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CONDITIONS OF SOLVABILITY OF THE DIRICHLET PROBLEM FOR DEGENERATE ANISOTROPIC ELLIPTIC SECOND-ORDER EQUATIONS WITH L^1-DATA

In this article, we deal with the Dirichlet problem for a class of degenerate anisotropic elliptic second-order equations with L^1-right-hand sides in a bounded open set of $\mathbb{R}^n$ ($n \geqslant 2$). This class is described by the presence of a set of exponents $q_1, \ldots, q_n$ and of a set of weighted functions $\nu_1, \ldots, \nu_n$ in growth and coercivity conditions on coefficients of the equations under consideration. On the basis of general theorems obtained in one of our recent work, we establish conditions of existence of T-solutions and W-solutions of the given problem for model cases where n is an even number, $q_1 = \ldots = q_{n/2} \leqslant q_{n/2+1} = \ldots = q_n$ and ν_i are some power weights.

Keywords: *degenerate anisotropic elliptic second-order equations, L^1-data, Dirichlet problem, T-solution, W-solution, conditions of existence of solutions.*

1. Introduction. In [8], the Dirichlet problem was studied for a class of degenerate anisotropic elliptic second-order equations with L^1-right-hand sides in a bounded open set Ω of $\mathbb{R}^n$ ($n \geqslant 2$). This class is described by the presence of a set of exponents $q_1, \ldots, q_n$ and of a set of weighted functions $\nu_1, \ldots, \nu_n$ in growth and coercivity conditions on coefficients of the equations under consideration. The exponents q_i charac-terize the rates of growth of the coefficients with respect to the corresponding derivatives of unknown function, and the functions ν_i characterize degeneration or singularity of the coefficients with respect to the spatial variable. This is the most general situation in comparison with works of other authors (cf. [1-6, 9]).

Observe that the initial assumptions on the exponents q_i and the functions ν_i in [8] are as follows: $q_i \in (1, n)$, $\nu_i : \Omega \to \mathbb{R}$, $\nu_i \geqslant 0$ in Ω, $\nu_i > 0$ a. e. in Ω, $\nu_i \in L^1_{\mathrm{loc}}(\Omega)$ and $(1/\nu_i)^{1/(q_i-1)} \in L^1(\Omega)$. On the basis of results obtained in [7], in [8] we proved, in particular, theorems on the existence of T-solutions and W-solutions to the investigated Dirichlet problem. The statements of these theorems contain additional conditions on the numbers q_i and the exponents of increased summability of functions $1/\nu_i$ and ν_i.

In the present article, we give equivalent formulations of the mentioned additional conditions on q_i and ν_i for model cases where n is an even number, $q_1 = \ldots = q_{n/2} \leqslant q_{n/2+1} = \ldots = q_n$ and ν_i are some power weights with degeneration or singularity in Ω.

2. Preliminaries. Let $n \in \mathbb{N}$, $n \geqslant 2$, Ω be a bounded open set of $\mathbb{R}^n$, and let for every $i \in \{1, \ldots, n\}$ we have $q_i \in (1, n)$. We set $q = \{q_i : i = 1, \ldots, n\}$.

Let for every $i \in \{1, \ldots, n\}$, ν_i be a nonnegative function on Ω such that $\nu_i > 0$ a. e. in Ω,

$$\nu_i \in L^1_{\mathrm{loc}}(\Omega), \qquad \left(\frac{1}{\nu_i}\right)^{1/(q_i-1)} \in L^1(\Omega). \tag{1}$$

We set $\nu = \{\nu_i : i = 1, \ldots, n\}$. We denote by $W^{1,q}(\nu, \Omega)$ the set of all functions

$u \in L^1(\Omega)$ such that for every $i \in \{1,\dots,n\}$ there exists the weak derivative $D_i u$ and $\nu_i|D_i u|^{q_i} \in L^1(\Omega)$.

