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ASYMPTOTIC ANALYSIS OF A VIBRATING SYSTEM CONTAINING STIFF-HEAVY AND FLEXIBLE-LIGHT PARTS

A model of a strongly inhomogeneous medium with simultaneous perturbation of the rigidity and mass density is studied. The medium has strongly contrasting physical characteristics in two parts with the ratio of rigidities being proportional to a small parameter ε. Additionally, the ratio of mass densities is of order ε^{-1}. We investigate the asymptotic behaviour of the spectrum and eigensubspaces as $\varepsilon \to 0$. Complete asymptotic expansions of eigenvalues and eigenfunctions are constructed and justified.

We show that the limit operator is nonself-adjoint in general and possesses two-dimensional Jordan cells in spite of the singular perturbed problem is associated with a self-adjoint operator in appropriated Hilbert space $\mathcal{L}_\varepsilon$. This may happen if the metric in which the problem is self-adjoint depends on small parameter ε in a singular way. In particular, it leads to a loss of completeness for the eigenfunction collection. We describe how root spaces of the limit operator approximate eigenspaces of the perturbed operator.

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Introduction.

We consider a model of strongly inhomogeneous medium consisting of two nearly homogeneous components. Assuming a strong contrast of the corres ponding stiffness coefficients $k_1 \ll k_2$, we get that their ratio k_1/k_2 has a small order, which we denote by ε. In general, the mass densities r_1 and r_2 in two parts could be quite different as well or could be the same. We model this assuming that the density ratio r_1/r_2 is proportional to ε^{-m}. We investigate how the resonance vibrations of the medium change if the parameter ε tends to 0. In the one-dimensional case we consider the spectral problem

$$\tfrac{d}{dx}\left(k_\varepsilon(x)\tfrac{du_\varepsilon}{dx}\right) + \lambda^\varepsilon\, r_\varepsilon(x)u_\varepsilon = 0 \quad \text{in } (a,b),$$

$$\alpha_1 u_\varepsilon'(a) + \alpha_0 u_\varepsilon(a) = 0, \qquad \beta_1 u_\varepsilon'(b) + \beta_0 u_\varepsilon(b) = 0,$$

where (a,b) is an interval in $\mathbb{R}$ containing the origin and

$$k_\varepsilon(x)=\begin{cases} k(x) & \text{for } x\in(a,0)\\ \varepsilon\,\varkappa(x) & \text{for } x\in(0,b),\end{cases} \quad r_\varepsilon(x)=\begin{cases} \varepsilon^{-m}r(x) & \text{for } x\in(a,0)\\ \rho(x) & \text{for } x\in(0,b).\end{cases} \tag{1}$$

Here k, r and $\varkappa$, ρ are smooth positive functions in intervals $[a,0]$ and $[0,b]$ respectively. At point $x=0$ of discontinuity of the coefficients we assume that transmission conditions $u_\varepsilon(-0)=u_\varepsilon(+0)$, $(ku_\varepsilon')(-0)=\varepsilon(\varkappa u_\varepsilon')(+0)$ hold.

Of course, the limit properties of spectrum depend on the power m characterizing the density ratio. Intuitively, we expect that for large values of m the mass density perturbation has to be dominating whereas for small m the rigidity perturbation has to be leading. Then it has to be at least one critical point m separating the cases. It appears to be truth exactly for $m=1$, when the mass density perturbation is strictly inverse to the stiffness one.

This paper is devoted to the critical case $m=1$. We consider the Dirichlet problem

$$(k(x)\,u_\varepsilon')' + \varepsilon^{-1}\lambda^\varepsilon\, r(x)\,u_\varepsilon = 0, \qquad x\in(a,0), \tag{2}$$
$$\varepsilon\,(\varkappa(x)\,u_\varepsilon')' + \lambda^\varepsilon\,\rho(x)\,u_\varepsilon = 0, \qquad x\in(0,b), \tag{3}$$
$$u_\varepsilon(-0)=u_\varepsilon(+0), \qquad (ku_\varepsilon')(-0)=\varepsilon\,(\varkappa u_\varepsilon')(+0), \tag{4}$$
$$u_\varepsilon(a)=0, \qquad u_\varepsilon(b)=0 \tag{5}$$

and investigate the asymptotic behavior of eigenvalues λ^ε and eigenfunctions u_ε as $\varepsilon\to 0$.

After a proper change of spectral parameter problem (2)-(5) can be represented as a problem with perturbation of the transmission conditions only (cf. the example with constant coefficients below). At first blush, the problem looks very simple. But the point is that the problem shows a complicated picture of the eigenspace bifurcation. In Section we prove that the limit behavior of the spectrum is described in terms of a nonself-adjoint operator that has in general multiple eigenvalues and two-dimensional root spaces. At the same time, (2)-(5) is associated with a self-adjoint operator in the weighted space $\mathcal{L}_\varepsilon$ with the following scalar product and norm

$$(\phi,\psi)_\varepsilon = \varepsilon^{-1}(r\phi,\,\psi)_{L_2(a,0)} + (\rho\phi,\,\psi)_{L_2(0,b)}, \qquad \|\phi\|_\varepsilon = \sqrt{(\phi,\phi)_\varepsilon}\,. \tag{6}$$

It is obvious that for each fixed $\varepsilon>0$ the spectrum of (2)-(5) is real, discrete and simple, $0<\lambda_1^\varepsilon<\lambda_2^\varepsilon<\cdots<\lambda_j^\varepsilon<\cdots\to\infty$ as $j\to\infty$ and the corresponding real-valued eigenfunctions $\{u_{\varepsilon,j}\}_{j=1}^\infty$ form an orthogonal basis in $\mathcal{L}_\varepsilon$. How may it happen? The metric in $\mathcal{L}_\varepsilon$ for which the perturbed problem is self-adjoint, depends on small parameter ε in a singular way. In Sections , we construct and justify the complete asymptotic expansions of eigenvalues and eigenfunctions. Therefore there exist pairs of closely adjacent eigenvalues λ_j^ε and $\lambda_{j+1}^\varepsilon$ being the bifurcation of double limit eigenvalues. Although the corresponding eigenfunctions $u_{\varepsilon,j}$ and $u_{\varepsilon,j+1}$ remain orthogonal in $\mathcal{L}_\varepsilon$ for all $\varepsilon > 0$, they make an infinitely small angle between them in $L_2(a, b)$ with the standard metric and stick together at the limit. In particular, it leads to the loss of completeness in $L_2(a, b)$ for the limit eigenfunction collection. Nevertheless both $u_{\varepsilon,j}$ and $u_{\varepsilon,j+1}$ converge to the same limit, a plane $\pi(\varepsilon)$ being the linear span of these eigenfunctions has regular asymptotic behaviour as $\varepsilon \to 0$. In fact, a root space π corresponding the double eigenvalue is the limit position of plane $\pi(\varepsilon)$ as $\varepsilon \to 0$, as is shown in Theorem 5. We actually prove that the completeness property of the perturbed eigenfunction collection passes into the completeness of eigenfunctions and adjoined functions of the limit nonself-adjoint operator.

This work was motivated by [1, Ch.8], where the similar problem for the Laplace operator has been considered. The authors have handled the limit operator as the direct sum of two self-adjoint operators that nevertheless does not entirely explain the bifurcation picture in perturbation theory of operators. The aim of this paper is to present more rigorous and detailed study of the case in operator framework.

Finally, let us remark that the vibrating systems with singularly perturbed stiffness and mass density have been considered in many papers. In the case of purely stiff models (with homogeneous mass density), the asymptotic behavior of spectra have been studied in [6] - [12]. Referring to problems with purely density perturbation often involving domain perturbations, we mention [13]- [18] with the latter including a broad literature overview in the area. Spectral properties of vibrating systems with mass entirely neglected in a subdomain were also studied in [19], [20]. The asymptotic results for the problems with simultaneous perturbations of mass density and stiffness were obtained in [21], [22].

1. Preliminaries.

We demonstrate an example where eigenvalue bifurcation is calculated explicitly. If all coefficients in (2), (3) are constant we get the eigenvalue problem

$$y_\varepsilon'' + \omega_\varepsilon^2 y_\varepsilon = 0, \quad x \in (a,0) \cup (0,b), \tag{7}$$

$$y_\varepsilon(a) = 0, \quad y_\varepsilon(b) = 0, \quad y_\varepsilon(-0) = y_\varepsilon(+0), \quad y_\varepsilon'(-0) = \varepsilon y_\varepsilon'(+0), \tag{8}$$

where $\omega_\varepsilon^2 = \varepsilon^{-1}\lambda^\varepsilon$. Then each non-zero solution can be represented by

$$y_\varepsilon = \begin{cases} A_\varepsilon \sin \omega_\varepsilon (x-a) & \text{for } x \in (a,0), \\ B_\varepsilon \sin \omega_\varepsilon (x-b) & \text{for } x \in (0,b), \end{cases}$$

with $\omega_\varepsilon > 0$ and $A_\varepsilon, B_\varepsilon \in \mathbb{R}$. By virtue of (8) we have

$$A_\varepsilon \sin \omega_\varepsilon a - B_\varepsilon \sin \omega_\varepsilon b = 0 \qquad \text{and} \qquad A_\varepsilon \cos \omega_\varepsilon a - \varepsilon B_\varepsilon \cos \omega_\varepsilon b = 0.$$

Looking for a non-zero solution of the algebraic system, we get the characteristic equation

$$\cos \omega_\varepsilon a \, \sin \omega_\varepsilon b = \varepsilon \sin \omega_\varepsilon a \, \cos \omega_\varepsilon b. \tag{9}$$

The latter easily gives existence of the limit $\omega_\varepsilon \to \omega$ as $\varepsilon \to 0$ such that

$$\cos \omega a \, \sin \omega b = 0. \tag{10}$$

Moreover, the root ω has to be positive. Obviously, if we suppose, contrary to our claim, that ω_ε goes to 0 as $\varepsilon \to 0$, then (9) can be written in the equivalent form

$$\frac{\cos \omega_\varepsilon a \, \sin \omega_\varepsilon b}{\cos \omega_\varepsilon b \, \sin \omega_\varepsilon a} = \varepsilon$$

for sufficiently small ε. A passage to the limit as $\varepsilon \to 0$ and $\omega_\varepsilon \to 0$ leads to a contradiction, because the left-hand side converges towards the negative number b/a.

