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THE STEFAN PROBLEM

The Stefan problem in its classical statement is a mathematical model of the process of propagation of heat in a medium with different phase states, e.g., in a medium with liquid and solid phases.The process of propagation of heat in each phase is described by the parabolic equations. In this work we prove the existence of the global classical solution in a two-phase multidimensional Stefan problem. We apply a method which consists of the following. First, we construct approximating problems, then we prove some uniform estimates and pass to the limit.

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1. Statement of the problem.

The Stefan problem in its classical statement is a mathematical model of the process of propagation of heat in a medium with different phase states, e.g., in a medium with liquid and solid phases. As a result of melting or crystallization, the domains occupied by the liquid and solid phases undergo certain changes. This unknown interface is called a free boundary. The process of propagation of heat in each phase is described by the heat equation.

Let $D \in \mathbb{R}^3$ - the bounded domain , which boundary consists of two $C^{2+\alpha}$ surfaces ∂D_1 and ∂D_2 such, that ∂D_1 contains inside ∂D_2 and limits domain D, $D_T = D \times (0,T)$. The problem is to find a function $u(x,t)$ and domains Ω_T, G_T, satisfying to following conditions:

$$\Delta u - a(u)\frac{\partial u}{\partial t} = 0 \text{ in } \Omega_T \cup G_T, \tag{1.1}$$

$$\Omega_T =\{(x,t) \in D_T : 0 < u(x,t) < 1\},\ G_T =\{(x,t) \in D_T : u(x,t) > 1\},$$

where $a(u)$ - step function $a_1 > 0$ в Ω_T and $a_2 > 0$ in G_T ; on the known boundary ∂D_T

$$u(x,t) = 0 \text{ on } \partial D_1 \times (0,T),\ \ u(x,t) = \varphi(x,t) > 1 \text{ on } \partial D_2 \times (0,T); \tag{1.2}$$

on the unknown (free) boundary $\gamma_T = D_T \cap \partial\Omega_T = D_T \cap \partial G_T$

$$u^-(x,t) = u^+(x,t) = 1,\ \sum_{i=1}^{3}\Big(\frac{\partial u^-}{\partial x_i} - \frac{\partial u^+}{\partial x_i}\Big)\cos(n,x_i) + \lambda\cos(n,t) = 0; \tag{1.3}$$

where λ is a positive constant, n is the normal to the surface γ_T directed to the side of increase of the function $u(x,t)$; $u^+(x,t), u^-(x,t)$ are the boundary values on the surface γ_T taken from the domains G_T, Ω_T respectively. The function $u(x,t)$ is interpreted as the temperature of the medium, γ_t is the interface between the liquid and solid phases, and $u(x,t) = 1$ is the temperature of melting. The initial conditions are

$$\begin{gathered} u(x,0) = \psi(x) > 0 \text{ in } D,\ \psi(x)\Big|_{\partial D_1} = 0, \\ \psi(x)\Big|_{\partial D_2} = \varphi(x,0) > 1, \\ \Omega_0 = \{x \in D : 0 < \psi(x) < 1\}, \\ G_0 = \{x \in D : \psi(x) > 1\},\ \gamma_0 = D \cap \partial\Omega_0. \end{gathered} \tag{1.4}$$

2.Construction of the approximating problem.

Assume that problem has a classical solution. Multiply the equation (1.1) by a smooth function $\eta(x,t)$ which vanishes on ∂D_T and integrate by parts:

$$\begin{gathered} \int\limits_{D_T}\Big(\nabla u(x,t)\nabla\eta(x,t) + a(u)\tfrac{\partial u}{\partial t}\eta(x,t) + \lambda\chi(u)\tfrac{\partial\eta}{\partial t}\Big)dxdt + \\ +\lambda\int\limits_{D}\chi(\psi)\eta(x,0)dx = 0, \end{gathered} \tag{2.1}$$

where

$$\chi(u) = \begin{cases} 1, \text{ if } u(x,t) < 1, \\ 0, \text{ if } u(x,t) > 1 \end{cases}.$$

For $\forall\varepsilon > 0$ we introduce a function $\chi_\varepsilon(\tau) \in C^\infty(\mathbb{R}^1)$:

$$\chi_\varepsilon(\tau) = 1\ \forall\tau \le 1-\varepsilon,\ \chi_\varepsilon(\tau) = 0\ \forall\tau \ge 1,\ \chi'_\varepsilon(\tau) \le 0,\ \chi_\varepsilon^{(n)}(x) \le \frac{c}{\varepsilon^n},$$