Let $\|\cdot\|_{1,q,\nu}$ be the mapping from $W^{1,q}(\nu,\Omega)$ into $\mathbb{R}$ such that for every function $u \in W^{1,q}(\nu,\Omega)$,

$$\|u\|_{1,q,\nu} = \int_\Omega |u|dx + \sum_{i=1}^n \left(\int_\Omega \nu_i|D_i u|^{q_i}dx\right)^{1/q_i}.$$

The mapping $\|\cdot\|_{1,q,\nu}$ is a norm in $W^{1,q}(\nu,\Omega)$, and, in view of the second inclusion of (1), the set $W^{1,q}(\nu,\Omega)$ is a Banach space with respect to the norm $\|\cdot\|_{1,q,\nu}$. Moreover, by virtue of the first inclusion of (1), we have $C_0^\infty(\Omega) \subset W^{1,q}(\nu,\Omega)$.

We denote by $\overset{\circ}{W}{}^{1,q}(\nu,\Omega)$ the closure of the set $C_0^\infty(\Omega)$ in the space $W^{1,q}(\nu,\Omega)$. Obviously, the set $\overset{\circ}{W}{}^{1,q}(\nu,\Omega)$ is a Banach space with respect to the norm induced by the norm $\|\cdot\|_{1,q,\nu}$. We observe that $C_0^1(\Omega) \subset \overset{\circ}{W}{}^{1,q}(\nu,\Omega)$.

We define

$$\overline{q} = \left(\frac{1}{n}\sum_{i=1}^n \frac{1}{q_i}\right)^{-1}$$

and for every $m \in \mathbb{R}^n$ such that $m_i > 0$, $i = 1,\dots,n$, we set

$$p_m = n\left(\sum_{i=1}^n \frac{1+m_i}{m_i q_i} - 1\right)^{-1}.$$

Observe that if $m \in \mathbb{R}^n$ and for every $i \in \{1,\dots,n\}$, $m_i \geqslant 1/(q_i - 1)$, then $p_m > 1$. Moreover, if $m \in \mathbb{R}^n$ and for every $i \in \{1,\dots,n\}$ we have $m_i \geqslant 1/(q_i-1)$ and $1/\nu_i \in L^{m_i}(\Omega)$, then the space $\overset{\circ}{W}{}^{1,q}(\nu,\Omega)$ is continuously imbedded into the space $L^{p_m}(\Omega)$. This fact follows from Proposition 2.8 of [7]. In turn, the mentioned proposition was established with the use of an imbedding result for the nonweighted anisotropic case [10].

Further, let for every $k > 0$, $T_k : \mathbb{R} \to \mathbb{R}$ be the function such that

$$T_k(s) = \begin{cases} s & \text{if } |s| \leqslant k, \\ k \operatorname{sign} s & \text{if } |s| > k. \end{cases}$$

We denote by $\overset{\circ}{\mathcal{T}}{}^{1,q}(\nu,\Omega)$ the set of all functions $u : \Omega \to \mathbb{R}$ such that for every $k > 0$, $T_k(u) \in \overset{\circ}{W}{}^{1,q}(\nu,\Omega)$. Note that $\overset{\circ}{W}{}^{1,q}(\nu,\Omega) \subset \overset{\circ}{\mathcal{T}}{}^{1,q}(\nu,\Omega)$.

For every $u : \Omega \to \mathbb{R}$ and for every $x \in \Omega$ we set

$$k(u,x) = \min\{l \in \mathbb{N} : |u(x)| \leqslant l\}.$$

Definition 1. Let $u \in \overset{\circ}{\mathcal{T}}{}^{1,q}(\nu,\Omega)$ and $i \in \{1,\dots,n\}$. Then $\delta_i u : \Omega \to \mathbb{R}$ is the function such that for every $x \in \Omega$, $\delta_i u(x) = D_i T_{k(u,x)}(u)(x)$.

Observe that if $u \in \overset{\circ}{\mathcal{T}}{}^{1,q}(\nu,\Omega)$ and $i \in \{1,\dots,n\}$, then for every $k > 0$, $D_i T_k(u) = \delta_i u \cdot 1_{\{|u|<k\}}$ a. e. in Ω (see [7, Proposition 2.4]).