If a and b are incommensurable number, then all roots of (10) are simple. In fact, multiple roots exist iff $2n|a| = (2l-1)b$ for certain natural l and n. Let us consider the case $a = -1$ and $b = 2$. Then the lowest positive root $\omega = \pi/2$ of (10) has multiplicity 2. On the other hand, equation (9) admits the factorization

$$\left(\cos \omega_\varepsilon - \sqrt{\frac{\varepsilon}{2+2\varepsilon}}\right)\left(\cos \omega_\varepsilon + \sqrt{\frac{\varepsilon}{2+2\varepsilon}}\right)\sin \omega_\varepsilon = 0.$$

Hence the lowest eigenvalues $\omega_{\varepsilon,1} = \frac{\pi}{2} - \arcsin\sqrt{\frac{\varepsilon}{2+2\varepsilon}}$, $\omega_{\varepsilon,2} = \frac{\pi}{2} + \arcsin\sqrt{\frac{\varepsilon}{2+2\varepsilon}}$ are closely adjacent and converge to the same limit

$\pi/2$. The corresponding eigenfunctions $y_{\varepsilon,1}$ and $y_{\varepsilon,2}$ are defined up to a constant factor as

$$y_{\varepsilon,j}(x) = \begin{cases} (-1)^j \sqrt{2\varepsilon/(1+\varepsilon)}\sin\omega_{\varepsilon,j}(x+1) & \text{for } x \in (-1,0), \\ \sin\omega_{\varepsilon,j}(x-2) & \text{for } x \in (0,2). \end{cases} \tag{11}$$

We see at once that the angle in $L_2(-1,2)$ between the eigenfunctions $y_{\varepsilon,1}$ and $y_{\varepsilon,2}$ is infinitely small as $\varepsilon \to 0$, because both eigenfunctions converge towards the same function

$$y_*(x) = \begin{cases} 0 & \text{for } x \in (-1,0), \\ \sin\dfrac{\pi}{2}(x-2) & \text{for } x \in (0,2). \end{cases}$$

The point of the example is that the collection of eigenfunctions $\{u_{\varepsilon,j}\}_{j=1}^{\infty}$ loses the completeness property at the limit on account of the double eigenvalues. We now turn to perturbed problem (2)-(5) in the general case. To shorten formulas below, we introduce notation $I_a = (a,0)$, $I_b = (0,b)$ and

$$K(x) = \begin{cases} k(x) & \text{for } x \in I_a \\ \varkappa(x) & \text{for } x \in I_b, \end{cases} \qquad R(x) = \begin{cases} r(x) & \text{for } x \in I_a \\ \rho(x) & \text{for } x \in I_b. \end{cases}$$

Proposition 1. *For each number $j \in \mathbb{N}$ eigenvalue λ_j^ε of (2)-(5) is a continuous function of $\varepsilon \in (0,1)$ and $c\,\varepsilon < \lambda_j^\varepsilon \le C_j\,\varepsilon$ with constants c, C_j being independent of ε.*

Proof. The continuity of eigenvalues with respect to the small parameter follows immediately from the mini-max principle

$$\lambda_j^\varepsilon = \min_{E_j} \max_{\substack{v\in E_j \\ v\neq 0}} \frac{\int_a^0 kv'^2\,dx + \varepsilon\int_0^b \varkappa v'^2\,dx}{\varepsilon^{-1}\int_a^0 rv^2\,dx + \int_0^b \rho v^2\,dx}, \tag{12}$$

where the minimum is taken over all the subspaces $E_j \subset H_0^1(a,b)$ with $\dim E_j = j$. We consider the eigenfunctions $v_1,\dots,v_j$ corresponding to the lowest eigenvalues $\mu_1,\dots,\mu_j$ of the problem

$$(\varkappa(x)v')' + \mu\rho(x)v = 0, \quad x \in I_b, \qquad v(0) = v(b) = 0. \tag{13}$$

Extending each v_k by zero to $(a,0)$ we get that the span $\mathcal{M}$ of $v_1,\dots,v_j$ is an j-dimensional subspace of $H_0^1(a,b)$. Then

$$\lambda_j^\varepsilon \le \max_{v\in\mathcal{M}} \frac{\int_a^0 kv'^2\,dx + \varepsilon\int_0^b \varkappa v'^2\,dx}{\varepsilon^{-1}\int_a^0 rv^2\,dx + \int_0^b \rho v^2\,dx} = \max_{v\in\mathcal{M}} \frac{\varepsilon\int_0^b \varkappa v'^2\,dx}{\int_0^b \rho v^2\,dx} = \varepsilon\mu_j, \tag{14}$$

which establishes the upper estimate. Next, by the same mini-max principle

$$\lambda_j^\varepsilon > \lambda_1^\varepsilon = \min_{H_0^1(a,b)} \frac{\int_a^0 kv'^2\,dx + \varepsilon\int_0^b \varkappa v'^2\,dx}{\varepsilon^{-1}\int_a^0 rv^2\,dx + \int_0^b \rho v^2\,dx} \geq$$

$$\geq \frac{k_*}{r_*}\min_{H_0^1(a,b)} \frac{\int_a^0 v'^2\,dx + \varepsilon\int_0^b v'^2\,dx}{\varepsilon^{-1}\int_a^0 v^2\,dx + \int_0^b v^2\,dx} = \frac{\varepsilon k_*\,\omega_{\varepsilon,1}^2}{r_*} \geq c\varepsilon,$$

where $k_* = \min_{x\in(a,b)} K(x)$, $r_* = \max_{x\in(a,b)} R(x)$ and $\omega_{\varepsilon,1}^2$ is the first eigenvalue of problem (7)-(8) with constant coefficients. It remains to note that $\omega_{\varepsilon,1} \to \pi/2$. □

2. Convergence Results and Properties of Limit Problem .

Let us consider the eigenvalue problem

$$\begin{cases} (K(x)u')' + \mu R(x)u = 0, & x\in I_a\cup I_b, \\ u(a)=0, \quad u(b)=0, \quad u(-0)=u(+0), & u'(-0)=0, \end{cases} \tag{15}$$

that will be referred to as *the limit spectral problem.* The spectrum of (15) is discrete and real (see Th. 1 below). We introduce the space $\mathcal{H} = \{f\in H_0^1(a,b)\colon f_a\in H^2(a,0) \text{ and } f_b\in H^2(0,b)\}$, where f_a and f_b are the restrictions of f to intervals I_a and I_b resp. Problem (15) admits the variational formulation: *to find $\mu\in\mathbb{C}$ and a nontrivial $u\in\mathcal{H}$ such that*

$$\int_a^b K\,u'\phi'\,dx + \varkappa(0)u'(+0)\phi(0) = \mu\int_a^b R\,u\phi\,dx \tag{16}$$

for all $\phi\in C_0^\infty(a,b)$. We first prove a conditional results.

Proposition 2. *Given eigenvalue λ^ε and the corresponding eigenfunction u_ε of (2)-(5), if $\varepsilon^{-1}\lambda^\varepsilon\to\mu^*$ and $u_\varepsilon\to u_*$ in H^2 weakly on each intervals I_a, I_b and u_* is different from zero, then μ^* is an eigenvalue of (15) with the eigenfunction u_*.*

Proof. We make a change of spectral parameter $\lambda^\varepsilon = \varepsilon\mu^\varepsilon$ in (2)-(5), whereat we can reduce equation (3) by the first order of ε. Then each pair $(\mu^\varepsilon, u_\varepsilon)$ satisfies the integral identity

$$\int_a^b K\,u_\varepsilon'\phi'\,dx + (1-\varepsilon)\varkappa(0)u_\varepsilon'(+0)\phi(0) = \mu_\varepsilon\int_a^b R\,u_\varepsilon\phi\,dx \tag{17}$$

for all $\phi \in C_0^\infty(a,b)$. The weak convergence of u_ε in $H^2(0,b)$ gives the convergence $u_\varepsilon \to u_*$ in $C^1(0,b)$, in particular, $u'_\varepsilon(+0) \to u'_*(+0)$ as well as $u'_\varepsilon(-0) \to 0$. Moreover, the limit function u_* belongs to $\mathcal{H}$, since each u_ε is a continuous function at $x=0$. A passage to the limit in (17) implies that pair (μ^*, u_*) satisfies identity (16). Recall that u_* is different from zero, which completes the proof. □

Before improving the convergent results, we first compute the spectrum of the limit problem. Let us introduce space $\mathcal{L} = L_2(r, I_a) \oplus L_2(\rho, I_b)$, where $L_2(g, I)$ is a weighted L_2-space with the norm $\|v\| = \left(\int_I g|v|^2\right)^{1/2}$. We consider two operators

$$A_1 = -\tfrac{1}{r}\tfrac{d}{dx}k\tfrac{d}{dx} \text{ in } L_2(r, I_a),$$
$$\mathcal{D}(A_1) = \big\{u \in H^2(I_a)\colon u(a) = 0,\ u'(0) = 0\big\},$$
$$A_2 = -\tfrac{1}{\rho}\tfrac{d}{dx}\varkappa\tfrac{d}{dx} \text{ in } L_2(\rho, I_b),$$
$$\mathcal{D}(A_2) = \big\{u \in H^2(I_b)\colon u(b) = 0\big\}.$$

