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ABOUT SOME MAPPINGS IN $\lambda(r)$ -REGULAR METRIC SPACES

It is formulated conditions on functions $Q(x)$ and boundaries of domains under which every Q -homeomorphism admits continuous or homeomorphic extension to the boundary in metric spaces with measures.

Keywords: *modulus, Q -homeomorphisms.*

1. Introduction. Mapping theory started in the 18th century. Beltrami, Caratheodory, Christoffel, Gauss, Hilbert, Liouville, Poincaré, Riemann, Schwarz, and so on all left their marks in this theory. Conformal mappings and their applications to potential theory, mathematical physics, Riemann surfaces, and technology played a key role in this development.

During the late 1920s and early 1930s, Grötzsch, Lavrentiev, and Morrey introduced a more general and less rigid class of mappings that were later named quasiconformal. The concept of Q -homeomorphism is a natural extension of the geometric definition of quasiconformality; see, e.g., [2]. The subject of Q -homeomorphisms is interesting on its own right and has applications to many classes of mappings. In particular, the theory of Q -homeomorphisms can be applied to mappings in local Sobolev classes (see, e.g., Sections 6.3 and 6.10 in [2]) to the mappings with finite length distortion (see Sections 8.6 and 8.7 in [2]) and to the finitely bi-Lipschitz mappings; see Section 10.6 in [2]. The main goal of the theory of Q -homeomorphisms is to clear up various interconnections between properties of the majorant $Q(x)$ and the corresponding properties of the mappings themselves.

2. Preliminaries. Given a set S in (X, d) and $\alpha \in [0, \infty)$, H^α denotes the α -dimensional Hausdorff measure of S in (X, d) , i.e.,

$$H^\alpha(S) = \sup_{\varepsilon > 0} H_\varepsilon^\alpha(S), \quad (1)$$

$$H_\varepsilon^\alpha(S) = \inf \sum_{i=1}^{\infty} \delta_i^\alpha, \quad (2)$$

where the infimum is taken over all countable collections of numbers $\delta_i \in (0, \varepsilon)$ such that some sets S_i in (X, d) with diameters δ_i cover S . Note that H^α is nonincreasing in the parameter α . The Hausdorff dimension of S is the only number $\alpha \in [0, \infty]$ such that $H^{\alpha'}(S) = 0$ for all $\alpha' > \alpha$ and $H^{\alpha''}(S) = \infty$ for all $\alpha'' < \alpha$.

Recall, for a given continuous path $\gamma : [a, b] \rightarrow X$ in a metric space (X, d) , that its length is the supremum of the sums

$$\sum_{i=1}^k d(\gamma(t_i), \gamma(t_{i-1}))$$

over all partitions $a = t_0 \leq t_1 \leq \dots \leq t_k = b$ of the interval $[a, b]$. The path γ is called *rectifiable* if its length is finite.

In what follows, (X, d, μ) denotes a space X with a metric d and a locally finite Borel measure μ . Given a family of paths Γ in X , a Borel function $\varrho : X \rightarrow [0, \infty]$ is called *admissible* for Γ , abbr. $\varrho \in \text{adm } \Gamma$, if

$$\int_{\gamma} \varrho ds \geq 1 \tag{3}$$

for all $\gamma \in \Gamma$.

An open set in X whose points can all be connected pairwise by continuous paths is called a *domain* in X .

The *modulus* of the path family Γ in D is given by the equality

$$M(\Gamma) = \inf_{\varrho \in \text{adm } \Gamma} \int_D \varrho^\alpha(x) d\mu(x). \tag{4}$$

In the case of the path family $\Gamma' = f\Gamma$, we take the Hausdorff dimension α' of the domain D' .

Let D and D' be domains with finite Hausdorff dimensions α and $\alpha' \geq 1$ in spaces (X, d, μ) and (X', d', μ') , and let $Q : D \rightarrow [0, \infty]$ be a measurable function. We say that a homeomorphism $f : D \rightarrow D'$ is a *Q-homeomorphism* if

$$M(f\Gamma) \leq \int_D Q(x) \cdot \varrho^\alpha(x) d\mu(x) \tag{5}$$

for every family Γ of paths in D and every admissible function ϱ for Γ .

A space (X, d, μ) is called (*Ahlfors*) α -*regular* if there is a constant $C \geq 1$ such that

$$C^{-1}r^\alpha \leq \mu(B_r) \leq Cr^\alpha \tag{6}$$

for all balls B_r in X with the radius $r < \text{diam } X$. A space (X, d, μ) is (*Ahlfors*) *regular* if it is (*Ahlfors*) α -regular for some $\alpha \in (1, \infty)$.