Definition 2. If $u \in \overset{\circ}{\mathcal{T}}{}^{1,q}(\nu,\Omega)$, then $\delta u : \Omega \to \mathbb{R}^n$ is the mapping such that for every $x \in \Omega$ and for every $i \in \{1,\dots,n\}$, $(\delta u(x))_i = \delta_i u(x)$.

3. General theorems on solvability of the Dirichlet problem. Let $c_1, c_2 > 0$, $g_1, g_2 \in L^1(\Omega)$, $g_1, g_2 \geqslant 0$ in Ω, and let for every $i \in \{1,\dots,n\}$, $a_i : \Omega \times \mathbb{R}^n \to \mathbb{R}$ be a Carathéodory function. We suppose that for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^n$,

$$\sum_{i=1}^n (1/\nu_i)^{1/(q_i-1)}(x)|a_i(x,\xi)|^{q_i/(q_i-1)} \leqslant c_1 \sum_{i=1}^n \nu_i(x)|\xi_i|^{q_i} + g_1(x),$$

$$\sum_{i=1}^n a_i(x,\xi)\xi_i \geqslant c_2 \sum_{i=1}^n \nu_i(x)|\xi_i|^{q_i} - g_2(x).$$

Moreover, we assume that for almost every $x \in \Omega$ and for every $\xi, \xi' \in \mathbb{R}^n$, $\xi \neq \xi'$,

$$\sum_{i=1}^n [\, a_i(x,\xi) - a_i(x,\xi')](\xi_i - \xi_i') > 0.$$

Let $f \in L^1(\Omega)$. We consider the following Dirichlet problem:

$$-\sum_{i=1}^n \frac{\partial}{\partial x_i} a_i(x, \nabla u) = f \quad \text{in } \ \Omega, \tag{2}$$

$$u = 0 \quad \text{on } \ \partial\Omega. \tag{3}$$

Definition 3. A T-solution of problem (2), (3) is a function $u \in \overset{\circ}{\mathcal{T}}{}^{1,q}(\nu,\Omega)$ such that:
(i) for every $i \in \{1,\dots,n\}$, $a_i(x,\delta u) \in L^1(\Omega)$;
(ii) for every function $w \in C_0^1(\Omega)$, $\displaystyle\int_\Omega \Big\{ \sum_{i=1}^n a_i(x,\delta u) D_i w \Big\} dx = \int_\Omega f w \, dx.$

Theorem 1. *Suppose that there exist $m, \sigma \in \mathbb{R}^n$ such that the following conditions are satisfied*:

$$\forall\, i \in \{1,\dots,n\}, \qquad m_i \geqslant 1/(q_i - 1), \ 1/\nu_i \in L^{m_i}(\Omega), \tag{4}$$

$$\forall\, i \in \{1,\dots,n\}, \qquad \sigma_i > 0, \ \frac{1}{\sigma_i} < 1 - \frac{(q_i-1)\overline{q}}{p_m(\overline{q}-1)}\,, \ \nu_i \in L^{\sigma_i}(\Omega). \tag{5}$$

Then there exists a T-solution of problem (2), (3).

Definition 4. A W-solution of problem (2), (3) is a function $u \in \overset{\circ}{W}{}^{1,1}(\Omega)$ such that:
(i) for every $i \in \{1,\dots,n\}$, $a_i(x,\nabla u) \in L^1(\Omega)$;
(ii) for every function $w \in C_0^1(\Omega)$, $\displaystyle\int_\Omega \Big\{ \sum_{i=1}^n a_i(x,\nabla u) D_i w \Big\} dx = \int_\Omega f w \, dx.$

Theorem 2. *Suppose that there exist $m, \sigma \in \mathbb{R}^n$ with positive coordinates such that the following conditions are satisfied*:

$$\forall\, i \in \{1,\dots,n\}, \qquad \frac{\overline{q}}{p_m(\overline{q}-1)} < q_i - 1 - \frac{1}{m_i}\,, \ 1/\nu_i \in L^{m_i}(\Omega), \tag{6}$$

$$\forall\, i \in \{1, \ldots, n\}, \qquad \frac{1}{\sigma_i} < 1 - \frac{(q_i - 1)\overline{q}}{p_m(\overline{q} - 1)}\,, \;\; \nu_i \in L^{\sigma_i}(\Omega). \tag{7}$$

Then there exists a W-solution of problem (2), (3).