For problem (15) we assign the matrix operator

$$\mathcal{A} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \text{ in } \mathcal{L},$$

$$\mathcal{D}(\mathcal{A}) = \big\{(u_1, u_2) \in \mathcal{D}(A_1) \oplus \mathcal{D}(A_2)\colon u_1(0) = u_2(0)\big\}.$$

The operator $\mathcal{A}$ is nonself-adjoint. Actually, it is easy to check that

$$\mathcal{A}^* = \begin{pmatrix} \hat{A}_1 & 0 \\ 0 & \hat{A}_2 \end{pmatrix},$$

$$\mathcal{D}(\mathcal{A}^*) = \big\{(v_1, v_2) \in \mathcal{D}(\hat{A}_1) \oplus \mathcal{D}(\hat{A}_2)\colon (kv_1')(0) = (\varkappa v_2')(0)\big\},$$

where $\hat{A}_1$ is the extension of operator A_1 to $\mathcal{D}(\hat{A}_1) = \big\{u \in H^2(a,0)\colon u(a) = 0\big\}$ and $\hat{A}_2$ is the restriction of A_2 to $\mathcal{D}(\hat{A}_2) = \big\{u \in \mathcal{D}(A_2)\colon u(0) = 0\big\}$. Let $\sigma(A)$ and $\varrho(A)$ denote the spectrum and the resolvent set of an operator A respectively. Let $\mathcal{R}_\mu(A)$ denote the resolvent $(A - \mu\mathcal{I})^{-1}$ of an operator A, where $\mathcal{I}$ is the identity operator in $\mathcal{L}$.

Definition. *Let u be an eigenvector of $\mathcal{A}$ with eigenvalue μ. A solution u_* to $(\mathcal{A}-\mu\mathcal{I})u_* = u$ is called an adjoined vector of $\mathcal{A}$ (corresponding to the eigenvalue μ).*

Theorem 1.

(i) $\sigma(\mathcal{A}) = \sigma(A_1) \cup \sigma(\hat{A}_2)$.

(ii) If μ *belongs to* $\sigma(\mathcal{A}) \setminus \big(\sigma(A_1) \cap \sigma(\hat{A}_2)\big)$, *then* μ *is a simple eigenvalue. If* $\mu \in \sigma(A_1) \cap \sigma(\hat{A}_2)$, *then* μ *has multiplicity* 2 *and the corresponding root space is generated by an eigenvector and an adjoined vector of* $\mathcal{A}$.

(iii) The set of eigenvectors and adjoined vectors of $\mathcal{A}$ *forms a complete system in* $\mathcal{L}$.

Proof.

(i) Let us consider the equation $(\mathcal{A}-\mu\mathcal{I})u = f$ for fixed $f \in \mathcal{L}$. In the coordinate representation we have $A_1\, u_1 - \mu u_1 = f_1$, $A_2\, u_2 - \mu u_2 = f_2$. If $\mu \notin \sigma(A_1)$, then $u_1 = \mathcal{R}_\mu(A_1) f_1$. In order to find u_2 we introduce the bounded intertwining operator $T_\mu \colon H^2(I_a) \to H^2(I_b)$ that solves the problem $(\varkappa\psi')' + \mu\rho\psi = 0$ in I_b, $\psi(0) = g(0)$, $\psi(b) = 0$ for each $g \in H^2(I_a)$. Note that T_μ is a well-defined operator for all $\mu \in \varrho(\hat{A}_2)$. Then $u_2 = T_\mu \mathcal{R}_\mu(A_1) f_1 + \mathcal{R}_\mu(\hat{A}_2) f_2$ and the resolvent of $\mathcal{A}$ can be written in the form

$$\mathcal{R}_\mu(\mathcal{A}) = \begin{pmatrix} \mathcal{R}_\mu(A_1) & 0 \\ T_\mu \mathcal{R}_\mu(A_1) & \mathcal{R}_\mu(\hat{A}_2) \end{pmatrix}. \tag{18}$$

From the explicit representation of $\mathcal{R}_\mu(\mathcal{A})$ it follows that sets $\sigma(\mathcal{A})$ and $\sigma(A_1) \cup \sigma(\hat{A}_2)$ coincide.

(ii) We suppose that $\mu \in \sigma(A_1) \setminus \sigma(\hat{A}_2)$. Then there exists an eigenvector $U_\mu = (u_1, T_\mu u_1)$, where u_1 is an eigenvector of A_1 and, that is the same, one is an eigenfunction of problem $(k\phi')' + \mu r\phi = 0$ in I_a, $\phi(a) = \phi'(0) = 0$. Note that μ is a simple eigenvalue of the problem. Indeed, $(\mathcal{A} - \mu\mathcal{I})U_\mu = 0$ follows from the evident equality $(A_2 - \mu\mathcal{I})T_\mu = 0$ for all $\mu \in \varrho(\hat{A}_2)$.

Suppose now that $\mu \in \sigma(\hat{A}_2) \setminus \sigma(A_1)$. Then operator $\mathcal{A}$ has the eigenvector $V_\mu = (0, u_2)$, where u_2 is an eigenvector of $\hat{A}_2$. In other words, u_2 is an eigenfunction of the Dirichlet problem (13). Note that each point of $\sigma(\hat{A}_2)$ is a simple eigenvalue. Furthermore, the first component u_1 must be zero, since $\mu \notin \sigma(A_1)$.

Finally we shall show that each point of intersection $\sigma(A_1) \cap \sigma(\hat{A}_2)$ is an eigenvalue of algebraic multiplicity 2. Obviously, vector $V_\mu = (0, u_2)$, which appears above, is an eigenvector of $\mathcal{A}$ in this case too. Next we consider the system

$$A_1 v_1 - \mu\, v_1 = 0, \quad A_2 v_2 - \mu\, v_2 = u_2 \tag{19}$$

determining adjoined vectors. If $v_1 = 0$, then v_2 must be a solution of the boundary value problem $(\varkappa\phi')' + \mu\rho\phi = -\rho u_2$ in I_b, $\phi(0) = \phi(b) = 0$, which is unsolvable. Actually, since $\mu \in \sigma(\hat{A}_2)$, by the Fredholm alternative the problem admits a solution iff $\int_0^b \rho|u_2|^2\,dx = 0$. This contradicts the fact that u_2 is an eigenvector of $\hat{A}_2$. Consequently we have to assume that v_1 is an eigenvector of A_1 and examine the problem $(\varkappa v_2')' + \mu\rho v_2 = -\rho u_2$ in I_b, $v_2(0) = v_1(0)$, $v_2(b) = 0$. Here the Fredholm alternative gives the solvability condition

$$\varkappa(0)u_2'(0)v_1(0) = -\int_0^b \rho\, u_2^2\,dx. \tag{20}$$

We satisfy one by normalization of v_1, because $u_2'(0)$ is different from zero. This condition assures the existence of v_2 and a solution $V_\mu^* = (v_1, v_2)$ of system (19). Vector V_μ^* is the adjoined vector of $\mathcal{A}$. Pair $\{V_\mu, V_\mu^*\}$ forms a basis in the root space that corresponds to μ.

The last statement of the theorem follows from the Keldysh theorem [3]. $\square$

We investigate the limit behaviour of eigenfunctions $u_{\varepsilon,n}$ normalized by conditions

$$\int_a^b R(x)\,u_{\varepsilon,j}^2(x)\,dx = 1, \qquad u_{\varepsilon,j}'(b) > 0. \tag{21}$$

Let us enumerate the eigenvalues of operator $\mathcal{A}$ in increasing order and repeat each eigenvalue according to its multiplicity: $\mu_1 \le \mu_2 \le \cdots \le \mu_j \le \cdots$. The next statement improves the conditional results of Proposition 2.

Theorem 2. *There exists a one-to-one correspondence between the set of eigenvalues $\{\lambda_j^\varepsilon\}_{j=1}^\infty$ of perturbed problem (2)-(5) and the spectrum of operator $\mathcal{A}$. Namely, $\varepsilon^{-1}\lambda_j^\varepsilon \to \mu_j$ as $\varepsilon \to 0$, for each $j \in \mathbb{N}$. Furthermore, a sequence of the corresponding eigenfunctions $u_{\varepsilon,j}$ converges in $H^1(a,b)$ towards the eigenfunction u with eigenvalue μ_j.*

Proof.