A space (X, d, μ) is *upper α -regular at a point* $x_0 \in X$ if there is a constant $C > 0$ such that

$$\mu(B(x_0, r)) \leq Cr^\alpha \tag{7}$$

for the balls $B(x_0, r)$ centered at $x_0 \in X$ with all radii $r < r_0$ for some $r_0 > 0$. A space (X, d, μ) is *upper α -regular* if condition (7) holds at every point $x_0 \in X$, see [2], p. 258.

Recall that a topological space is *connected space* if it is impossible to split it into two non-empty open sets. Compact connected spaces are called *continua*. A topological space T is said to be *path connected* if any two points x_1 and x_2 in T can be joined by a path $\gamma : [0, 1] \rightarrow T$, $\gamma(0) = x_1$ and $\gamma(1) = x_2$. A *domain* in T is an open path connected set in T . A domain D in a topological space T is called *locally connected at a point* $x_0 \in \partial D$ if, for every neighborhood U of the point x_0 , there is its neighborhood $V \subseteq U$ such that

$V \cap D$ is connected, [1], c. 232. Similarly, we say that a domain D is *locally path connected at a point* $x_0 \in \partial D$ if, for every neighborhood U of the point x_0 , there is its neighborhood $V \subseteq U$ such that $V \cap D$ is path connected.

The boundary of D is *weakly flat at a point* $x_0 \in \partial D$ if, for every number $P > 0$ and every neighborhood U of the point x_0 , there is its neighborhood $V \subset U$ such that

$$M(\Delta(E, F; D)) \geq P \tag{8}$$

for all continua E and F in D intersecting ∂U and ∂V .

We also say that the boundary of the domain G is *strongly accessible at a point* $x_0 \in \partial G$, if, for every neighborhood U of the point x_0 , there are a compact set $E \subset G$, a neighborhood $V \subset U$ of the point x_0 , and a number $\delta > 0$ such that

$$M(\Delta(E, F; G)) \geq \delta$$

for every continuum F in G intersecting ∂U and ∂V .

Finally, we say that the boundary ∂D is *weakly flat and strongly accessible* if the corresponding properties hold at every point of the boundary, see [2].

We start first from the following general statement, see Lemma 13.3 and Theorem 13.3 in [2], (cf. also Lemma 3 in [3]).

Lemma 1. *Let a domain D be locally path connected at a point $x_0 \in \partial D$, let $\overline{D'}$ be compact, and let $f : D \rightarrow D'$ be a Q -homeomorphism such that $\partial D'$ is strongly accessible at least at one point of the cluster set*

$$C(x_0, f) = \{y \in X' : y = \lim_{k \rightarrow \infty} f(x_k), x_k \rightarrow x_0, x_k \in D\}, \tag{9}$$

$Q : X \rightarrow (0, \infty)$ is a measurable function satisfying the condition

$$\int_{D(x_0, \varepsilon, \varepsilon_0)} Q(x) \cdot \psi_{x_0, \varepsilon}^\alpha(d(x, x_0)) d\mu(x) = o(I_{x_0}^\alpha(\varepsilon)) \tag{10}$$

as $\varepsilon \rightarrow 0$, where

$$D(x_0, \varepsilon, \varepsilon_0) = \{x \in D : \varepsilon < d(x, x_0) < \varepsilon_0\},$$

$\varepsilon(x_0) \in (0, d(x_0))$, $d(x_0) = \sup_{x \in D} d(x, x_0)$, and $\psi_{x_0, \varepsilon}(t)$ is a family of nonnegative measurable (by Lebesgue) functions on $(0, \infty)$ such that

$$I_{x_0, \varepsilon_0}(\varepsilon) := \int_\varepsilon^{\varepsilon_0} \psi_{x_0, \varepsilon}(t) dt < \infty \quad \forall \varepsilon \in (0, \varepsilon_0). \tag{11}$$

Then f can be extended to the point x_0 by continuity in (X', d') .