Theorems 1 and 2 were proved in [8].

4. Equivalent statements of conditions of solvability of the Dirichlet problem for some model cases. We suppose that $n = 2l$ where $l \in \mathbb{N}$, and let α and β be numbers such that

$$1 < \alpha \leqslant \beta < n. \tag{8}$$

We assume that

$$q_i = \alpha \quad \text{if} \quad i \in \{1, \ldots, l\}, \tag{9}$$

$$q_i = \beta \quad \text{if} \quad i \in \{l + 1, \ldots, n\}. \tag{10}$$

Since $n \geqslant 2$ and $\beta > 1$, we have

$$1 \leqslant \frac{\beta n}{2\beta + n - 2} < \beta,$$

and since, by (8), $\alpha \leqslant \beta$, from (9) and (10) it follows that

$$\forall\, i \in \{1, \ldots, n\}, \qquad \alpha \leqslant q_i \leqslant \beta. \tag{11}$$

Moreover, in view of the definition of the number $\overline{q}$ and (9) and (10), we have

$$\frac{1}{\overline{q}} = \frac{1}{2}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right). \tag{12}$$

Finally, if $m \in \mathbb{R}^n$, and $m_i > 0$, $i = 1, \ldots, n$, from the definitions of p_m and $\overline{q}$ it follows that

$$\frac{1}{p_m} > \frac{1}{\overline{q}} - \frac{1}{n}\,. \tag{13}$$

Proposition 1. *Suppose that for every $i \in \{1, \ldots, n\}$, $\nu_i \equiv 1$. Then the following assertions are equivalent*:

(a) *there exist $m, \sigma \in \mathbb{R}^n$ such that conditions* (4) *and* (5) *are satisfied*;

(b) *the inequality*

$$\frac{\beta n}{2\beta + n - 2} < \alpha \tag{14}$$

is valid.

Proof. Suppose that assertion (a) holds. Then, by virtue of condition (5) and assumption (10),

$$\frac{\beta - 1}{p_m} < 1 - \frac{1}{\overline{q}}\,. \tag{15}$$

Since, in view of condition (4), $m_i > 0$, $i = 1, \ldots, n$, using (13) and (15), we obtain

$$\frac{n}{\overline{q}} < 1 + \frac{n - 1}{\beta}\,. \tag{16}$$

Hence, using equality (12), we conclude that inequality (14) is valid. Thus, assertion (b) holds.

Conversely, let assertion (b) hold. Then from (12) and (14) we obtain that inequality (16) is valid, and therefore, $(\overline{q}-1)/(\beta-1) > (n-\overline{q})/n$. Taking this inequality into account, we fix a number t such that

$$t \geqslant \frac{1}{\alpha - 1}\,, \tag{17}$$

$$\frac{1}{t} < \frac{\overline{q}-1}{\beta-1} - \frac{n-\overline{q}}{n}\,. \tag{18}$$

From (18) it follows that

$$1 - \left(1 - \frac{\overline{q}}{n} + \frac{1}{t}\right)\frac{\beta-1}{\overline{q}-1} > 0.$$

Taking this inequality into account, we fix $s > 0$ such that

$$\frac{1}{s} < 1 - \left(1 - \frac{\overline{q}}{n} + \frac{1}{t}\right)\frac{\beta-1}{\overline{q}-1}\,. \tag{19}$$