For the perturbed problem (2)-(5) we assign the matrix operator in $\mathcal{L}$

$$\mathcal{A}_\varepsilon = \begin{pmatrix} \hat{A}_1 & 0 \\ 0 & A_2 \end{pmatrix}, \ \mathcal{D}(\mathcal{A}_\varepsilon) = \big\{\ (u_1, u_2) \in \mathcal{D}(\hat{A}_1) \oplus \mathcal{D}(A_2)\colon$$
$$u_1(0) = u_2(0), \quad (ku_1')(0) = \varepsilon(\varkappa u_2')(0)\big\}.$$

Clearly, if μ_ε belongs to $\sigma(\mathcal{A}_\varepsilon)$, then $\varepsilon\mu_\varepsilon$ is an eigenvalue of (2)-(5). Let us solve the equation $(\mathcal{A}_\varepsilon - \mu\mathcal{I})u = f$ for $f = (f_1, f_2) \in \mathcal{L}$ and $\mu \in \varrho(\mathcal{A}_\varepsilon)$. Similarly to the previous theorem we obtain $u_1 = \mathcal{R}_\mu(A_1)f_1 + \varepsilon S_\mu u_2$, $u_2 = T_\mu u_1 + \mathcal{R}_\mu(\hat{A}_2)f_2$, where $S_\mu\colon H^2(I_b) \to H^2(I_a)$ is a bounded intertwining operator that solves the problem $(k\psi')' + \mu r\psi = 0$ in I_a, $\psi(a) = 0$ and $(k\psi')(0) = (\varkappa g')(0)$ for each $g \in H^2(I_b)$. This yields that

$$\begin{pmatrix} I & -\varepsilon S_\mu \\ -T_\mu & I \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \mathcal{R}_\mu(A_1)f_1 \\ \mathcal{R}_\mu(\hat{A}_2)f_2 \end{pmatrix}, \tag{22}$$

where the matrix operator in the left-hand side is invertible as a small perturbation of the invertible one. Letting $\varepsilon \to 0$ we can assert that

$$\mathcal{R}_\mu(\mathcal{A}_\varepsilon) = \begin{pmatrix} I & -\varepsilon S_\mu \\ -T_\mu & I \end{pmatrix}^{-1} \begin{pmatrix} \mathcal{R}_\mu(A_1) & 0 \\ 0 & \mathcal{R}_\mu(\hat{A}_2) \end{pmatrix} \to$$

$$\to \begin{pmatrix} I & 0 \\ T_\mu & I \end{pmatrix} \begin{pmatrix} \mathcal{R}_\mu(A_1) & 0 \\ 0 & \mathcal{R}_\mu(\hat{A}_2) \end{pmatrix}.$$

Hence, $\mathcal{R}_\mu(\mathcal{A}_\varepsilon) \to \mathcal{R}_\mu(\mathcal{A})$ in the uniform operator topology as $\varepsilon \to 0$, which establishes a number-by-number convergence of the corresponding eigenvalues [3, Th. 3.1].

Next we prove existence of the limit for the eigenfunctions under normalization condition (21). We conclude from (17) that

$$\int_a^b K(x)u_\varepsilon'^2(x)\,dx + (1-\varepsilon)\varkappa(0)u_\varepsilon'(+0)u_\varepsilon(+0) = \mu_\varepsilon.$$

For each ν there exists a twice differentiable solution $\psi(x,\nu)$ of equation $(\varkappa v')' + \nu\rho\, v = 0$ in I_b that satisfies conditions $v(b) = 0$, $v'(b) = 1$. Moreover, $\psi(x,\nu)$ is an analytic function with respect to the second argument for each fixed x [2, Th.1.5]. In particular, $\psi(x,\mu^\varepsilon) \to \psi(x,\mu)$ in $C^2(0,b)$ as $\mu^\varepsilon \to \mu$. Then there exits constant β_ε such that $u_\varepsilon(x) = \beta_\varepsilon\psi(x,\mu^\varepsilon)$. Moreover, β_ε is bounded as $\varepsilon \to 0$, which is due to condition (21). Therefore the values $u_\varepsilon(+0)$ and $u_\varepsilon'(+0)$ are bounded with respect to ε. Consequently we have $\int_a^b K(x)u_\varepsilon'^2(x)\,dx \le \mu_\varepsilon + (1-\varepsilon)\varkappa(0)|u_\varepsilon'(+0)u_\varepsilon(+0)| \le M$. Then finally the sequence $\{u_\varepsilon\}_{\varepsilon>0}$ is precompact in the weak topology of $H^1(a,b)$. Let us consider a subsequence $u_{\varepsilon'}$ such that $u_{\varepsilon'} \to u$ in $H^1(a,b)$ weakly. We get $u_{\varepsilon'}(x) = \beta_{\varepsilon'}\psi(x,\mu^{\varepsilon'}) \to \beta\psi(x,\mu) = u(x)$ in $C^2(0,b)$ for certain β. Note that

$\beta > 0$, which is due to (21). Moreover, $u'_{\varepsilon'}(+0) \to u'(+0)$ as $\varepsilon' \to 0$. A passage to the limit in (17) implies that partial weak limit u satisfies the identity

$$\int_a^b K(x)u'\phi'\,dx + \varkappa(0)u'(+0)\phi(0) = \mu \int_a^b R(x)u\phi\,dx$$

for all $\phi \in C_0^\infty(a,b)$. Moreover, u is different from zero, since

$$\int_a^b R|u|^2\,dx = 1.$$

Consequently each weakly convergent subsequence of $\{u_\varepsilon\}_{\varepsilon>0}$ tends to u, where u is an eigenfunction of (15) that corresponds to the eigenvalue μ and satisfies conditions $\|u\|_{L^2(R,(a,b))} = 1$ and $u'(b) > 0$. Then the same conclusion can be drawn for the entire sequence. □

Remark 1. *In some cases value $\varepsilon^{-1}\lambda^\varepsilon$ doesn't actually depend on ε. The latter takes place if and only if the three-points problem*

$$\begin{cases} (K(x)u')' + \mu R(x)u = 0 \text{ for } x \in I_a \cup I_b, \\ u(a) = u(b) = u'(-0) = u'(+0) = 0 \end{cases} \tag{23}$$

has an eigenfunction u that is continuous at $x = 0$ *(for a certain eigenvalue μ). This situation is possible, for instance, in the case $a = -b$ when there exists even eigenfunction of the Dirichlet problem on $(-b, b)$. Then a trivial verification shows that $\lambda^\varepsilon = \varepsilon\mu$ is an eigenvalue of (2)-(5) with the eigenfunction $u_\varepsilon = u$ for all $\varepsilon \in (0, 1]$.*

Corollary 1. *Restrictions of eigenfunction $u_{\varepsilon,j}$ to the intervals I_a and I_b converge towards the corresponding restrictions of eigenfunction u in $H^2(a,0)$ and $H^2(0,b)$ respectively.*

Proof. Set $u_\varepsilon = u_{\varepsilon,j}$. We consider equation (2) in the form $u''_\varepsilon = -k'k^{-1}u'_\varepsilon - \mu_\varepsilon rk^{-1}u_\varepsilon$ in I_a. Then from Theorem 2 we have

$$u''_\varepsilon \to -k'k^{-1}u' - \mu rk^{-1}u \quad \text{in} \quad L^2(a,0), \tag{24}$$

where u is an eigenfunction of (15). From (15) it follows that the limit (24) is exactly the second derivative of the limiting eigenfunction in I_a. The proof for interval I_b is the same. □

3. Formal Asymptotic Expansions of Eigenvalues and Eigenfunctions.

3.1. Asymptotics of Simple Eigenvalues. In this section we construct the complete asymptotic expansions of eigenvalues λ^ε and eigenfunctions u_ε. We begin with the examination of eigenvalues λ_j^ε for which the limit $\mu = \lim\limits_{\varepsilon\to 0} \lambda_j^\varepsilon/\varepsilon$ is a simple eigenvalue of operator $\mathcal{A}$. Clearly, μ depends on j, which we do not indicate for the sake of notation simplicity. The asymptotic expansions of the eigenvalues and the corresponding eigenfunctions are represented by

$$\lambda^\varepsilon \sim \varepsilon\,(\mu + \varepsilon\nu_1 + \cdots + \varepsilon^n\nu_n + \cdots), \tag{25}$$

$$u_\varepsilon(x) \sim \begin{cases} y_0(x) + \varepsilon y_1(x) + \cdots + \varepsilon^n y_n(x) + \cdots & \text{for } x \in I_a, \\ z_0(x) + \varepsilon z_1(x) + \cdots + \varepsilon^n z_n(x) + \cdots & \text{for } x \in I_b, \end{cases} \tag{26}$$

where μ is an arbitrary eigenvalue of limit problem (15). Then

$$u(x) = \begin{cases} y_0(x) & \text{for } x \in I_a, \\ z_0(x) & \text{for } x \in I_b \end{cases} \tag{27}$$

is the corresponding eigenfunction of (15) as it follows from Th. 2. Since in this section we treat only the simple eigenvalues μ, according to Th. 1 we only consider here two possible situations: $\mu \in \sigma(A_1)\backslash\sigma(\hat{A}_2)$ and $\mu \in \sigma(\hat{A}_2)\backslash\sigma(A_1)$.

3.1.1. Case $\mu \in \sigma(A_1)\backslash\sigma(\hat{A}_2)$. We fix the corresponding eigenfunction y_0 of operator A_1 such that $\int\limits_a^0 r y_0^2\,dx = 1$ and $y_0(0) > 0$. Since μ doesn't belong to the spectrum of $\hat{A}_2$ there exists a unique solution z_0 to the problem

$$(\varkappa z_0')' + \mu\rho z_0 = 0 \quad \text{in} \quad I_b, \qquad z_0(0) = y_0(0), \quad z_0(b) = 0. \tag{28}$$

An easy computation shows that the next terms of the expansions are unique solutions to the recurrent sequence of problems

$$\begin{cases} (k y_n')' + \mu r y_n = -\nu_n r y_0 - r \sum\limits_{j=1}^{n-1} \nu_j\, y_{n-j} \quad \text{in} \quad I_a, \\ y_n(a) = 0, \quad (k y_n')(0) = (\varkappa z_{n-1}')(0), \quad \int_a^0 r y_n y_0\,dx = 0, \end{cases} \tag{29}$$

$$\begin{cases} (\varkappa z_n')' + \mu\rho z_n = -\rho \sum\limits_{j=1}^{n} \nu_j\, z_{n-j} \quad \text{in} \quad I_b, \\ z_n(0) = y_n(0), \quad z_n(b) = 0 \end{cases} \tag{30}$$

with $\nu_n = -(\varkappa z'_{n-1})(0)y_0(0)$ for $n = 1, 2, \dots$. The last formula for ν_n is obtained as the solvability condition of (29). Note that all solutions y_n, z_n are smooth functions.