Theorem 1. *Let D be locally path connected at all its boundary points and \overline{D} compact, D' with a weakly flat boundary, and let $f : D \rightarrow D'$ be a Q -homeomorphism with $Q \in L_\mu^1(D)$. Then the inverse homeomorphism $g = f^{-1} : D' \rightarrow D$ admits a continuous extension $\bar{g} : \overline{D'} \rightarrow \overline{D}$.*

3. Basic results. Let $\lambda : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be an increasing function, $\mathbb{R}^+ = [0, \infty)$. We will say that a space (X, d, μ) is *upper $\lambda(r)$ -regular at a point $x_0 \in X$* if there is a constant $C > 0$ such that

$$\mu(B(x_0, r)) \leq C\lambda(r) \quad (12)$$

for the balls $B(x_0, r)$ centered at $x_0 \in X$ with all radii $r < r_0$ for some $r_0 > 0$. We will also say that a space (X, d, μ) is *upper $\lambda(r)$ -regular* if condition (12) holds at every point $x_0 \in X$.

Lemma 2. *Let D be a domain in a space (X, d, μ) that is upper $\lambda(r)$ -regular at the point $x_0 \in \overline{D}$ and λ is increasing function. If, for every nonnegative function $Q : D \rightarrow \mathbb{R}^+$, condition*

$$\limsup_{\varepsilon \rightarrow 0} \frac{1}{\mu(B(x_0, \varepsilon) \cap D)} \int_{B(x_0, \varepsilon) \cap D} Q(x) d\mu(x) < \infty \quad (13)$$

holds, then

$$\int_{D \cap A(x_0, \varepsilon, \varepsilon_0)} \frac{Q(x) d\mu(x)}{\lambda(d(x, x_0))} = O\left(\log \frac{1}{\varepsilon}\right) \quad (14)$$

as $\varepsilon \rightarrow 0$ and some $\varepsilon_0 \in (0, \delta_0)$, where $\delta_0 = \min(e^{-1}, d_0)$, $d_0 = \sup_{x \in D} d(x, x_0)$, $A(x_0, \varepsilon, \varepsilon_0) = \{x \in X : \varepsilon < d(x, x_0) < \varepsilon_0\}$, $B_r = B(x_0, r) = \{x \in X : d(x, x_0) < r\}$.

Proof. Choose $\varepsilon_0 \in (0, \delta_0)$ such that the function Q is integrable in $D_0 = D \cap B_0$ with respect to the measure μ , where $B_0 = B(x_0, \varepsilon_0)$, and

$$s_0 = \sup_{r \in (0, \varepsilon_0)} \frac{1}{\mu(D_r)} \int_{D_r} Q(x) d\mu(x) < \infty,$$

$D_r = D \cap B(x_0, r)$. Further, let $\varepsilon < 2^{-1}\varepsilon_0$, $\varepsilon_k = 2^{-k}\varepsilon_0$, $A_k = \{x \in X : \varepsilon_{k+1} \leq d(x, x_0) < \varepsilon_k\}$, $B_k = B(x_0, \varepsilon_k)$. Choose a natural number N such that $\varepsilon \in [\varepsilon_{N+1}, \varepsilon_N)$. Then $D \cap A(x_0, \varepsilon, \varepsilon_0) \subset \Delta(\varepsilon) := \bigcup_{k=0}^N \Delta_k$, where $\Delta_k = D \cap A_k$ and

$$\begin{aligned} \eta(\varepsilon) &= \int_{\Delta(\varepsilon)} \frac{Q(x) d\mu(x)}{\lambda(d(x, x_0))} \leq \sum_{k=0}^N \int_{\Delta_k} \frac{Q(x) d\mu(x)}{\lambda(d(x, x_0))} \leq \\ &\leq \sum_{k=0}^N \frac{1}{\lambda(\varepsilon_{k+1})} \int_{B(\varepsilon_k) \cap D} Q(x) d\mu(x) \leq s_0 \cdot \sum_{k=0}^N \frac{\mu(B_k \cap D)}{\lambda(\varepsilon_{k+1})}. \end{aligned}$$

In pursuance of upper $\lambda(r)$ -regularity we obtain that $\mu(B_k) \leq C \cdot \lambda(\varepsilon_k)$ and

$$\eta(\varepsilon) \leq C \cdot s_0 \cdot \sum_{k=0}^N \frac{\lambda(\varepsilon_k)}{\lambda(\varepsilon_{k+1})} \leq C \cdot s_0 \cdot N.$$

Since $N < \log_2 \frac{\varepsilon_0}{\varepsilon} < \log_2 \frac{1}{\varepsilon} = \frac{\log \frac{1}{\varepsilon}}{\log 2}$, then

$$\int_{D \cap A(x_0, \varepsilon, \varepsilon_0)} \frac{Q(x) d\mu(x)}{\lambda(d(x, x_0))} \leq \frac{C \cdot s_0}{\log 2} \log \frac{1}{\varepsilon}.$$

So, we complete the proof.