Let

$$a_\varepsilon(x) = a_1 + \chi_\varepsilon(x)(a_2 - a_1).$$

Define the function $\{u^\varepsilon(x,t)\}$ as solutions of the following problem:

$$\Delta u^\varepsilon(x,t) - a_\varepsilon(u^\varepsilon(x,t))\frac{\partial u^\varepsilon(x,t)}{\partial t} = -\lambda \frac{\partial \chi_\varepsilon(u^\varepsilon(x,t))}{\partial t} \text{ in } D_T \quad (2.2)$$

$$u^\varepsilon(x,t) = 0 \text{ on } \partial D_1 \times [0,T), \quad u^\varepsilon(x,t) = \varphi(x,t) \text{ on } \partial D_2 \times [0,T); \quad (2.3)$$

$$u^\varepsilon(x,0) = \psi(x) \text{ in } D, \ \psi(x) = 0 \text{ on } \partial D_1, \ \psi(x) = \varphi(x,0) > 1 \text{ on } \partial D_2. \quad (2.4)$$

Then, as is well known [1], takes place the statement

Theorem 2.1. Let

$$\psi(x) \in C^{l+\alpha}(\overline{D}), \ \varphi(x,t) \in H^{l+\alpha,\frac{l+\alpha}{2}}(\overline{D_T}), 0 < \alpha < 1,$$

and assume that the corresponding consistency conditions hold at $t = 0, x \in \partial D$. Then this problem is solvable and

$$\left\| u^\varepsilon(x,t) \right\|_{H^{l+\alpha,\frac{l+\alpha}{2}}(\overline{D_T})} \le \frac{c}{M(\varepsilon)}, \quad (2.5)$$

where positive constant c do not depend on ε,$M(\varepsilon) \to 0$, if $\varepsilon \to 0$.

The equation (2.2) we shall transform to a form

$$\Delta u^\varepsilon(x,t) - \frac{\partial}{\partial t} \int\limits_0^{u^\varepsilon(x,t)} [a_\varepsilon(\tau) - \lambda\chi'_\varepsilon(\tau)]d\tau = 0.$$

We divide the cylinder D_T by the planes $t = k\tau, k = 1,2,...N, N\tau = T$, integrate equation (2.2) with respect to the variable t, from $(k-1)\tau$ to $k\tau$.

$$\int\limits_{(k-1)\tau}^{k\tau} \Delta u^\varepsilon(x,\tau)d\tau - \int\limits_{u^\varepsilon_{k-1}(x)}^{u^\varepsilon_k(x)} [a_\varepsilon(\tau) - \lambda\chi'_\varepsilon(\tau)]d\tau = 0,$$

where $u^\varepsilon_k(x) \equiv u^\varepsilon(x,k\tau)$. After simple transformations we obtain

$$\triangle u^\varepsilon_k(x) - \beta^\varepsilon_k(x)\frac{\partial u^\varepsilon_k(x)}{\partial \overline{t}} = f^\varepsilon_k(x), \quad (2.6)$$

$$u^\varepsilon_k(0) = 0, \ \ u^\varepsilon_k(l) = \varphi(k\tau) = \varphi_k, \ \ u^\varepsilon_0(x) = \psi(x). \quad (2.7)$$

где

$$\frac{\partial u^\varepsilon_k(x)}{\partial \overline{t}} = \frac{u^\varepsilon_k(x) - u^\varepsilon_{k-1}(x)}{\tau},$$

$$\beta_k^\varepsilon(x) = \int_0^1 \{k_\varepsilon[u_{k-1}^\varepsilon + \tau(u_k^\varepsilon - u_{k-1}^\varepsilon)] - \lambda\chi_\varepsilon'[u_{k-1}^\varepsilon + \tau(u_k^\varepsilon - u_{k-1}^\varepsilon)]\}d\tau,$$

$$f_k^\varepsilon(x) = \frac{1}{\tau}\int_{(k-1)\tau}^{k\tau} [\Delta u^\varepsilon(x,k\tau) - \Delta u^\varepsilon(x,t]dt.$$

3. The fundamental solutions and its properties.

For studying of the problem (2.4), (2.5) we need in the integral representation of the solution. Let $K_R(x_0)$ be ball with center at the point x_0 and the radius R and

$$\Gamma_{n-k+1}(|x-x_0|) = \frac{i\tau}{2\pi a_n}\oint_{\partial L} \frac{(\sinh\sqrt{z}R)^{-1}\sinh\sqrt{z}(R-|x-x_0|)dz}{4\pi|x-x_0|(1-\frac{z\tau}{a_n})(1-\frac{z\tau}{a_{n-1}})(1-\frac{z\tau}{a_k})}, \tag{3.1}$$

where

$$L = \{z = \xi + i\eta : Rez > -\frac{\pi^2}{R^2}, |z| < \varrho\},$$

$$\left(\frac{a_1}{\tau},0\right), \left(\frac{a_2}{\tau},0\right), ... \left(\frac{a_n}{\tau},0\right) \in L, \quad a_i > 0, i = 1,2,...n.$$

Property 1.