Remark 2. *It might happened that $z'_0(0) = 0$ (cf. the proof of Th. 2). In this case function u defined by (27) is exactly an eigenfunction of the perturbed problem for each $\varepsilon \in (0, 1]$. Then the construction of asymptotics is interrupted and we can state that there exists an eigenvalue $\lambda^\varepsilon = \varepsilon\mu$ for all $\varepsilon > 0$. The corresponding eigenfunction*

$$u_\varepsilon(x) = \begin{cases} y_0(x) & \text{for } x \in I_a, \\ z_0(x) & \text{for } x \in I_b \end{cases}$$

doesn't depend on ε.

3.1.2. Case $\mu \in \sigma(\hat{A}_2)\backslash\sigma(A_1)$ This situation immediately implies $y_0 = 0$ (cf. the proof of Th. 1, part (ii)). We fix the corresponding eigenfunction z_0 of $\hat{A}_2$ such that $\int\limits_0^b \rho z_0^2\, dx = 1$ and $z'_0(0) > 0$. A trivial verification shows that the next terms of expansions (26) are the unique smooth solutions to the problems

$$\begin{cases} (ky'_n)' + \mu r y_n = -r \sum\limits_{j=1}^{n-1} \nu_j\, y_{n-j} \quad \text{in} \;\; I_a, \\ y_n(a) = 0, \quad (ky'_n)(0) = (\varkappa z'_{n-1})(0), \end{cases}$$
$$\begin{cases} (\varkappa z'_n)' + \mu\rho z_n = -\nu_n \rho z_0 - \rho \sum\limits_{j=1}^{n-1} \nu_j\, z_{n-j} \quad \text{in} \;\; I_b, \\ z_n(0) = y_n(0), \quad z_n(b) = 0, \quad \int_0^b \rho z_n z_0\, dx = 0, \end{cases} \tag{31}$$

with $\nu_n = -(\varkappa z'_0)(0)y_n(0)$ for $n = 1, 2, \dots$. Such choice of ν_n assures the solvability of (31).

3.2. Asymptotics of Double Eigenvalues. In this subsection we treat the case when for two successive eigenvalues λ^ε_j and $\lambda^\varepsilon_{j+1}$ the corresponding ratios $\varepsilon^{-1}\lambda^\varepsilon_j$ and $\varepsilon^{-1}\lambda^\varepsilon_{j+1}$ converge to the same limit μ. It is obvious that μ must belong to the intersection $\sigma(A_1)\cup\sigma(\hat{A}_2)$. Let us assume that the eigenvalues and the corresponding eigenfunctions admit expansions

$$\lambda^\varepsilon \sim \varepsilon\,(\mu + \sqrt{\varepsilon}\nu_1 + \varepsilon\nu_2 + \cdots), \tag{32}$$

$$u_\varepsilon(x) \sim \begin{cases} \sqrt{\varepsilon}\, w_1(x) + \varepsilon\, w_2(x) + \cdots & \text{for } x \in (a, 0), \\ v_0(x) + \sqrt{\varepsilon}\, v_1(x) + \varepsilon\, v_2(x) + \cdots & \text{for } x \in (0, b), \end{cases} \tag{33}$$

because the eigenvectors of operator $\mathcal{A}$ that correspond to double eigenvalues μ have the form $V_\mu = (0, v_0)$ (see Th. 1). Substituting (32), (33) into the perturbed problem we obtain

$$(\varkappa v_0')' + \mu\rho v_0 = 0 \quad \text{in} \quad I_b, \qquad v_0(0) = v_0(b) = 0, \tag{34}$$

$$(k w_1')' + \mu r w_1 = 0 \quad \text{in} \quad I_a, \qquad w_1(a) = w_1'(0) = 0. \tag{35}$$

We fix $\mu \in \sigma(A_1) \cup \sigma(\hat{A}_2)$ and introduce the functions

$$U(x) = \begin{cases} 0 & \text{for } x \in I_a \\ v(x) & \text{for } x \in I_b \end{cases}, \qquad U_*(x) = \begin{cases} w_*(x) & \text{for } x \in I_a \\ v_*(x) & \text{for } x \in I_b \end{cases} \tag{36}$$

that correspond to the eigenvector and adjoined vector of $\mathcal{A}$ (cf. vectors V_μ and V_μ^* in Th. 1). Here v is an eigenfunction of (34) such that $\int_0^b \rho v^2\, dx = 1$, $v'(0) > 0$ and adjoined vector U_* is chosen such that $(U, U_*)_{L_2(R,(a,b))} = 0$. We also introduce an eigenfunction w of (35) such that $\int_a^0 r w^2\, dx = 1$ and $w(0) > 0$. It follows that $v_0 = \alpha v$ and $w_1 = \beta w$ with certain constants α and β. In addition, α must be different from zero. The next problems to solve are

$$\begin{cases} (\varkappa v_1')' + \mu\rho v_1 = -\nu_1 \alpha \rho v \quad \text{in} \quad I_b, \\ v_1(0) = \beta w(0), \quad v_1(b) = 0, \end{cases} \tag{37}$$

$$\begin{cases} (k w_2')' + \mu r w_2 = -\nu_1 \beta r w \quad \text{in} \quad I_a, \\ w_2(a) = 0, \quad k(0) w_2'(0) = \alpha \varkappa(0) v'(0). \end{cases} \tag{38}$$

In general case both problems (37) and (38) are unsolvable, since μ belongs to the spectra $\sigma(A_1)$ and $\sigma(\hat{A}_2)$ at one time. Hence we have to apply Fredholm's alternative for both the problems. After multiplying equations (38) and (37) by eigenfunctions v and w respectively and integrating by parts, one yields the common solvability condition:

$$\begin{pmatrix} 0 & \omega \\ \omega & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = -\nu_1 \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \tag{39}$$

where $\omega = (\varkappa w v')(0)$ is positive. Since the first component of vector $\gamma = (\alpha, \beta)$ must be different from zero, $-\nu_1$ is an eigenvalue of the matrix in (39). Therefore if either $\nu_1 = \omega$ and $\gamma = (1, -1)$ or $\nu_1 = -\omega$ and $\gamma = (1, 1)$, then problems (37), (38) admit solutions. Moreover, functions $\nu_1 w_*$ and $\nu_1 v_*$ solve problems (35) and (37) respectively for both values of ν_1. Actually these problems imply immediately

$(\mathcal{A}-\mu)U_* = \omega U$. In other words, the first corrector is an adjoined vector of $\mathcal{A}$ that corresponds to the eigenvector ωU. It causes no confusion that we use the same letters U, U_* to designate a function of $L_2(a,b)$ and a vector in $\mathcal{L}$.

Summarizing, we formally demonstrate that there exists a pair of closely adjacent eigenvalues λ_j^ε and $\lambda_{j+1}^\varepsilon$ that admit the asymptotic expansions

$$\lambda_j^\varepsilon = \varepsilon\mu - \varepsilon^{3/2}\omega + O(\varepsilon^2), \quad \lambda_{j+1}^\varepsilon = \varepsilon\mu + \varepsilon^{3/2}\omega + O(\varepsilon^2), \quad \text{as } \varepsilon \to 0.$$

As of asymptotics of eigenfunctions we have

$$u_{\varepsilon,j}(x) = U(x) - \sqrt{\varepsilon}\,\omega U_*(x) + O(\varepsilon),$$

$$u_{\varepsilon,j+1}(x) = U(x) + \sqrt{\varepsilon}\,\omega U_*(x) + O(\varepsilon).$$

These eigenfunctions subtend an infinitely small angle in L^2-space as $\varepsilon \to 0$. Hence $u_{\varepsilon,j}$ and $u_{\varepsilon,j+1}$ stick together at the limit. The latter gives rise to the loss of completeness of the limit eigenfunction system.

Suppose that $\nu_1 = \omega$ and $\gamma = (1,-1)$. Then we will denote by V_1 and W_2 such solutions of the problems that $\int_0^b \rho V_1 v\,dx = 0$ and $\int_a^0 r W_2 w\,dx = 0$. We see at once that $-V_1$ and $-W_2$ are solutions of (37), (38) for $\nu_1 = -\omega$ and $\gamma = (1,1)$.