Now, let $m, \sigma \in \mathbb{R}^n$ be such that

$$m_i = t, \quad i = 1, \dots, n, \tag{20}$$

$$\sigma_i = s, \quad i = 1, \dots, n. \tag{21}$$

Using (11), (17) and (20) along with the fact that $\nu_i \equiv 1$, $i = 1, \dots, n$, we establish that condition (4) is satisfied. At the same time, using (20), we find that $\overline{q}/p_m = 1 - \overline{q}/n + 1/t$. This and (19) imply that

$$\frac{1}{s} < 1 - \frac{(\beta-1)\overline{q}}{p_m(\overline{q}-1)}\,. \tag{22}$$

Using (11), (21) and (22) along with the fact that $\nu_i \equiv 1$, $i = 1, \dots, n$, we establish that condition (5) is satisfied. Thus, assertion (a) holds. The proposition is proved.

Proposition 2. *Suppose that for every $i \in \{1, \dots, n\}$, $\nu_i \equiv 1$. Then the following assertions are equivalent*:

(a) *there exist $m, \sigma \in \mathbb{R}^n$ with positive coordinates such that conditions* (6) *and* (7) *are satisfied*;

(b) *the inequality*

$$\max\left\{\frac{\beta n}{2\beta + n - 2}\,, \frac{(3n-2)\beta}{(2\beta-1)n}\right\} < \alpha \tag{23}$$

is valid.

Proof. Suppose that assertion (a) holds. Then, by virtue of condition (6) and assumption (9), $\frac{\overline{q}}{p_m(\overline{q}-1)} < \alpha - 1$. Hence, using (13), we get $\frac{1}{\overline{q}} < 1 - \frac{n-1}{n\alpha}$. This and (12) imply that

$$(3n-2)\beta < (2\beta-1)n\alpha. \tag{24}$$

Moreover, by condition (7) and assumption (10), we have $(\beta-1)/p_m < 1-1/\overline{q}$. Hence, using (12) and (13), we obtain that $\beta n < (2\beta+n-2)\alpha$. This and (24) lead to the conclusion that inequality (23) is valid. Thus, assertion (b) holds.

Conversely, let assertion (b) hold. Then from equality (12) and the inequality $(3n-2)\beta < (2\beta-1)n\alpha$ we deduce that

$$\frac{n-\overline{q}}{n(\overline{q}-1)} < \alpha - 1. \tag{25}$$

Moreover, from equality (12) and the inequality $\beta n < (2\beta+n-2)\alpha$ we infer that $n/\overline{q} < 1+(n-1)/\beta$. Therefore, $(n-\overline{q})/n < (\overline{q}-1)/(\beta-1)$. Taking into account this inequality and (25), we fix $t>0$ such that

$$\frac{1}{t} < \frac{\overline{q}-1}{\overline{q}}\left[\alpha - 1 - \frac{n-\overline{q}}{n(\overline{q}-1)}\right], \tag{26}$$

$$\frac{1}{t} < \frac{\overline{q}-1}{\beta-1} - \frac{n-\overline{q}}{n}\,. \tag{27}$$

From (26) it follows that

$$\left(\frac{1}{\overline{q}} - \frac{1}{n} + \frac{1}{\overline{q}t}\right)\frac{\overline{q}}{\overline{q}-1} < \alpha - 1 - \frac{1}{t}\,, \tag{28}$$

and from (27) we get

$$1 - \left(1 - \frac{\overline{q}}{n} + \frac{1}{t}\right)\frac{\beta-1}{\overline{q}-1} > 0.$$

Taking the latter inequality into account, we fix $s>0$ such that

$$\frac{1}{s} < 1 - \left(1 - \frac{\overline{q}}{n} + \frac{1}{t}\right)\frac{\beta-1}{\overline{q}-1}\,. \tag{29}$$

Now, let $m,\sigma \in \mathbb{R}^n$ be such that

$$m_i = t, \quad i = 1,\dots,n, \tag{30}$$

$$\sigma_i = s, \quad i = 1,\dots,n. \tag{31}$$

Using (30), we obtain that

$$\frac{1}{p_m} = \frac{1}{\overline{q}} - \frac{1}{n} + \frac{1}{t\overline{q}}\,. \tag{32}$$