Let $|x-x_0| \neq 0$, then

$$\Delta\Gamma_{n-k+1} - a_k\frac{\Gamma_{n-k+1}-\Gamma_{n-k}}{\tau} = 0, \ \forall k = 1...(n-1),$$

$$\Delta\Gamma_1 - a_n\frac{\Gamma_1}{\tau} = 0, \ \Gamma_1(|x-x_0|) = \frac{\sinh\sqrt{\frac{a_n}{\tau}}(R-|x-x_0|)}{4\pi|x-x_0|\sinh\sqrt{\frac{a_n}{\tau}}}. \tag{3.2}$$

Property 2. There is the estimation

$$\int_{K_R(x_0)} \frac{\Gamma_n(|x-x_0|)}{\tau}dx \leq \int_{K_R(x_0)} \frac{\Gamma_{n-1}(|x-x_0|)}{\tau}dx \leq ...$$

$$\leq \int_{K_R(x_0)} \frac{\Gamma_1(|x-x_0|)}{\tau}dx \leq \frac{1}{a_n}\big(1 - \frac{\sqrt{\frac{a_n}{\tau}}R}{\sinh\sqrt{\frac{a_n}{\tau}}R}\big) \leq 1. \tag{3.3}$$

Property 3. Let $K_\delta(x_0)$ denote the ball with its center at the point x_0 and radius δ. Then

$$\lim_{\delta\to 0}\oint_{\partial K_\delta(x_0} \frac{\partial\Gamma_{m-k+1}}{\partial n}ds = \begin{cases} 1, \text{ if } k = m, \\ 0, \text{ if } k \neq m, \end{cases}$$

where n is the inner normal. All reduced above property can be received by immediate calculations.

Property 4. Let $\{v_k(x) \in C^2(D)\}$ and they satisfy the following equations:

$$\Delta v_k - \frac{a_k v_k - a_{k-1} v_{k-1}}{\tau} = -\frac{f_k - f_{k-1}}{\tau},$$

then there is an integral representation

$$v_m(x_0) = \int\limits_{K_R(x_0)} \frac{(a_0 v_0 - f_0)\Gamma_m(|x-x_0|}{\tau} dx - \sum_{k=1}^{m} \int\limits_{\partial K_R(x_0)} v_k \frac{\partial \Gamma_{m-k+1}}{\partial n} ds +$$

$$+ \sum_{k=1}^{m} \int\limits_{K_R(x_0)} f_k \frac{\Gamma_{m-k+1} - \Gamma_{m-k}}{\tau} dx. \tag{3.4}$$

This integral representation follows from the previous properties of the fundamental solutions and Green's for elliptic equations.

Property 5. The functions $\{\Gamma_{m-k+1}(|x-x_0|) - \Gamma_{m-k}(|x-x_0|)\}$ change the sign on the interval $0 < |x - x_0| < R$ no more than once and

$$\left|\frac{\partial \Gamma_{k-1}(r)}{\partial r}\right|_{r=R} \le \left|\frac{\partial \Gamma_k(r)}{\partial r}\right|_{r=R}. \tag{3.5}$$

This property follows from property1 and from principle of the maximum.

Property 6. We will denote by $r_{k,k-1}$ the points, where the functions $\Gamma_k(r) - \Gamma_{k-1}(r), \quad k = 1, 2...n$, are equal to zero. Then we have the inequality
$r_{k,k-1} \le r_{k+1,k}$, and

$$r_{2,1} = \sqrt{\tau} \frac{\ln a_{n-1} - \ln a_n}{\sqrt{a_{n-1}} - \sqrt{a_n}} + o(\sqrt{\tau}) \ \ \tau \to 0, \text{ if } a_{n-1} \ne a_n, \tag{3.6}$$

$$r_{2,1} = \sqrt{\tau} \frac{2}{a_n}, \text{ if } a_{n-1} = a_n.$$

Proof. The functions $\Gamma_k(r) - \Gamma_{k-1}(r)$ satisfy the equation

$$\Delta(\Gamma_{k+1}(|x-x_0|) - \Gamma_k(|x-x_0|)) - a_{m-k} \frac{\Gamma_{k+1}(|x-x_0|) - \Gamma_k(|x-x_0|)}{\tau} =$$

$$= -a_{m-k+1} \frac{\Gamma_k(|x-x_0|) - \Gamma_{k-1}(|x-x_0|)}{\tau}. \tag{3.7}$$