From now on we distinct two branches of expansions (32)

$$\begin{aligned} \lambda_-^\varepsilon &\sim \varepsilon(\mu - \sqrt{\varepsilon}\omega + \varepsilon\nu_2^- + \dots + \varepsilon^{n/2}\nu_n^- + \dots), \\ \lambda_+^\varepsilon &\sim \varepsilon(\mu + \sqrt{\varepsilon}\omega + \varepsilon\nu_2^+ + \dots + \varepsilon^{n/2}\nu_n^+ + \dots), \end{aligned} \tag{40}$$

and the corresponding branches of (33) are

$$\begin{aligned} &u_\varepsilon^\pm(x) \sim \\ &\sim \begin{cases} \mp\sqrt{\varepsilon}\,w(x) \pm \varepsilon\,w_2^\pm(x) + \dots + \varepsilon^{n/2}w_n^\pm(x)\dots, x \in I_a, \\ v(x) \pm \sqrt{\varepsilon}\,v_1^\pm(x) + \varepsilon v_2^\pm(x) + \dots + \varepsilon^{n/2}v_n^\pm(x)\dots, x \in I_b. \end{cases} \end{aligned} \tag{41}$$

All coefficients are endowed with indexes + or − if they depend on the choice of the sign of the first corrector $\nu_1 = \pm\omega$. Note that the high order correctors in (40), (41) have to be calculated separately for both the branches. We now turn to the case $\nu_1 = \omega$ and find coefficients ν_n^+, w_n^+ and v_n^+. To shorten notation, we omit upper index "+" for a while. Next, we see that problems (37) and (38) admit many solutions

$v_1 = V_1 + \alpha_1 v$ and $w_2 = W_2 + \beta_1 w$, where α_1, β_1 are constants. These constants can be obtained from the consistency of problems

$$\begin{cases} (\varkappa v_2')' + \mu\rho v_2 = -\nu_1\rho\,(V_1 + \alpha_1 v) - \nu_2\rho v, \quad x \in I_b \\ v_2(0) = W_2(0) + \beta_1 w(0), \quad v_2(b) = 0, \end{cases} \tag{42}$$

$$\begin{cases} (kw_3')' + \mu r w_3 = -\nu_1 r\,(W_2 + \beta_1 w) - \nu_2 r w_1, \quad x \in I_a \\ w_3(a) = 0, \quad k(0)w_3'(0) = \varkappa(0)\,(V_1 + \alpha_1 v)'\,(0). \end{cases} \tag{43}$$

The solvability conditions for problems (42) and (43), which arrive from Fredholm's alternatives, can be represented as a linear algebraic system

$$\begin{pmatrix} \nu_1 & \omega \\ \omega & \nu_1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} (\varkappa W_2 v')(0) + \nu_2 \\ (\varkappa w V_1)'(0) - \nu_2 \end{pmatrix}. \tag{44}$$

The system has solution if and only if $\nu_2 = \frac{1}{2}\left(\varkappa w V_1{}' - \varkappa W_2 v'\right)(0)$. After the solvability condition is satisfied, system (44) has a partial solution $\alpha_1 = \beta_1 = \frac{1}{2\omega}\left(\varkappa w V_1' + \varkappa W_2 v'\right)(0)$ and problems (42) and (43) admit solutions V_2 and W_3 such that $\int\limits_0^b \rho V_2 v\,dx = 0$ and

$$\int\limits_a^0 r W_3 w\,dx = 0$$

. Therefore, all other solutions of (42) and (43) allow the representation $v_2 = V_2 + \alpha_2 v$ and $w_3 = W_3 + \beta_2 w$ with real constants α_2, β_2.

We construct the general terms of expansions (40) and (41) as solutions to the problems

$$\begin{cases} (\varkappa v_n')' + \mu\rho v_n = -\rho \sum\limits_{j=1}^{n} \nu_j v_{n-j}, \quad x \in I_b, \\ v_n(0) = w_n(0), \quad v_n(b) = 0, \end{cases} \tag{45}$$

$$\begin{cases} (kw_{n+1}')' + \mu r w_{n+1} = -r \sum\limits_{j=1}^{n} \nu_j w_{n+1-j}, \quad x \in I_a, \\ w_{n+1}(a) = 0, \quad (kw_{n+1})'(0) = (\varkappa v_{n-1})'(0), \end{cases} \tag{46}$$

with

$$v_{n-1} = V_{n-1} + \alpha_{n-1} v \qquad \text{and} \qquad w_n = W_n + \beta_{n-1} w, \tag{47}$$

where V_{n-1} and W_n are solutions of the previous problems chosen accordingly to the orthogonality conditions $\int\limits_0^b \rho V_{n-1} v\, dx = 0$ and $\int\limits_a^0 r W_n w\, dx = 0$, $n \geq 2$. Constants α_{n-1} and β_{n-1} we find from the solvability conditions for (45) and (46) given by

$$\begin{pmatrix} \nu_1 & \omega \\ \omega & \nu_1 \end{pmatrix}\begin{pmatrix} \alpha_{n-1} \\ \beta_{n-1} \end{pmatrix}=\begin{pmatrix} (\varkappa W_n v')(0) + \sum\limits_{j=2}^{n-1} \nu_j \alpha_{n-j} + \nu_n \\ \left(\varkappa w V'_{n-1}\right)(0) + \sum\limits_{j=2}^{n-1} \nu_j \beta_{n+1-j} - \nu_n \end{pmatrix}. \quad (48)$$

The latter has a solution if and only if $\nu_n = \frac{1}{2}\left(\varkappa w V'_{n-1} - \varkappa W_n v'\right)(0)$. Then system (48) has a partial solution

$$\alpha_{n-1} = \beta_{n-1} = \frac{1}{2\omega}\left(\varkappa w V'_{n-1} + \varkappa W_n v'\right)(0) + \frac{1}{\omega}\sum_{j=2}^{n-1} \nu_j \alpha_{n-j}.$$

Substituting the constants into (47) we finish the general step of recurrent algorithm. Hence, after coming back our natation we obtain all coefficients ν_n^+, v_n^+ and w_n^+ of series (40) and (41).

Similarly, we can construct the coefficients ν_n^-, v_n^- and w_n^- of series (40) and (41). Then, by induction we get that for any natural n the coefficients satisfy relations $\nu_n^- = (-1)^n \nu_n^+$, $v_n^- = (-1)^n v_n^+$ and $w_n^- = (-1)^n w_n^+$.

4. Justification of Asymptotic Expansions.

Let $\mathcal{L}_\varepsilon$ be he weighted L_2-space with the scalar product and norm given by (6). We also introduce space $\mathcal{H}_\varepsilon$ as the Sobolev space $H_0^1(a,b)$ with scalar product and norm

$$\langle \phi, \psi \rangle_\varepsilon = \int_a^0 k\phi'\,\psi'\,dx + \varepsilon \int_0^b \varkappa\phi'\,\psi'\,dx, \qquad \|\phi\|_{\mathcal{H}_\varepsilon} = \sqrt{\langle \phi, \phi \rangle_\varepsilon}. \quad (49)$$

It is easily seen that

$$c\|\phi\| \leq \|\phi\|_\varepsilon \leq C\varepsilon^{-1/2}\|\phi\|, \qquad c\varepsilon^{1/2}\|\phi\|_1 \leq \|\phi\|_{\mathcal{H}_\varepsilon} \leq C\|\phi\|_1, \quad (50)$$

where $\|\cdot\|$ and $\|\cdot\|_1$ are standard norms in $L_2(a,b)$ and $H_0^1(a,b)$ respectively.

For the sake of completeness, we introduce here below the classical result on quasimodes. Let A be a self-adjoint operator in Hilbert space H with domain $\mathcal{D}(A)$ and $\sigma > 0$.

Definition. *We will say that pair* $(\mu, u) \in \mathbb{R} \times \mathcal{D}(A)$ *is a quasimode with accuracy to* σ *for operator* A *if* $\|(A-\mu I)u\|_H \le \sigma$ *and* $\|u\|_H = 1$.

Lemma 1 (Vishik and Lyusternik). *Suppose that the spectrum of* A *is discrete. If* (μ, u) *is a quasimode of* A *with accuracy to* σ*, then interval* $[\mu - \sigma, \mu + \sigma]$ *contains an eigenvalue of* A*. Furthermore, if segment* $[\mu - d, \mu + d]$, $d > 0$*, contains one and only one eigenvalue* λ *of* A*, then* $\|u - v\|_H \le 2d^{-1}\sigma$*, where* v *is an eigenfunction of* A *with eigenvalue* λ, $\|v\|_H = 1$. *[4, 5]*

4.1. Simple Spectrum. We will denote by $\Lambda_{\varepsilon,n} = \varepsilon\,(\mu + \varepsilon\nu_1 + \cdots + \varepsilon^n \nu_n)$ and

$$U_{\varepsilon,n}(x) = \begin{cases} y_0(x) + \varepsilon y_1(x) + \cdots + \varepsilon^n y_n(x) & \text{for } x \in I_a \\ z_0(x) + \varepsilon z_1(x) + \cdots + \varepsilon^n z_n(x) & \text{for } x \in I_b \end{cases}$$

the partial sums of series (25), (26). The perturbed problem is associated with self-adjoint operator $A_\varepsilon = -\frac{1}{r_\varepsilon}\frac{d}{dx}k_\varepsilon\frac{d}{dx}$ in $\mathcal{L}_\varepsilon$ with the domain $\mathcal{D}(A_\varepsilon) = \{f \in \mathcal{H}\colon (kf')(-0) = \varepsilon(\varkappa f')(+0)\}$, where coefficients k_ε, r_ε are given by (1) for $m = 1$.

Theorem 3. *If* $\mu_j \in \sigma(A_1)\backslash\sigma(\hat{A}_2)$*, then eigenfunction* $u_{\varepsilon,j}$ *of* (2)-(5) *with eigenvalue* λ_j^ε *converges in* $H^1(a,b)$ *towards the function*

$$u(x) = \begin{cases} y(x) & \textit{for } x \in I_a \\ z(x) & \textit{for } x \in I_b, \end{cases}$$

where y *is an eigenfunction of the problem* $(ky')' + \mu r y = 0$ *in* I_a, $y(a) = y'(0) = 0$ *with eigenvalue* μ_j*, and* z *is a unique solution of the problem* $(\varkappa z')' + \mu_j\,\rho z = 0$ *in* I_b, $z(0) = y(0)$, $z(b) = 0$.