As before, here (X, d, μ) and (X', d', μ') are spaces with metrics d and d' and locally finite Borel measures μ and μ' , and D and D' are domains in X and X' with finite Hausdorff dimensions α and $\alpha' > 1$, respectively.

Theorem 2. *Let X be upper $\lambda(r)$ -regular with*

$$\lambda(\varepsilon) = o\left(\varepsilon^\alpha \log^{\alpha-1} \frac{1}{\varepsilon}\right) \quad (15)$$

as $\varepsilon \rightarrow 0$, D is locally path connected, $\overline{D'}$ be compact and $\partial D'$ strongly accessible. If

$$\limsup_{\varepsilon \rightarrow 0} \frac{1}{\mu(B(x_0, \varepsilon) \cap D)} \int_{B(x_0, \varepsilon) \cap D} Q(x) d\mu(x) < \infty, \quad (16)$$

then any Q -homeomorphism $f : D \rightarrow D'$ admits a continuous extension in (X', d') .

Proof. By elementary argument, we see that condition (15) implies that

$$\int_0^{\varepsilon_0} \frac{dt}{\lambda^{\frac{1}{\alpha}}(t)} = \infty. \quad (17)$$

Choosing in Lemma 1 $\psi(t) = \frac{1}{\lambda^{1/\alpha}(t)}$ and combining conclusion Lemma 2, we show, by L'Hospital rule, that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{D \cap A(x_0, \varepsilon, \varepsilon_0)} \frac{Q(x) d\mu(x)}{\lambda(d(x, x_0))} / \left(\int_{\varepsilon}^{\varepsilon_0} \frac{dt}{\lambda^{1/\alpha}(t)} \right)^\alpha &\leq \lim_{\varepsilon \rightarrow 0} c \log \frac{1}{\varepsilon} / \left(\int_{\varepsilon}^{\varepsilon_0} \frac{dt}{\lambda^{1/\alpha}(t)} \right)^\alpha = \\ &= \gamma \lim_{\varepsilon \rightarrow 0} \frac{\lambda(\varepsilon)}{\varepsilon^\alpha} \log^{1-\alpha} \frac{1}{\varepsilon}, \end{aligned}$$

where $\gamma = \frac{c}{\alpha^\alpha}$. According to (15)

$$\lim_{\varepsilon \rightarrow 0} \int_{D \cap A(x_0, \varepsilon, \varepsilon_0)} \frac{Q(x) d\mu(x)}{\lambda(d(x, x_0))} / \left(\int_{\varepsilon}^{\varepsilon_0} \frac{dt}{\lambda^{1/\alpha}(t)} \right)^\alpha = 0.$$

Since conditions of Lemma 1 are hold then exist extending to x_0 by continuity.

Combining the theorems 1 and 2, we obtain the following theorem.

Theorem 3. *Let X be upper $\lambda(r)$ -regular at a point $x_0 \in \partial D$ where D and D' have weakly flat boundaries, let \overline{D} and $\overline{D'}$ be compact, and let $Q : X \rightarrow (0, \infty)$ be a function of the class $L_\mu^1(D)$ with condition (16). If*

$$\lambda(\varepsilon) = o\left(\varepsilon^\alpha \log^{\alpha-1} \frac{1}{\varepsilon}\right) \quad (18)$$

as $\varepsilon \rightarrow 0$ then any Q -homeomorphism $f : D \rightarrow D'$ is extended to a homeomorphism $\bar{f} : \bar{D} \rightarrow \bar{D}'$.

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О некоторых отображениях в $\lambda(r)$ -регулярных метрических пространствах.

Сформулированы условия на функцию $Q(x)$ и границы областей, при которых любой Q -гомеоморфизм допускает непрерывное или гомеоморфное продолжение на границу в метрических пространствах с мерами.

Ключевые слова: модули, Q -гомеоморфизмы

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Про деякі відображення у $\lambda(r)$ -регулярних метричних просторах.

Сформульовано умови на функцію $Q(x)$ та межі областей, за якими будь-який Q -гомеоморфізм допускає неперервне або гомеоморфне продовження на межу у метричних просторах із мірами.

Ключові слова: модулі, Q -гомеоморфізми

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