$ and $f\in L^2(G_0^{2d})$. Then

$$||u||_{V^2_{2,0}(G_0^d)}\le c\left(||u||_{L^2(G_0^{2d})}+||f||_{L^2(G_0^{2d})}\right). \qquad (4.14)$$

Proof. Let us fix $d>0$, such that the inequality (4.4) would be fulfilled. We take $\rho\in(0,d/2)$ and $\varsigma\in(0,1)$. Let us introduce the cutoff function $\eta\in C^2(G_0^{2\rho})$, such that $\eta(x)\equiv1$ if $r(x)\in[0,\varsigma\rho]$, $0\le\eta(x)\le1$ if $r(x)\in(\varsigma\rho,\varsigma'\rho)$, $\eta(x)\equiv0$ when $r(x)\in[\varsigma'\rho,\infty)$, $|\eta_x|\le4/((1-\varsigma)\varepsilon)$, $|\eta_{xx}|\le16/((1-\varsigma)^2\varepsilon^2)$, where $\varsigma'=(1+\varsigma)/2$. Now, if $v=\eta u$ we apply the estimate (4.4) to the solution v of the (4.2) with $\alpha=0$

$$\begin{aligned}||u_{xx}||_{L^2(G_0^{\varsigma\rho})}&\le c_1\left(||u||_{L^2(G_0^{\varsigma'\rho})}+||u_x||_{L^2(G_0^{\varsigma'\rho})}+||f_1||_{L^2(G_0^{\varsigma'\rho})}\right)=\\&=c_1\left(||u||_{L^2(G_0^{\varsigma'\rho})}+||u_x||_{L^2(G_0^{\varsigma'\rho})}+\right.\\&\left.+||a_{ij}(2u_{x_i}\eta_{x_j}+u\eta_{x_ix_j})+a_iu\eta_{x_i}+f\eta||_{L^2(G_0^{\varsigma'\rho})}\right)\le\\&\le c_2\left(||f||_{L^2(G_0^{\varsigma'\rho})}+\frac{1}{(1-\varsigma)\rho}||u_x+r^{-1}u||_{L^2(G_0^{\varsigma'\rho})}+\right.\\&\left.+\frac{1}{(1-\varsigma)^2\rho^2}||u||_{L^2(G_0^{\varsigma'\rho})}\right).\end{aligned}$$

Rewriting this inequality in the form

$$\begin{aligned}&\sup_{0<\varsigma<1}(1-\varsigma)^2\rho^2||u_{xx}||_{L^2(G_0^{\varsigma\rho})}\le\\&\le c_3\left(\rho^2||f||_{L^2(G_0^{\rho})}+\sup_{0<\varsigma<1}(1-\varsigma)\rho||u_x||_{L^2(G_0^{\varsigma'\rho})}+\sup_{0<\varsigma<1}||u||_{L^2(G_0^{\varsigma'\rho})}\right)=\\&=c_3\left(\rho^2||f||_{L^2(G_0^{\rho})}+\right.\\&\left.+2\sup_{1/2<\varsigma'<1}(1-\varsigma')\rho||u_x||_{L^2(G_0^{\varsigma'\rho})}+2\sup_{1/2<\varsigma'<1}||u||_{L^2(G_0^{\varsigma'\rho})}\right)\le\\&\le c_4\left(\rho^2||f||_{L^2(G_0^{\rho})}+\sup_{0<\varsigma<1}(1-\varsigma)\rho||u_x||_{L^2(G_0^{\varsigma\rho})}+\sup_{0<\varsigma<1}||u||_{L^2(G_0^{\varsigma\rho})}\right)\le\end{aligned}$$

from the interpolation inequality (see (7.61), example 7.19 [3])

$$\begin{aligned}&\le c_5\left(\rho^2||f||_{L^2(G_0^{\rho})}+\sup_{0<\varsigma<1}(1-\varsigma)\rho\left(\varepsilon(1-\varsigma)\rho||u_x||_{L^2(G_0^{\varsigma\rho})}+\right.\right.\\&\left.\left.+\varepsilon^{-1}(1-\varsigma)^{-1}\rho^{-1}||u||_{L^2(G_0^{\varsigma\rho})}\right)+\sup_{0<\varsigma<1}||u||_{L^2(G_0^{\varsigma\rho})}\right).\end{aligned}$$

Hence, choosing $\varepsilon > 0$ sufficiently small, we can write

$$||u_{xx}||_{L^2(G_0^{\varsigma\rho})} \leq \frac{c_6}{(1-\varsigma)^2\rho^2}\left(||f||_{L^2(G_0^{\rho})} + ||u||_{L^2(G_0^{\varsigma\rho})}\right).$$

Taking $\varsigma = 1/2$ and using (3.5), we arrive to the sought estimate (4.14).