If $z'(0) = 0$*, then* $\lambda_j^\varepsilon = \varepsilon\mu_j$ *and* $u_{\varepsilon,j} = u$ *for all* $\varepsilon > 0$*. Otherwise* λ_j^ε *and* $u_{\varepsilon,j}$ *admit asymptotics expansions* (25), (26) *obtained in 3.1.1 for* $\mu = \mu_j$*. Moreover, the estimates of remainder terms hold*

$$\left|\varepsilon^{-1}\lambda_j^\varepsilon - (\mu_j + \varepsilon\nu_1 + \cdots + \varepsilon^n\nu_n)\right| \le c_n\varepsilon^{n+1}, \tag{51}$$

$$\|u_{\varepsilon,j} - \vartheta_\varepsilon U_{\varepsilon,n}\|_{H^1(a,b)} \le C_n\varepsilon^{n+1}, \tag{52}$$

where ϑ_ε *is a normalizing multiplier with strictly positive limit as* $\varepsilon \to 0$.

Proof. The case $z'(0)=0$ was considered in Remarks 1 and 2. Suppose that $z'(0)\neq 0$. We first check that the the series being constructed in give us the quasimodes with accuracy to an arbitrary order. It follows from (29), (30) that

$$\left|r_\varepsilon^{-1}(k_\varepsilon U'_{\varepsilon,n})'+\Lambda_{\varepsilon,n}U_{\varepsilon,n}\right|\leq c_n\varepsilon^{n+2} \tag{53}$$

in $[a,b]$ uniformly, $U_{\varepsilon,n}(a)=U_{\varepsilon,n}(b)=0$, $U_{\varepsilon,n}(-0)=U_{\varepsilon,n}(+0)$ and

$$\beta_{\varepsilon,n}=(kU'_{\varepsilon,n})(-0)-\varepsilon(\varkappa U'_{\varepsilon,n})(+0)=O(\varepsilon^{n+1}),\quad \varepsilon\to 0. \tag{54}$$

Note that $U_{\varepsilon,n}$ doesn't belong to the domain of A_ε since $\beta_{\varepsilon,n}$ is different from zero in the general case. Set $\phi(x)=x(\frac{x}{a}-1)$ for $x\in(a,0)$ and $\phi(x)=0$ elsewhere. Then $V_{\varepsilon,n}=U_{\varepsilon,n}+\beta_{\varepsilon,n}\phi$ belongs to $\mathcal{D}(A_\varepsilon)$ and a simple computation gives $\|A_\varepsilon V_{\varepsilon,n}-\Lambda_{\varepsilon,n}V_{\varepsilon,n}\|_\varepsilon\leq c_n\varepsilon^{n+3/2}$. Hence $(\Lambda_{\varepsilon,n},V_{\varepsilon,n}/\|V_{\varepsilon,n}\|_\varepsilon)$ is a quasimode of operator A_ε with accuracy to $c_n\varepsilon^{n+2}$ because $\|V_{\varepsilon,n}\|_\varepsilon=O(\varepsilon^{-1/2})$. According to the Vishik-Lyusternik Lemma there exists an eigenvalue λ^ε of A_ε such that $|\lambda^\varepsilon-\Lambda_{\varepsilon,n}|\leq c_n\varepsilon^{n+2}$, which establishes (51). Moreover, there exists an unique eigenvalue $\lambda^\varepsilon=\lambda_j^\varepsilon$ with such asymptotics by Theorem 2. Next, for a certain $d>0$ segment $[\Lambda_{\varepsilon,n}-d\varepsilon,\Lambda_{\varepsilon,n}+d\varepsilon]$ contains one and only one eigenvalue of A_ε. Repeated application of Lemma 1 enables us to write $\left\|\|u_\varepsilon\|_\varepsilon^{-1}\cdot u_\varepsilon-\|V_{\varepsilon,n}\|_\varepsilon^{-1}\cdot V_{\varepsilon,n}\right\|_\varepsilon\leq 2c_nd^{-1}\varepsilon^{n+1}$, where $u_\varepsilon=u_{\varepsilon,j}$. Hence, by (50)

$$\left\|u_\varepsilon-\frac{\|u_\varepsilon\|_\varepsilon}{\|V_{\varepsilon,n}\|_\varepsilon}V_{\varepsilon,n}\right\|_\varepsilon\leq\frac{2c_n}{d}\|u_\varepsilon\|_\varepsilon\varepsilon^{n+1}\leq C_n\varepsilon^{n+1/2}$$

and $\vartheta_\varepsilon=\frac{\|u_\varepsilon\|_\varepsilon}{\|V_{\varepsilon,n}\|_\varepsilon}$ converges to 1 by Theorem 2.

Pair $(\lambda^\varepsilon,u_\varepsilon)$ satisfies identity $\langle u_\varepsilon,\psi\rangle_\varepsilon=\lambda^\varepsilon(u_\varepsilon,\psi)_\varepsilon$ for all $\psi\in H_0^1(a,b)$. Similarly,

$$\langle V_{\varepsilon,n},\psi\rangle_\varepsilon=\Lambda_{\varepsilon,n}(V_{\varepsilon,n},\psi)_\varepsilon+\alpha_\varepsilon(\psi),$$

where $|\alpha_\varepsilon(\psi)|\leq c\varepsilon^{n+1/2}\|\psi\|_{\mathcal{H}_\varepsilon}$. The latter gives

$$\begin{aligned}&\|u_\varepsilon-\vartheta_\varepsilon V_{\varepsilon,n}\|_{\mathcal{H}_\varepsilon}\leq\\&\leq\Lambda_{\varepsilon,n}\|u_\varepsilon-\vartheta_\varepsilon V_{\varepsilon,n}\|_\varepsilon+|\lambda^\varepsilon-\Lambda_{\varepsilon,n}|\,\|u_\varepsilon\|_\varepsilon+|\alpha_\varepsilon(u_\varepsilon-\vartheta_\varepsilon V_{\varepsilon,n})|\leq\\&\leq 2\mu_j\,C_n\varepsilon^{n+3/2}+c_n\|u_\varepsilon\|\,\varepsilon^{n+3/2}+c\varepsilon^{n+1/2}\|u_\varepsilon-\vartheta_\varepsilon V_{\varepsilon,n}\|_{\mathcal{H}_\varepsilon}\end{aligned}$$

and consequently $\|u_\varepsilon - \vartheta_\varepsilon V_{\varepsilon,n}\|_{\mathcal{H}_\varepsilon} \le C_n \varepsilon^{n+3/2}$. From this and (50) we thus get estimate (52). □

The same proof works for the rest part of the simple spectrum of $\mathcal{A}$.

Theorem 4. *If $\mu_j \in \sigma(\hat{A}_2)\backslash\sigma(A_1)$, then eigenfunction $u_{\varepsilon,j}$ of (2)-(5) with eigenvalue λ_j^ε converges towards function*

$$u(x) = \begin{cases} 0 & \text{for } x \in I_a, \\ z(x) & \text{for } x \in I_b \end{cases}$$

in $H^1(a,b)$, where z is an eigenfunction of the problem $(\varkappa z')' + \mu\,\rho z = 0$ in I_b, $z(0) = 0$, $z(a) = 0$ with eigenvalue μ_j. Moreover λ_j^ε and $u_{\varepsilon,j}$ admit asymptotic expansions (25), (26) obtained in 3.1.2 for $\mu = \mu_j$ with the estimates of remainder terms

$$\left|\varepsilon^{-1}\lambda_j^\varepsilon - (\mu_j + \varepsilon\nu_1 + \cdots + \varepsilon^n\nu_n)\right| \le c_n\varepsilon^{n+1},$$

$$\|u_{\varepsilon,j} - \vartheta_\varepsilon U_{\varepsilon,n}\|_{H^1(a,b)} \le C_n\varepsilon^{n+1}.$$

Here ϑ_ε is a normalizing multiplier that converges to a positive constant as $\varepsilon \to 0$.

4.2. Double Spectrum. We introduce the partial sums of (40), (41)

$$\Lambda^\pm_{\varepsilon,n} = \varepsilon(\mu_j \pm \varepsilon^{1/2}\omega + \varepsilon\nu_2^\pm + \cdots + \varepsilon^{n/2}\nu_n^\pm), \tag{55}$$

$$U^\pm_{\varepsilon,n} = \begin{cases} \mp\varepsilon^{1/2}w + \varepsilon w_2^\pm + \cdots + \varepsilon^{n/2}w_n^\pm & \text{for } x \in I_a \\ v + \varepsilon^{1/2}v_1^\pm + \cdots + \varepsilon^{n/2}v_n^\pm & \text{for } x \in I_b \end{cases} \tag{56}$$

with all coefficients constructed in Section for certain double eigenvalue $\mu = \mu_j = \mu_{j+1}$. Set $V^\pm_{\varepsilon,n} = U^\pm_{\varepsilon,n} + \beta^\pm_{\varepsilon,n}\phi$, where $\beta^-_{\varepsilon,n}$ and $\beta^+_{\varepsilon,n}$ are residuals in condition (4) for $U^-_{\varepsilon,n}$ and $U^+_{\varepsilon,n}$ respectively defined similarly as in (54). Moreover, $\beta^\pm_{\varepsilon,n} = O(\varepsilon^{(n+1)/2})$ as $\varepsilon \to 0$.

Analysis similar to that in the proof of Theorem 3 leads to the following result.

Proposition 3. *The pairs $(\Lambda^-_{\varepsilon,n}, V^-_{\varepsilon,n}/\|V^-_{\varepsilon,n}\|_\varepsilon)$ and $(\Lambda^+_{\varepsilon,n}, V^+_{\varepsilon,n}/\|V^+_{\varepsilon,n}\|_\varepsilon)$ are quasimodes of operator A_ε with accuracy to $c_n\varepsilon^{n/2}$.*

Proposition 4. *There exist two closely adjacent eigenvalues λ^-_ε and λ^+_ε of (2)-(5) with the asymptotics*

$$\frac{\lambda^\pm_\varepsilon}{\varepsilon} = \mu_j \pm \sqrt{\varepsilon}\omega + \varepsilon\nu_2^\pm + \cdots + \varepsilon^{n/2}\nu_n^\pm + O(\varepsilon^{(n+1)/2}), \tag{57}$$

where μ_j is a double eigenvalue of operator $\mathcal{A}$ and ω, $\nu_k^\pm$ were defined in Sec. .