This along with (11), (28) and (30) implies that for every $i \in \{1,\dots,n\}$,

$$\frac{\overline{q}}{p_m(\overline{q}-1)} < q_i - 1 - \frac{1}{m_i}\,,$$

and obviously, for every $i \in \{1, \dots, n\}$ we have $1/\nu_i \in L^{m_i}(\Omega)$. Therefore, condition (6) is satisfied. Moreover, from (11), (29), (31) and (32) we derive that for every $i \in \{1, \dots, n\}$,

$$\frac{1}{\sigma_i} < 1 - \frac{(q_i - 1)\overline{q}}{p_m(\overline{q} - 1)},$$

and obviously, for every $i \in \{1, \dots, n\}$ we have $\nu_i \in L^{\sigma_i}(\Omega)$. Therefore, condition (7) is satisfied. Thus, assertion (a) holds. The proposition is proved.

Remark 1. The necessary requirements for the validity of inequality (23) are as follows:

$$\beta > 2 - 1/n, \qquad \alpha > 2 - \frac{5n - 4}{n^2 + 2n - 2}.$$

In fact, from (23) and the initial assumption $\alpha \leqslant \beta$ it follows that $(3n - 2)/n < 2\beta - 1$. Hence, $\beta > 2 - 1/n$. Moreover, by virtue of (23), we have $n/\alpha < 2 + (n - 2)/\beta$. This and the inequality $\beta > 2 - 1/n$ imply that $\alpha > 2 - (5n - 4)/(n^2 + 2n - 2)$.

Remark 2. Observe that if $\beta > 2 - 1/n$, then

$$\max\left\{\frac{n}{2\beta + n - 2}, \frac{3n - 2}{(2\beta - 1)n}\right\} < 1.$$

Taking this into account, we obtain that if $\beta > 2 - 1/n$,

$$\max\left\{\frac{n}{2\beta + n - 2}, \frac{3n - 2}{(2\beta - 1)n}\right\} < \varepsilon \leqslant 1$$

and $\alpha = \beta\varepsilon$, then inequality (23) is valid.

Let θ denote the origin in $\mathbb{R}^n$, i. e. $\theta \in \mathbb{R}^n$, and for every $i \in \{1, \dots, n\}$, $\theta_i = 0$.

Proposition 3. *Let $\theta \in \Omega$, $\gamma, \tau > 0$, and suppose that the following assertions hold*:

$$\text{if } i \in \{1, \dots, l\}, \text{ then for every } x \in \Omega \text{ we have } \nu_i(x) = |x|^\gamma, \tag{33}$$

$$\text{if } i \in \{l + 1, \dots, n\}, \text{ then for every } x \in \Omega \text{ we have } \nu_i(x) = |x|^\tau. \tag{34}$$

Then the following assertions are equivalent:

(a) *there exist $m, \sigma \in \mathbb{R}^n$ such that conditions* (4) *and* (5) *are satisfied*;

(b) *the inequalities*

$$\gamma < n(\alpha - 1), \qquad \tau < n(\beta - 1), \tag{35}$$

$$\frac{\gamma}{\alpha} + \frac{\tau}{\beta} < \frac{\alpha(2\beta + n - 2) - \beta n}{\alpha(\beta - 1)} \tag{36}$$

are valid.

