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ON THE NORMS OF THE MEANS OF SPHERICAL fOURIER SUMS[1]

The spherical Fourier sums $S_R(f,x) = \sum_{\|k\|\leqslant R} \widehat{f}(k)\, e^{ik\cdot x}$ of a periodic functions in m variables, the strong means $\left(\frac{1}{n}\sum_{j=0}^{n-1} |S_j(f,x)|^p\right)^{1/p}$ and the strong integral means $\left(\left(\int_0^R |s_r(f,x)|^p dx\right)/R\right)^{1/p}$ of these sums for $p \geq 1$ are considered. In contrast to the one-dimensional case treated by Hardy and Littlewood, for $m \geq 2$ the norms of the corresponding operators in the space L_∞ are not bounded. The sharp order of growth of these norms is found. The upper and lower bounds differ by a factor depending only on the dimension m.

A sufficient condition on the function ensuring the uniform strong p-summability of its Fourier series is given.

Keywords: *Multiple Fourier series, spherical sums, strong means.*

1. Introduction. Given a function f integrable on the cube $\mathbb{T}^m = [-\pi,\pi]^m$, the spherical partial sum $S_R(f,x)$ of the Fourier series of f, has the form (hereafter $k \in \mathbb{Z}^m$)

$$S_R(f,x) = \sum_{\|k\|\leqslant R} \widehat{f}(k)\, e^{ik\cdot x}, \quad \text{where } \widehat{f}(k) = \frac{1}{(2\pi)^m}\int_{\mathbb{T}^m} f(u)\, e^{-ik\cdot u} du.$$

This is the convolution f with the spherical Dirichlet kernal

$$D_R(x) = \frac{1}{(2\pi)^m}\sum_{\|k\|\leqslant R} e^{-ik\cdot x}.$$

The L_1-norm

$$\mathcal{L}_R = \int_{\mathbb{T}^m} |D_R(x)|\, dx = \sup_{|f|\leqslant 1} |S_R(f,0)|$$

of this kernel is called the Lebesgue constant. In the multidimensional case ($m > 1$) the two-sided bound $A_m R^{\frac{m-1}{2}} \leqslant \mathcal{L}_R \leqslant B_m R^{\frac{m-1}{2}}$ is valid for $R \geqslant 1$ (see [1], [2]).

The strong spherical means of the Fourier series are defined as follows (hereafter $p \geqslant 1$ and $n \in \mathbb{N}$)

$$H_{n,p}(f,x) = \Big(\frac{1}{n}\sum_{j=0}^{n-1} |S_j(f,x)|^p\Big)^{\frac{1}{p}} = \frac{1}{n^{\frac{1}{p}}} \sup_{|\varepsilon|_q\leqslant 1} \Big|\sum_{j=0}^{n-1} \varepsilon_j S_j(f,x)\Big|.$$

Here, supremum is taken over all collections $\varepsilon = \{\varepsilon_0, \ldots, \varepsilon_{n-1}\}$ of real numbers satisfying the condition

$$|\varepsilon|_q = \big(|\varepsilon_0|^q + \ldots + |\varepsilon_{n-1}|^q\big)^{\frac{1}{q}} \leqslant 1$$

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(q is the conjugate exponent: $\frac{1}{p}+\frac{1}{q}=1$).

Our purpose is to estimate the norms of the corresponding operators, i.e., the quantities

$$H_{n,p}=\sup_{|f|\leqslant 1} H_{n,p}(f,0)=\frac{1}{n^{\frac{1}{p}}}\sup_{|\varepsilon|_q\leqslant 1}\int_{\mathbb{T}^m}\Big|\sum_{j=0}^{n-1}\varepsilon_j D_j(x)\Big|dx.$$

Note that, by Hölder's inequality, the means $H_{n,p}(f)$, and hence the norms $H_{n,p}$, increase with p.

The notion of strong summability of Fourier series was introduced by Hardy and Littlewood [3] in the one-dimensional case more one hundred years ago. They proved that for $m=1$ and a fixed p, the norms $H_{n,p}$ are bounded.

In the multidimensional case, the situation is different. For $m\geqslant 3$ the norms $H_{n,p}$ not bounded, being of order $n^{\frac{m-1}{2}-\min\{\frac{1}{2},\frac{1}{p}\}}$ (см. [4], [5]). In the two-dimensional case, the results were not complete: this two-sided bound still holds true for any fixed $p>2$, while for $p\in[1,2]$ only the upper bound $H_{n,p}\leqslant c\sqrt{\ln(n+1)}$ was known [see 6,7].