We construct an integral representation at the center of the sphere $K_{r_{k,k+1}}(x_0)$

$$\Gamma_{k+1}(0) - \Gamma_k(0) =$$

$$= \int\limits_{K_{r_{k,k+1}}(x_0)} a_{m-k+1} \tfrac{\Gamma_k(|x-x_0|)-\Gamma_{k-1}(|x-x_0|)}{\tau} \Gamma_1(|x-x_0|)dx =$$

$$= - \int\limits_{\partial K_{r_{k,k+1}}(x_0)} (\Gamma_{k+1}(|x-x_0|) - \Gamma_k(|x-x_0|) \frac{\partial \Gamma_1}{\partial n} ds,$$

$$\Gamma_1(|x-x_0|) = \frac{\sinh \sqrt{\frac{a_{m-k}}{\tau}} (r_{k+1,k} - |x-x_0|)}{4\pi |x-x_0| \sinh \sqrt{\frac{a_{m-k}}{\tau}} r_{k+1,k}}.$$

Taking into account that the function $\Gamma_{k+1}(r) - \Gamma_k(r)$ is equal to zero at the points $r = 0, r_{k+1,k} = 0$, we obtain

$$0 = \int\limits_{K_{r_{k,k+1}}(x_0)} a_{m-k+1} \frac{\Gamma_k(|x-x_0|) - \Gamma_{k-1}(|x-x_0|)}{\tau} \Gamma_1(|x-x_0|)dx.$$

If $r_{k,k-1} > r_{k+1,k}$, then this equality is impossible.

Formula (3.1) implies

$$4\pi|x-x_0|\Gamma_1(|x-x_0|) = e^{-|x-x_0|\sqrt{\frac{a_m}{\tau}}} + O\left(e^{-R\sqrt{\frac{a_m}{\tau}}}\right),$$

$$4\pi|x-x_0|\Gamma_2(|x-x_0|) = \frac{a_{m-1}}{a_{m-1}-a_m} \left(e^{-|x-x_0|\sqrt{\frac{a_m}{\tau}}} - e^{-|x-x_0|\sqrt{\frac{a_{m-1}}{\tau}}}\right) +$$
$$+ O\left(e^{-R\sqrt{\frac{a_{min}}{\tau}}}\right), \quad \text{if} \ \ a_{m-1} \neq a_m,$$

$$4\pi|x-x_0|\Gamma_2(|x-x_0|) = \frac{|x-x_0|}{2\sqrt{a_m \tau}} e^{-|x-x_0|\sqrt{\frac{a_m}{\tau}}} + O\left(e^{-R\sqrt{\frac{a_m}{\tau}}}\right)$$

if $a_{m-1} = a_m$. From here we obtain

$$r_{2,1} = \sqrt{\tau} \frac{\ln a_{m-1} - \ln a_m}{\sqrt{a_{m-1}} - \sqrt{a_m}} + o(\tau) \ \ \tau \to 0.$$

In particular, if $a_{m-1} = a_m$, then

$$r_{2,1} = \sqrt{\tau} \frac{2}{\sqrt{a_m}}.$$

The function $\Gamma_2(r)-\Gamma_1(r)$ changes the sign once. Therefore, as follows from the equations (3.7), the functions $\Gamma_k(r)-\Gamma_{k-1}(r)$ change the sign once too. It means that the inequalities (3.6) hold.