Proof. From Proposition 3 and the Vishik-Lyusternik Lemma it follows that there exists at least one eigenvalue of A_ε in each $\varepsilon^{n/2}$-vicinity of $\Lambda_{\varepsilon,n}^-$ and $\Lambda_{\varepsilon,n}^+$. Moreover, $|\lambda_\varepsilon^\pm - \Lambda_{\varepsilon,n}^\pm| \le c_n\varepsilon^{n/2}$. Evidently, eigenvalues λ_ε^-, λ_ε^+ are different, because $\Lambda_{\varepsilon,n}^+ - \Lambda_{\varepsilon,n}^- \ge \omega\varepsilon^{3/2}$ and $\varepsilon^{n/2}$-vicinities of $\Lambda_{\varepsilon,n}^-$ and $\Lambda_{\varepsilon,n}^+$ don't intersect for $n > 3$ and sufficient small ε. In fact, $|\lambda_\varepsilon^+ - \lambda_\varepsilon^-| \ge c\varepsilon^{3/2}$ for certain positive c. We conclude from $|\lambda_\varepsilon^\pm - \Lambda_{\varepsilon,n+3}^\pm| \le c_{n+3}\varepsilon^{(n+3)/2}$ that

$$\left|\tfrac{\lambda_\varepsilon^\pm}{\varepsilon} - (\mu_j \pm \sqrt{\varepsilon}\omega + \dots + \varepsilon^{\frac{n}{2}}\nu_n^\pm)\right| \le$$

$$\le c_{n+3}\varepsilon^{\frac{n+1}{2}} + \textstyle\sum_{k=1}^3 \varepsilon^{\frac{n+k}{2}}|\nu_{n+k}^\pm| \le C_n\varepsilon^{\frac{n+1}{2}},$$

which establishes (57). □

We consider two planes in $L_2(a,b)$. Let π be the root subspace that corresponds to double eigenvalue μ_i and $\pi(\varepsilon)$ be the linear span of two eigenfunctions u_ε^- and u_ε^+ that correspond to eigenvalues λ_ε^- and λ_ε^+. These eigenfunctions as above are normalized by (21).

Theorem 5. *The root subspace π is the limit position of plane $\pi(\varepsilon)$ as $\varepsilon \to 0$ that is to say $\|P_{\pi(\varepsilon)} - P_\pi\| \to 0$, where $P_{\pi(\varepsilon)}$ and P_π are the orthogonal projectors onto planes $\pi(\varepsilon)$ and π.*

Proof. Nevertheless both eigenfunction u_ε^- and u_ε^+ converge to the same limit being the eigenfunction of $\mathcal{A}$ with eigenvalue μ_j, the π_ε has regular asymptotic behaviour as $\varepsilon \to 0$. We choose new $L_2(R,(a,b))$-orthogonal basis in $\pi(\varepsilon)$: $f_\varepsilon = \frac{1}{2}(u_\varepsilon^+ + u_\varepsilon^-)$, $g_\varepsilon = \frac{1}{2\omega\sqrt{\varepsilon}}(u_\varepsilon^+ - u_\varepsilon^-)$.

By Theorem 2 the first vector f_ε converges in L_2 towards eigenfunction $U \in \pi$ given by (36). Next, function g_ε solves the problem

$$\begin{cases} (kg_\varepsilon')' + \dfrac{\lambda_\varepsilon^+}{\varepsilon} rg_\varepsilon = \dfrac{\lambda_\varepsilon^- - \lambda_\varepsilon^+}{2\omega\varepsilon\sqrt{\varepsilon}} ru_\varepsilon^- \quad \text{in } I_a, \\ (\varkappa g_\varepsilon')' + \dfrac{\lambda_\varepsilon^+}{\varepsilon}\rho g_\varepsilon = \dfrac{\lambda_\varepsilon^- - \lambda_\varepsilon^+}{2\omega\varepsilon\sqrt{\varepsilon}}\rho u_\varepsilon^- \quad \text{in } I_b, \\ g_\varepsilon(a) = 0, \quad g_\varepsilon(b) = 0, \\ g_\varepsilon(-0) = g_\varepsilon(+0), \quad (kg_\varepsilon')(-0) = \varepsilon(\varkappa g_\varepsilon')(+0). \end{cases}$$

Since $\varepsilon^{-1}\lambda_\varepsilon^+ \to \mu_j$, $\varepsilon^{-3/2}(\lambda_\varepsilon^+ - \lambda_\varepsilon^-) \to 2\omega$ by (57) and the right-hand side is orthogonal to the eigenfunction u_ε^+ in $\mathcal{L}_\varepsilon$, one obtains that norms $\|g_\varepsilon\|_{H^2(a,0)}$ and $\|g_\varepsilon\|_{H^2(0,b)}$ are bounded as $\varepsilon \to 0$. Taking into account Corollary 1 we can assert that each converging subsequence $g_{\varepsilon'}$ converges as $\varepsilon \to 0$ towards a solution of the problem

$$\begin{cases} (kg')' + \mu_j rg = 0 \quad \text{in } I_a, & (\varkappa g')' + \mu_j \rho g = -\rho v \quad \text{in } I_b, \\ g(a) = 0, \quad g(b) = 0, \quad g(-0) = g(+0), \quad g'(-0) = 0, \end{cases}$$

because u_ε^- converges to eigenfunction U, which equals v in I_b and vanishes in I_a. Hence, all partial limits of the second basis vector g_ε have to be the adjoined vectors corresponding to the eigenvalue μ_j. In fact, by orthogonality of f_ε and g_ε these limits belong to the line $\{\alpha U_* \mid \alpha \in \mathbb{R}\} \subset \pi$, which is orthogonal to U (see (36) for definition of U_*). $\square$

Indeed, in previous statements $\lambda_\varepsilon^- = \lambda_j^\varepsilon$, $\lambda_\varepsilon^+ = \lambda_{j+1}^\varepsilon$ and $u_\varepsilon^- = u_{\varepsilon,j}$, $u_\varepsilon^+ = u_{\varepsilon,j+1}$, by Theorem 2. Next theorem summarizes all information on bifurcation of the double spectrum.

Theorem 6. *Let $\mu_j \in \sigma(A_1) \cap \sigma(\hat{A}_2)$ be a double eigenvalue with eigenfunction U and adjoined function U_* given by (36), $\mu_j = \mu_{j+1}$. Then both eigenfunction $u_{\varepsilon,j}$ and $u_{\varepsilon,j+1}$ converge to the same eigenfunction U and the difference $\frac{1}{\sqrt{\varepsilon}}(u_{\varepsilon,j+1} - u_{\varepsilon,j})$ converges to adjoined function γU_* for certain $\gamma \neq 0$. Besides, $\lambda_\varepsilon^- = \lambda_j^\varepsilon$, $\lambda_\varepsilon^+ = \lambda_{j+1}^\varepsilon$ and $u_{\varepsilon,j}$, $u_{\varepsilon,j+1}$ admit asymptotic expansions (40), (41) derived in Section 3.2 for $\mu = \mu_j$. The estimates of remainder terms hold*

$$\left|\varepsilon^{-1}\lambda_\varepsilon^\pm - \left(\mu_j \pm \sqrt{\varepsilon}\omega + \varepsilon\nu_2^\pm + \cdots + \varepsilon^{n/2}\nu_n^\pm\right)\right| \le c_n^\pm \varepsilon^{(n+1)/2}, \qquad (58)$$

$$\|u_{\varepsilon,j} - \vartheta_\varepsilon^- U_{\varepsilon,n}^-\|_{H^1(a,b)} \le C_n^- \varepsilon^{\frac{n+1}{2}}, \ \|u_{\varepsilon,j+1} - \vartheta_\varepsilon^+ U_{\varepsilon,n}^+\|_{H^1(a,b)} \le C_n^+ \varepsilon^{\frac{n+1}{2}}, \qquad (59)$$

where $\vartheta_\varepsilon^\pm$ are normalizing multipliers with strictly positive limit as $\varepsilon \to 0$.

Proof. It remains to prove estimates (59). From (58) and Theorem 2 it may be concluded that for certain $d > 0$ and $n \ge 2$ interval $[\Lambda_{\varepsilon,n}^- - d\varepsilon^2, \Lambda_{\varepsilon,n}^- + d\varepsilon^2]$ contains eigenvalue λ_j^ε only. In view of Prop. 3 and the Vishik-Lyusternik Lemma, we have

$$\left\| u_{\varepsilon,j} - \frac{\|u_{\varepsilon,j}\|_\varepsilon}{\|V_{\varepsilon,n}^-\|_\varepsilon} V_{\varepsilon,n}^- \right\|_\varepsilon \le \frac{2c_n}{d\varepsilon^2}\|u_\varepsilon\|_\varepsilon \varepsilon^{n/2} \le C_n \varepsilon^{\frac{n-5}{2}}.$$

As in the proof of Theorem 3 we can obtain $\|u_{\varepsilon,j} - \vartheta_\varepsilon^- U_{\varepsilon,n}^-\|_{H^1(a,b)} \leq C_n \varepsilon^{\frac{n-4}{2}}$. Since all the coefficients of sum $U_{\varepsilon,n}^-$ are bounded in $H^1(a,b)$, the first estimate (59) follows from the last inequality with n replaced by $n+5$. The same proof works for $u_{\varepsilon,j+1}$. $\Box$

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