2. The main result for $H_{n,p}$. We prove [8] that [2]

$$H_{n,p}\asymp\begin{cases} n^{\frac{m-1}{2}-\min\{\frac{1}{2},\frac{1}{p}\}} & \text{if } m\geqslant 3,\ p\geqslant 1;\\ n^{\frac{1}{2}-\frac{1}{p}}\min^{\frac{1}{p}}\left\{\ln(n+1),\frac{1}{p-2}\right\} & \text{if } m=2,\ p\geqslant 2;\\ \sqrt{\ln(n+1)} & \text{if } m=2,\ p\in[1,2].\end{cases}$$

Note that in the two-dimensional case, for "large" p (with $p-2$ greater that a fixed positive number) the factor $\min^{\frac{1}{p}}\left\{\ln(n+1),\frac{1}{p-2}\right\}$ can be replaced by 1.

3. The main lemma.

Lemma 1. *Let $a_1,\dots,a_n$ be nonnegative numbers. Then*

$$\sup_{\varepsilon_j=\pm 1}\int_0^{\pi}\Big|\sum_{j=1}^{n}\varepsilon_j a_j\cos\Big(jt+\frac{\pi}{4}\Big)\Big|\frac{dt}{\sqrt{t}}\gg\sqrt{\frac{\ln n}{n}}\sum_{j=1}^{n}a_j.$$

With $a_j=\sqrt{j}$ this immediately implies the desired relation $H_{n,1}\gg\sqrt{\ln n}$.

4. The norms of the integral means of spherical Fourier sums. In addition to the means $H_{n,p}(f)$, it is also natural to consider their integral analogs defined by the equality

$$\mathscr{H}_{R,p}(f,x)=\Big(\frac{1}{R}\int_0^R|S_r(f,x)|^p dr\Big)^{\frac{1}{p}}.$$

Note that averaging over the radius was used systematically in many papers, for example, in the study of the Riesz sums of multiple Fourier series.

Our goal is to estimate of the norms the corresponding operators, i.s., the quantities

[2]The notion $\alpha_n\asymp\beta_n$ means that $\alpha_n=O(\beta_n)$ and $\beta_n=O(\alpha_n)$ simultaneously. Instead of $\alpha_n=O(\beta_n)$ we also write $\alpha_n\ll\beta_n$. Hereafter, the constants in the corresponding inequalities valid for all n may depend only on the dimension m.

$$\mathscr{H}_{R,p} = \sup_{|f|\leqslant 1} \mathscr{H}_{R,p}(f,0).$$

In view of Hölder's inequality, the means $\mathscr{H}_{R,p}(f)p$ increase with p and, therefore, so do the norms $\mathscr{H}_{R,p}$. In [7] they were estimated from above. It is easy to see that, to do this, it suffices to consider $\mathscr{H}_{R,p}$ only for integer values R.

We prove [9] that the norms $\mathscr{H}_{n,p}$ и $H_{n,p}$ have the same order. More precisely,

$$\mathscr{H}_{n,p} \asymp H_{n,p} \asymp \begin{cases} n^{\frac{m-1}{2}-\min\{\frac{1}{2},\frac{1}{p}\}} & \text{if } m \geqslant 3,\ p \geqslant 1; \\ n^{\frac{1}{2}-\frac{1}{p}} \min^{\frac{1}{p}}\{\ln(n+1), \frac{1}{p-2}\} & \text{if } m=2,\ p>2; \\ \sqrt{\ln(n+1)} & \text{if } m=2,\ p\in[1,2]. \end{cases}$$

5. Estimating $\mathscr{H}_{n,p}$ from below. We begin with two-dimensional case. For $1 \leqslant p \leqslant 2$

$$\mathscr{H}_{n,p} = H_{n,p} + O(1) \geqslant \alpha_m \sqrt{\ln(n+1)}.$$

In addition, for $p > 2$ the relations
$H_{n,p} \asymp n^{\frac{1}{2}-\frac{1}{p}} \min^{\frac{1}{p}}\{\ln(n+1), \frac{1}{p-2}\}$ и $\mathscr{H}_{n,p} = H_{n,p} + O\big(n^{\frac{1}{2}-\frac{1}{p}}\big)$
imply
$\mathscr{H}_{n,p} \geqslant \beta_m n^{\frac{1}{2}-\frac{1}{p}} \min^{\frac{1}{p}}\{\ln(n+1), \frac{1}{p-2}\}$
at least for p, sufficiently close to 2.

We need following result for $p \geqslant 2$.

Further, $R > 1$, $0 < \delta \leqslant 1$; $f_{R,\delta}$ is a function, equal to $\cos\big(R\|x\| - \frac{\pi}{4}(m+1)\big)$ for $\|x\| \leqslant \delta$ and zero for other $x \in \mathbb{T}^m$; $B(r)$ is the ball of radius r centered at zero.