Property 7. We have following estimate

$$\left|\frac{\partial\Gamma_N}{\partial n}\right| \le M_1\left\{\frac{1}{q^N R}\exp\{-M_2\frac{R}{\sqrt{\tau}}\}+\exp\left\{-\frac{T\pi^2}{R^2 a_{\max}}\right\}\right\}, \tag{3.8}$$

where $q\ge 2$, and positive constants M_1, M_2 do not depend on N,τ,R.
Proof. Let us estimate integral

$$\left.\frac{\partial\Gamma_N}{\partial n}\right|_{\partial K_R(x_0)} = \frac{-i\tau}{2\pi a_N}\oint\limits_L \frac{\sqrt{z}}{2\pi R\sinh(\sqrt{z}R)}\cdot\frac{dz}{(1-\frac{z\tau}{a_1})(1-\frac{z\tau}{a_2})...(1-\frac{z\tau}{a_N})}, \tag{3.9}$$

where ∂L is the boundary of the domain

$$L=\{z:\varrho=|z|<\frac{(1+q)\max\limits_{1\le k\le N}a_k}{\tau},$$

$$Rez=b_0>-\frac{\pi^2}{R^2}, b_0<0, \varrho>\frac{\max\limits_{1\le k\le N}a_k}{\tau}\}.$$

Let us represent the integral (3.9) as a sum of two terms: I_1 and I_2, where I_1 denotes the integral along the part of the curve ∂L which is an arch of a circle, and I_2 denotes the integral along the part of the contour which lies inside the straight line $Rez=b_0$. Let us estimate the integral I_1. The estimates

$$\left|(1-\frac{z\tau}{a_1})(1-\frac{z\tau}{a_2})...(1-\frac{z\tau}{a_N})\right| \ge \left|\frac{|z|\tau}{a_{\max}}-1\right|^N \ge q^N,$$

$$\left|\sinh(\sqrt{z}R)\right| \ge \sinh[\sqrt{|z|}R\cos(argz/2)] = \sinh[\sqrt{|z|}R\cos\varphi],$$

where $\varphi\to\frac{\pi}{4}$, if $\tau\to 0$, imply

$$|I_1|\le c_1\frac{1}{q^N R}\exp\{-c_2\frac{R}{\sqrt{\tau}}\},$$

where the constants c_1 and c_2 do not depend on τ. Let us now estimate the integral I_2. As $Rez=b_0$, we obtain

$$\left|(1-\frac{z\tau}{a_1})(1-\frac{z\tau}{a_2})...(1-\frac{z\tau}{a_N})\right| \ge \left(1+\tau\frac{|b_0|}{a_{\max}}\right)^{\frac{T}{\tau}}.$$

Assume $b_0 = -\frac{\pi^2}{2R^2}$. Then $\left|\frac{\sqrt{|z|}R}{\sinh\sqrt{z}R}\right| \leq c_4$. Thus we obtain

$$|I_2| \leq c_5 \exp\left\{-\frac{T\pi^2}{R^2 a_{\max}}\right\}.$$

The constants c_3, c_4, c_5 do not depend on τ.

4. Uniform estimates. Passage to the limit.

Theorem 4.1. *Let*

$$\psi(x) \in C^{l+\alpha}(\overline{D}),\ \varphi(x,t) \in H^{l+\alpha,\frac{l+\alpha}{2}}(\overline{D_T}), 0 < \alpha < 1,\ l \geq 3,$$

and assume that the corresponding consistency conditions hold at $t = 0, x \in \partial D$. *Then exists the constant* c, *which not depend* ε, *that the following estimate holds:*

$$\max_{x\in\overline{D}} \left|\frac{\partial u^\varepsilon(x,t)}{\partial t}\right| + \max_{\overline{D_T}\setminus\{\omega^\varepsilon(x,t)\cap\omega^\varepsilon(x,0)\}} \left|\frac{\partial u^\varepsilon(x,t)}{\partial x}\right| \leq c, \tag{4.1}$$

where $\omega^\varepsilon(x,t) = \{(x,t) \in D_T : 1 < u^\varepsilon(x,t) < 1+\varepsilon\}$.

Let x_0 be an arbitrary point in D. Denote by $K_R(x_0)$ the sphere with the center at the point x_0 of radius $R = \tau^\sigma, K_R(x_0) \subset D$. Let's transform the equation (2.6) to the following form

$$\Delta\left(\frac{\partial u_k^\varepsilon(x)}{\partial \overline{t}}\right) - \frac{\beta_k^\varepsilon(x_0)}{\tau}\frac{\partial u_k^\varepsilon(x)}{\partial \overline{t}} + \frac{\beta_{k-1}^\varepsilon(x_0)}{\tau}\frac{\partial u_{k-1}^\varepsilon(x)}{\partial \overline{t}} =$$

$$= \frac{\partial[f_k^\varepsilon(x) - f_{k-1}^\varepsilon(x)]}{\partial \overline{t}} + + \frac{\beta_k^\varepsilon(x) - \beta_k^\varepsilon(x_0)}{\tau}\frac{\partial u_k^\varepsilon(x)}{\partial \overline{t}} - \frac{\beta_{k-1}^\varepsilon(x) - \beta_{k-1}^\varepsilon(x_0)}{\tau}\frac{\partial u_{k-1}^\varepsilon(x)}{\partial \overline{t}}.$$