Theorem 4.4. Let $u(x)$ be a strong solution of problem (1.1)–(1.3) and assumptions (2.2)–(0.0) are satisfied with $\mathcal{A}(r)$ Dini continuous at zero. Suppose, in addition $f \in V^0_{2,1}(G)$ and there exist real numbers $s > 0$, $k_s \geq 0$ such that $k_s = \sup\limits_{\rho>0} \rho^{-s}||f||_{V^0_{2,1}(G_0^{\rho})}$. Then there are $d > 0$ and a constant $c > 0$ depends only on ν, μ, $\mathcal{A}(d)$, s, λ, G and on the quantity $\int\limits_0^d t^{-1}\mathcal{A}(t)dt$ such that $\forall\rho \in (0, d)$

$$||u||_{V^2_{2,1}(G_0^{\rho})} \leq c\left(||u||_{L^2(G)} + ||f||_{V^0_{2,1}(G)} + k_s\right)\cdot \begin{cases} \rho^{\lambda}, & s > \lambda, \\ \rho^{\lambda}\ln^{3/2}(1/\rho), & s = \lambda, \\ \rho^{s}, & s < \lambda. \end{cases} \tag{4.15}$$

Proof. We consider the equation (4.3) with $\eta \equiv 1$ $(v \equiv u)$. Let us now multiply both parts of the (4.3) by $r^{-1}u$ and integrate over G_0^{ρ}; twice having applied the formula of integration by parts. As a result we have

$$\int\limits_{\Omega}\left(\rho u\frac{\partial u}{\partial r} + \frac{u^2}{2}\right)d\Omega - \int\limits_{G_0^{\rho}} r^{-1}u_x^2 dx =$$
$$= \int\limits_{G_0^{\rho}} r^{-1}u\left((a_{ij}(x) - a_{ij}(0))u_{x_ix_j} + a_i(x)u_{x_i} + a(x)u\right)dx.$$

Let $U(\rho) := \int\limits_{G_0^{\rho}} r^{-1}u_x^2 dx$. From the assumption (0.0), estimates (3.6), (3.8) and the Cauchy inequality we obtain for $\forall\delta > 0$

$$U(\rho) \leq \tfrac{\rho}{2\lambda}V'(\rho) + c_1\mathcal{A}(\rho)\int\limits_{G_0^{\rho}} ru_{xx}^2 dx +$$
$$+ c_2\mathcal{A}(\rho)U(\rho) + \tfrac{\delta}{2}U(\rho) + \tfrac{1}{2\delta}||f||^2_{V^0_{2,1}(G_0^{\rho})}$$

If we take into account (4.4) and condition on the function f, we get

$$U(\rho) \leq \tfrac{\rho}{2\lambda}U'(\rho) + c_3\mathcal{A}(\rho)U(2\rho) +$$
$$+ c_4(\mathcal{A}(\rho) + \delta)U(\rho) + c_5\tfrac{1}{\delta}k_s^2\rho^{2s}, \quad \forall\delta > 0,\ \rho \in (0, d). \tag{4.16}$$

Moreover, because of (4.14) in virtue of the obvious embedding $V^0_{2,0}(G^\rho_0) \subset V^0_{2,1}(G^\rho_0)$, we have the initial condition $U(d) \equiv U_0 < \infty$. The estimate (4.15) follow from (4.16), in the same way as (4.3.43) [1] from (4.3.47) [1] (see also (4.2.43) and (4.2.47) [2]).

5. The estimate of the solution modulus.

Theorem 5.1. *Let $u(x)$ be a strong solution of problem (1.1)–(1.3) and let the assumptions of theorem 4.4 be satisfied. Then there are $d > 0$ and a constant $c > 0$, depends on the same values as constant c in the theorem 4.4, such that for $\forall x \in G^d_0$*

$$|u(x)| \leq c\left(||u||_{L^2(G)} + ||f||_{V^0_{2,1}(G)} + k_s\right)\cdot\begin{cases} r^\lambda, & s > \lambda, \\ r^\lambda \ln^{3/2}(1/r), & s = \lambda, \\ r^s, & s < \lambda. \end{cases} \quad (5.1)$$

Proof. Let us introduce the function

$$\psi(\rho) = \begin{cases} \rho^\lambda, & s > \lambda, \\ \rho^\lambda \ln^{3/2}(1/\rho), & s = \lambda, \\ \rho^s, & s < \lambda, \end{cases}$$

for $\rho \in (0, d)$. We make the transformation $x = \rho x'$, $u(x) = v(\rho x') = \psi(\rho)w(x')$. By the Sobolev Imbedding theorems (see (7.30) [3]) $W^{2,2}(G^1_{1/2}) \subset C^0(G^1_{1/2})$ and we have

$$\sup_{G^1_{1/2}} |w(x')| \leq c_1||w||_{W^{2,2}(G^1_{1/2})}$$

Returning to the variables x, u considering the inequality (4.15), we have for $\forall\rho \in (0, d)$

$$\sup_{G^\rho_{\rho/2}} \psi^{-1}(\rho)|u(x)| \leq c_2\psi^{-2}(\rho)||u||_{V^2_{2,1}(G^\rho_{\rho/2})} \leq c_3(||u||_{L^2(G)}+||f||_{V^0_{0,1}(G)}+k_s)$$

Putting now $r = 2\rho/3$, we obtain finally the desired estimate.