Lemma 2. *If δ is sufficiently small and the product $R\delta$ is sufficiently large (the boundaries depend only the dimension m), then the function $f_{R,\delta}$ satisfies the inequality*

$$S_r(f_{R,\delta},0) = \int_{B(\delta)} f_{R,\delta}(x) D_r(x)\, dx \geqslant c_m\, (r\delta)^{\frac{m-1}{2}} \quad \textit{for all } r,\ |r-R| \leqslant 1/\delta.$$

For the coefficient c_m we can take the fraction $\pi^{\frac{m-1}{2}}/2\Gamma(1+\frac{m}{2})$.

In view [1] $|S_r(f,0)| \leqslant \int_{\mathbb{T}^m} |D_r(x)|\, dx = O\big(r^{\frac{m-1}{2}}\big)$, if $|f| \leqslant 1$, then this implies that, for $f_{R,\delta}$,not only the spherical sum $S_R(f_{R,\delta},0)$, but also all the sums $S_r(f_{R,\delta},0)$ for $R - 1/\delta \leqslant r \leqslant R + 1/\delta$ are the largest possible (among bounded functions).

For $p \geqslant 2$ the lemma 2 (for $R = n$ and small δ) immediately yields the lower bound:

$$\mathscr{H}_{n,p} = \sup_{|f|\leqslant 1} \left(\frac{1}{n} \int_0^n \big|S_r(f,0)\big|^p dr \right)^{\frac{1}{p}} \geqslant \frac{1}{n^{\frac{1}{p}}} \int_{n-1}^n S_r(f_{n,\delta},0)\, dr \geqslant c_m \delta^{\frac{m-1}{2}} n^{\frac{m-1}{2}-\frac{1}{p}}.$$

In the two-dimensional case, for "large" $p > 2$ (the difference $p - 2$ is bounded away from zero), the above relations are equivalent to the inequality

$$\mathscr{H}_{n,p} \geqslant \beta_m n^{\frac{1}{2}-\frac{1}{p}} \min^{\frac{1}{p}}\Big\{\ln(n+1), \frac{1}{p-2}\Big\},$$

which, for "small" $p > 2$, was established at the beginning. Thus, for $p > 2$ the required lower bound for the norms $\mathscr{H}_{n,p}$ is obtained for all $m = 2, 3, \ldots$. Recall that, at the beginning of this section, the lower bound was also obtained for $m = 2$ and $1 \leqslant p \leqslant 2$.

6. On p-summation of the Fourier series in the strong sense. The estimates of the norms $\mathscr{H}_{R,p}$, established above, by standard arguments lead to condition on a continuous periodic function f that ensures the uniform strong p-summability of the Fourier series, i.e., the uniform (with respect to $x \in \mathbb{R}^m$) convergence to zero as $R \to +\infty$ of the quantities

$$\frac{1}{R}\int_0^R |S_r(f,x) - f(x)|^p dr.$$

To do this, we shall need the notion of best uniform approximation of a function f, which is defined by the equality

$$E_R(f) = \min_M \|f - M\|_C, \qquad \text{where } M(x) = \sum_{\|k\|\leqslant R} c_k\, e^{ik\cdot x}.$$

Corollary. *Let f be a continuous and 2π-periodic (in each variable) function in $\mathbb{R}^m$ such thah $\mathscr{H}_{R,p}\, E_R(f) \underset{R\to+\infty}{\longrightarrow} 0$. Then*

$$\max_x \frac{1}{R}\int_0^R |S_r(f,x) - f(x)|^p dr \underset{R\to+\infty}{\longrightarrow} 0.$$

By Jackson's theorem, the best approximation $E_R(f)$ is majorized by the modulus of smoothness of the functions f. Therefore, in the two-dimensional, for $1 \leqslant p \leqslant 2$, the condition

$$\omega_f(t) = o(1/\sqrt{|\ln t|})$$

on the modulus of continuity of the function f is sufficient for the uniform strong p-summability of the spherical sums and, for a fixed $p > 2$, it ensures the more restrictive Lipschitz condition

$$\omega_f(t) = o(|t|^{\frac{1}{2}-\frac{1}{p}}).$$

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О. И. Кузнецова

О нормах средних сферических сумм Фурье.

Сферическая сумма Фурье $S_R(f,x) = \sum_{\|k\|\leqslant R} \widehat{f}(k)\, e^{ik\cdot x}$ периодической функции m переменных, ее сильные средние и сильные интегральные средние рассмотрены при $p \geq 1$. В отличие от одномерного случая, рассмотренного Харди и Литвудом, при $m \geq 2$ нормы соответствующих операторов в пространстве L_∞ не ограничены. Найден точный порядок роста этих норм. Оценки сверху и снизу различаются на коэффициенты, зависящие лишь от размерности . Получено достаточное условие на функцию, обеспечивающее равномерную сильную суммируемость ее ряда Фурье.

Ключевые слова: *Кратные ряды Фурье, сферические суммы, сильные средние*

Ин-т прикл. математики и механики, Донецк *Received 02.10.15*
kuznets@iamm.su