In order to obtain the estimate for the functions $\frac{\partial u_k^\varepsilon(x)}{\partial \overline{t}}$ we will use the integral representation (Property 4). It will give

$$\frac{\partial u_n^\varepsilon(x_0)}{\partial \overline{t}} = -\int\limits_{K_R(x_0)} \Delta u_0^\varepsilon(x)\frac{\Gamma_n(|x-x_0|)}{\tau}dx -$$

$$\sum_{k=1}^{n}\int\limits_{\partial K_R(x_0)}\frac{\partial u_k^\varepsilon}{\partial \bar t}\frac{\partial \Gamma_{n-k+1}}{\partial n}ds-$$

$$-\sum_{k=1}^{n}\int\limits_{K_R(x_0)}\frac{\partial (f_k^\varepsilon(x)-f_{k-1}^\varepsilon(x))}{\partial \bar t}\Gamma_{n-k+1}(|x-x_0|)dx-$$

$$-\sum_{k=1}^{n}\int\limits_{K_R(x_0)}\left\{\frac{\beta_k^\varepsilon(x)-\beta_k^\varepsilon(x_0)}{\tau}\frac{\partial u_k^\varepsilon(x)}{\partial \bar t}-\frac{\beta_{k-1}^\varepsilon(x)-\beta_{k-1}^\varepsilon(x_0)}{\tau}\frac{\partial u_{k-1}^\varepsilon(x)}{\partial \bar t}\right\}\times$$

$$\times\Gamma_{n-k+1}(|x-x_0|)dx=I_1+I_2+I_3+I_4.$$

Let's estimate every term. From (3.4) follows

$$|I_1|\le\int\limits_{K_R(x_0)}\frac{1}{\tau}|\Delta u_0^\varepsilon(x)|\Gamma_n(|x-x_0|)dx\le\max_{x\in\overline{D}}|\Delta u_0^\varepsilon(x)|\frac{1}{a_n}.$$

Let assume $0<\sigma<\frac{1}{2}$. Then from (3.8) follows

$$|I_2|\le\frac{Tc_1}{\tau M(\varepsilon)}\left\{\frac{1}{q^N R}\exp\{-c_2\frac{R}{\sqrt{\tau}}\}+\exp\left\{-\frac{T\pi^2}{R^2a_{\max}}\right\}\right\}\le$$

$$\le\frac{Tc_1}{\tau M(\varepsilon)}\left\{\frac{1}{q^N\tau^\sigma}\exp\{-c_2\frac{\tau^\sigma}{\sqrt{\tau}}\}+\exp\left\{-\frac{T\pi^2}{\tau^{2\sigma}a_{\max}}\right\}\right\}\le\frac{c(\sigma)}{M(\varepsilon)}\tau,$$

where $c(\sigma)$ do not depend ε.

From (2.5), (3.4) follows

$$|I_3|\le\sum_{k=1}^{n}\int\limits_{K_R(x_0)}|f_k^\varepsilon(x)|\left|\frac{\Gamma_{n-k+1}(|x-x_0|)-\Gamma_{n-k}(|x-x_0|)}{\tau}\right|dx\le$$

$$\le\int\limits_{K_{r_{2,1}}(x_0)}\max_{x\in\overline{D},1\le k\le N}|f_k^\varepsilon(x)|\frac{\Gamma_1(|x-x_0|)}{\tau}dx+$$

$$+\int\limits_{K_R\setminus K_{r_{2,1}}(x_0)}\max_{x\in\overline{D},1\le k\le N}|f_k^\varepsilon(x)|\frac{1}{\tau|x-x_0|}dx\le c\left(\frac{\tau}{M(\varepsilon)}+\frac{\tau^{2\sigma}}{M(\varepsilon)}\right),$$

where constant c do not depend ε,τ.