6. Remarks and examples.

Remark 6.1. The solution of (1.1)–(1.3) can be taken as a function from $W^{2,2}_{\text{loc}}(\overline{G} \setminus \ell) \cap C^0(\overline{G})$. Then from [8] we obtain $W^{2,2}_{\text{loc}}(\overline{G} \setminus \mathcal{O}) \cap C^0(\overline{G})$.

Remark 6.2. The number λ that is defined by (3.4) cannot in general be expressed as explicit functions of ϖ_1 and ϖ_2. There are a few examples (see below), where λ can be calculated directly. They shows that the exponent λ in (5.1) cannot be increased.

Example 6.3. Let $\Omega = \{\omega \colon \omega_1 \in [0, \varpi_1],\ \omega_2 \in [0, \varpi_2]\}$, where $\varpi_1 = \dfrac{\pi \cos^2 \varpi_2}{1 - 3\cos^2 \varpi_2}$, $\varpi_2 = \dfrac{5\pi}{12}$ is domain on the unit sphere S^2. Then $\vartheta_0 = 4(\gamma+2)(\gamma+3)$ is the smallest positive eigenvalue of the eigenvalue problem (3.1)–(3.3) ($\gamma = \dfrac{\pi}{2\varpi_1}$) and $\lambda = \gamma + 2$. Let us consider the function

$$u(x) = r^{\gamma+2} \cos \gamma\omega_1 \sin^\gamma \omega_2 (\cos^2 \omega_2 - \cos^2 \varpi_2)$$

in $G = \{(r, \omega) \colon 0 < r < \infty,\ \omega \in \Omega\}$. It is the solution of (1.1)–(1.3) for Laplacian.

Example 6.4. Let $\Omega = \{\omega \colon \omega_1 \in [0, \varpi_1],\ \omega_2 = \pi/2\}$, where $\varpi_1 \in (0, 2\pi)$. Then $\vartheta_0 = 4(\gamma+1)(\gamma+2)$ is the smallest positive eigenvalue of the eigenvalue problem (3.1)–(3.3) ($\gamma = \dfrac{\pi}{2\varpi_1}$) and $\lambda = \gamma + 1$. Let us consider the function

$$u(x) = r^{\gamma+1} \cos \gamma\omega_1 \sin^\gamma \omega_2 \cos \omega_2$$

in domain $G = \{(r, \omega) \colon 0 < r < \infty,\ \omega \in \Omega\}$. It is the solution of (1.1)–(1.3) for Laplacian.

Example 6.5. Let γ and domain G be defined as in the example 6.4 and let

$$u(x) = r^{\gamma+1} \ln(1/r) \cos \gamma\omega_1 \sin^\gamma \omega_2 \cos \omega_2.$$

The function u satisfies in the domain G_0^d following equations

$$a_{ij}(x) u_{x_i x_j} :=$$

$$= \left(\delta_i^j - \frac{2\gamma+3}{\gamma(\gamma+1)\ln(1/r)} \left(\frac{2\gamma+1}{2\gamma+3} \cdot \delta_i^j - \frac{x_i x_j}{r^2} \right) \right) u_{x_i x_j} = 0, \tag{6.1}$$

$$\Delta u = -a_i(x) u_{x_i} := -\frac{(2\gamma+3)x_i}{r^2((\gamma+1)\ln(1/r) - 1)} u_{x_i}, \tag{6.2}$$

$$\Delta u = -a(x) u := -\frac{2\gamma+3}{r^2 \ln(1/r)} u, \tag{6.3}$$

$$\Delta u = f(x) := -(2\gamma+3) r^{\gamma-1} \cos \gamma\omega_1 \sin^\gamma \omega_2 \cos \omega_2. \tag{6.4}$$

If $d < e^{(6\gamma+7)/(\gamma^2+\gamma)}$, then the equation (6.1) is uniformly elliptic with ellipticity constants $\nu = 1 - \dfrac{6\gamma+7}{\gamma(\gamma+1)\ln(1/d)}$, $\mu = 1 + \dfrac{4\gamma+8}{\gamma(\gamma+1)\ln(1/d)}$. Furthermore, $\mathcal{A}(r) = \dfrac{6\gamma+9}{\gamma(\gamma+1)\ln(1/r)}$. If $d < e^{-1}$, then for the equation (6.2) we have $\mathcal{A}(r) = \dfrac{6\gamma+9}{\gamma\ln(1/r)}$ and for the equation (6.3) we have $\mathcal{A}(r) = \dfrac{2\gamma+3}{\ln(1/r)}$. In all these cases $\int\limits_0^d r^{-1}\mathcal{A}(r)dr = +\infty$, that is the leading coefficients of the (6.1) and the lower order coefficients of (6.2), (6.3) are continuous but not Dini continuous at zero. From the explicit form of the solution $u(x)$ we have $|u(x)| \le cr^{\gamma+1-\varepsilon} = cr^{\lambda-\varepsilon}$, for all $\varepsilon > 0$, $x \in G_0^d$. Thus the assumptions about the coefficients are essential. In the case of (6.4) all assumptions on the coefficients are satisfied, but $||f||_{V^0_{2,1}(G_0^\rho)} \le c\rho^s$ with $s = \lambda$. This verifies the importance of conditions of our theorems.

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