From (3.5), (3.6) follows

$$|I_4|\le\sum_{k=1}^{n}\int\limits_{K_R(x_0)}|\beta_k^\varepsilon(x)-\beta_k^\varepsilon(x_0)|\times$$

$$\times\left|\frac{\partial u_k^\varepsilon(x)}{\partial \bar t}\right|\left|\frac{\Gamma_{n-k+1}(|x-x_0|)-\Gamma_{n-k}(|x-x_0|)}{\tau}\right|dx+$$

$$+\sum_{k=1}^{n}\int\limits_{K_R\setminus K_{r_{2,1}}(x_0)}|\beta_k^\varepsilon(x)-\beta_k^\varepsilon(x_0)|\times$$

$$\times|\tfrac{\partial u_k^\varepsilon(x)}{\partial \overline{t}}|\left|\tfrac{\Gamma_{n-k+1}(|x-x_0|)-\Gamma_{n-k}(|x-x_0|)}{\tau}\right|dx\le$$

$$\le\int\limits_{K_{r_{2,1}}(x_0)}\tfrac{c_1|x-x_0|}{\varepsilon^2M(\varepsilon)}\tfrac{\Gamma_1(|x-x_0|)}{\tau}dx+$$

$$+\int\limits_{K_R\setminus K_{r_{2,1}}(x_0)}\tfrac{c_1|x-x_0|}{\varepsilon^2M(\varepsilon)}\tfrac{1}{\tau|x-x_0|}dx\le\tfrac{c}{\varepsilon^2M(\varepsilon)}(\tau^{1/2}+\tau^{3\sigma-1}).$$

In result we shall receive

$$\left|\frac{\partial u_n^\varepsilon(x_0)}{\partial \overline{t}}\right|\le c_1\max_{x\in\overline{D}}|\Delta u_0^\varepsilon(x)|+c_2\frac{\tau^{3\sigma-1}}{\varepsilon^2M(\varepsilon)},$$

where constants c_1, c_2 do not depend from τ, ε. Let us assume that $\tau^{3\sigma-1}<\varepsilon^2M(\varepsilon), 1/3<\sigma<1/2$. Then

$$\left|\frac{\partial u^\varepsilon(x_0,t)}{\partial t}\right|\le\frac{\partial u_k^\varepsilon(x_0)}{\partial \overline{t}}+\left|\frac{1}{\tau}\int\limits_{(k-1)\tau}^{k\tau}\left\{\frac{\partial u^\varepsilon(x,t)}{\partial t}-\frac{\partial u^\varepsilon(x,\eta)}{\partial t}\right\}d\eta\right|\le$$

$$\le c_1\max_{x\in\overline{D}}|\Delta u_0^\varepsilon(x)|+c_2\frac{\tau^{3\sigma-1}}{\varepsilon^2M(\varepsilon)}+c_3\frac{\tau}{M(\varepsilon)}\le c_4,$$

where the constant c_4 do not depend from ε.

Near of boundary of domain D the equation (2.6) become the linear equation with the constant coefficients. Therefore the appropriate estimate can be easily received.

We differentiate the equation (2.6) with respect to one of the variables x_i.

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial u_k^\varepsilon(x)}{\partial x}\right)-\frac{b_\varepsilon(u_k^\varepsilon(x))u_k^{\varepsilon\prime}(x)-b_\varepsilon(u_{k-1}^\varepsilon(x))u_{k-1}^{\varepsilon\prime}(x)}{\tau}=\frac{\partial f_k^\varepsilon(x)}{\partial x}.$$

We use the property 4. It gives

$$u_n^{\varepsilon\prime}(x_0)=\int\limits_{K_R(x_0)}\beta_0^\varepsilon u_0^{\varepsilon\prime}(x)\tfrac{\Gamma_n(|x-x_0|)}{h}dx-$$

$$-\sum_{k=1}^{n}\left\{\int\limits_{\partial K_R(x_0)}u_k^{\varepsilon\prime}(x)\tfrac{\partial\Gamma_{n-k+1}}{\partial n}ds-\int\limits_{K_R(x_0)}f_k^{\varepsilon\prime}(x)\Gamma_{n-k+1}dx\right\}- \tag{4.3}$$

$$-\sum_{k=1}^{n}\int\limits_{K_R(x_0)}(\beta_k^\varepsilon(x_0)-\beta_k^\varepsilon(x))u_k^{\varepsilon\prime}(x)\tfrac{\Gamma_{m-k+1}(|x-x_0|)-\Gamma_{m-k}(|x-x_0|)}{h}dx$$

From this integral representation, applying the same reasoning as above, we obtain the second part of the estimate (4.1). The estimation (4.2) can be proved similarly.

Let's return to integral representation (4.3) and we will estimate from below the first term. We will assume, that $\frac{\partial\psi(x)}{\partial x} \neq 0$ everywhere in D. For definiteness we will assume $\frac{\partial\psi(x)}{\partial x} > 0$. Then

$$\int\limits_{K_R(x_0)} \beta^\varepsilon(\psi(x_0))\tfrac{\partial u_0(x)}{\partial x}\tfrac{\Gamma_n(|x-x_0|)}{\tau}dx \geq$$

$$\geq a_{min}\min_{\overline{D}}(\psi'(x) \int\limits_{K_R(x_0)} \tfrac{\Gamma_n(|x-x_0|)}{\tau}dx =$$

$$= a_{min}\min_{\overline{D}}(\psi'(x) \left\{ \int\limits_{\partial K_R(x_0)} \sum_{k=1}^{n-1} \tfrac{1}{\beta_k^\varepsilon(x_0)} \tfrac{\partial\Gamma_{n-k+1}(|x-x_0|)}{\partial n} dx + \right.$$

$$\left. + \int\limits_{K_R(x_0} \tfrac{\Gamma_1(|x-x_0|)}{\tau}dx. \right\} = I_1 + I_2.$$

From (3.8) follows

$$|I_1| \leq \frac{Tc}{\tau M(\varepsilon)}\left\{\frac{1}{q^N\tau^\sigma}\exp\{-c_2\frac{\tau^\sigma}{\sqrt{\tau}}\} + \exp\left\{-\frac{T\pi^2}{\tau^{2\sigma}a_{\max}}\right\}\right\} \leq \frac{c(\sigma)}{M(\varepsilon)}\tau,$$

where $c, c(\sigma)$ do not depend ε. From (3.3) follows

$$|I_2| = \int\limits_{K_R(x_0)} \frac{\Gamma_1(|x-x_0|)}{\tau}dx = \frac{1}{\beta_n^\varepsilon(x_0)}\Bigl(1 - \frac{\sqrt{\frac{\beta_n^\varepsilon(x_0)}{\tau}}\tau^\sigma}{\sinh\sqrt{\frac{\beta_n^\varepsilon(x_0)}{\tau}}\tau^\sigma}\Bigr)$$

Similarly to the previous theorem it is possible to prove that all other terms in the received integral representation (4.3) have limits equal to zero when $\sigma \to 0$. From here follows

Theorem 4.2. *Let*

$$\psi(x) \in C^{2+\alpha}(\overline{D}),\ \varphi(x,t) \in H^{2+\alpha,\frac{2+\alpha}{2}}(\overline{D_T}),\ \min_{\overline{D}}|\nabla\psi(x)| > 0,$$

$(0 < \alpha < 1)$ *and assume that the corresponding consistency conditions hold at* $t = 0, x \in \partial D$. *Then exists the constant* c, *which not depend* ε, *that the following estimate holds:*

$$|\nabla u^\varepsilon(x,t)| \geq c > 0 \ \ \forall (x,t) \in D_T \setminus \omega_\varepsilon(x,t).$$

Let the function $\eta(x,t) \in C^{2,1}(\overline{D_T})$ be equal to zero on $(\partial D \times (0,T)) \cup (D \times (t = T))$. We multiply (2.2) by $\eta(x,t)$, integrate it over D_T. After simple transformations we obtain

$$\int\limits_{D_T} \left(\nabla u^\varepsilon(x,t) \nabla \eta(x,t) + a(u) \tfrac{\partial u^\varepsilon}{\partial t} \eta(x,t) + \lambda \chi_\varepsilon(u^\varepsilon) \tfrac{\partial \eta}{\partial t} \right) dxdt+$$

$$+\lambda \int\limits_D \chi_\varepsilon(\psi) \eta(x,0) dx = 0,$$

The possibility of passage to the limit follows from the statements proved above. As a result we obtain

Theorem 4.3. *Let the following conditions be satisfied:*

$$\psi(x) \in C^{2+\alpha}(\overline{D}), \varphi(x,t) \in H^{2+\alpha,1+\alpha/2}(\overline{D_T}),\ \min_{\overline{D}} |\nabla \psi(x)| > 0,$$

$(0 < \alpha < 1)$ *and we will assume that corresponding consistency conditions hold at* $t = 0, x \in \partial D$. *Then* $\forall T > 0$ *there exists a solution of the problem (1.1)-(1.4) and*

$$u(x,t) \in C(\overline{D_T}) \cap \left(H^{2+\alpha,1+\alpha/2}(\overline{\Omega_T} \setminus \gamma_0) \times H^{2+\alpha,1+\alpha/2}(\overline{G_T} \setminus \gamma_0) \right),$$

the free boundary is given by the graph of a function from $H^{2+\alpha,1+\alpha/2}$ *class.*

In this work existence of classical solution is proved at more natural limitations, than in work [2].

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