Proof. Suppose that assertion (a) holds. Let $i \in \{1, \dots, l\}$. From condition (4) and assumption (9) it follows that $m_i \geqslant 1/(\alpha - 1)$ and $1/\nu_i \in L^{m_i}(\Omega)$. The latter inclusion along with the inclusion $\theta \in \Omega$ and (33) implies that $m_i < n/\gamma$. From the given inequalities for m_i we deduce that the first inequality of (35) is valid. Now, let $i \in$

$\{l+1,\dots,n\}$. From condition (4) and assumption (10) it follows that $m_i \geqslant 1/(\beta-1)$ and $1/\nu_i \in L^{m_i}(\Omega)$. The latter inclusion along with the inclusion $\theta \in \Omega$ and (34) implies that $m_i < n/\tau$. From the inequalities obtained for m_i we infer that the second inequality of (35) is valid. Moreover, owing to the above considerations, we have

$$\forall\, i \in \{1,\dots,l\}, \qquad \frac{1}{m_i} > \frac{\gamma}{n}\,, \tag{37}$$

$$\forall\, i \in \{l+1,\dots,n\}, \qquad \frac{1}{m_i} > \frac{\tau}{n}\,. \tag{38}$$

Using (9), (10), (37) and (38), we get

$$\frac{1}{p_m} > \frac{1}{\overline{q}} - \frac{1}{n} + \frac{1}{2n}\left(\frac{\gamma}{\alpha} + \frac{\tau}{\beta}\right). \tag{39}$$

Next, from condition (5) and assumption (10) it follows that $(\beta-1)/p_m < 1 - 1/\overline{q}$. This and (39) imply that

$$(\beta-1)\left[\frac{1}{\overline{q}} - \frac{1}{n} + \frac{1}{2n}\left(\frac{\gamma}{\alpha} + \frac{\tau}{\beta}\right)\right] < 1 - \frac{1}{\overline{q}}\,.$$

Hence, using (12), we deduce that inequality (36) is valid. Thus, assertion (b) holds.

Conversely, suppose that assertion (b) holds. From inequalities (35) and (36) it follows that $\frac{\gamma}{n(\alpha-1)} < 1$, $\frac{\tau}{n(\beta-1)} < 1$ and $\alpha(\beta-1)\left(\frac{\gamma}{\alpha} + \frac{\tau}{\beta}\right)[\alpha(2\beta+n-2) - \beta n]^{-1} < 1$. Taking these inequalities into account, we fix a number $\varepsilon < 1$ such that

$$\frac{\gamma}{n(\alpha-1)} < \varepsilon, \qquad \frac{\tau}{n(\beta-1)} < \varepsilon, \tag{40}$$

$$\alpha(\beta-1)\left(\frac{\gamma}{\alpha} + \frac{\tau}{\beta}\right)[\alpha(2\beta+n-2) - \beta n]^{-1} < \varepsilon. \tag{41}$$

Using (12) and (41), we establish that

$$1 - \frac{\overline{q}(\beta-1)}{\overline{q}-1}\left[\frac{1}{\overline{q}} - \frac{1}{n} + \frac{1}{2n\varepsilon}\left(\frac{\gamma}{\alpha} + \frac{\tau}{\beta}\right)\right] > 0.$$

Taking this inequality into account, we fix $s > 0$ such that

$$\frac{1}{s} < 1 - \frac{\overline{q}(\beta-1)}{\overline{q}-1}\left[\frac{1}{\overline{q}} - \frac{1}{n} + \frac{1}{2n\varepsilon}\left(\frac{\gamma}{\alpha} + \frac{\tau}{\beta}\right)\right]. \tag{42}$$

Now, let m, $\sigma \in \mathbb{R}^n$ be such that

$$m_i = \frac{n\varepsilon}{\gamma}, \qquad i = 1,\dots,l, \tag{43}$$

$$m_i = \frac{n\varepsilon}{\tau}, \qquad i = l+1,\dots,n, \tag{44}$$

$$\sigma_i = s, \qquad i = 1, \dots, n. \tag{45}$$

Using (9), (10), (33), (34),(40), (43), (44) and the inequality $\varepsilon < 1$, we establish that condition (4) is satisfied. At the same time, by virtue of (9), (10), (43) and (44), we have

$$\frac{1}{p_m} = \frac{1}{\overline{q}} - \frac{1}{n} + \frac{1}{2n\varepsilon}\left(\frac{\gamma}{\alpha} + \frac{\tau}{\beta}\right).$$

Taking into account this equality and using (11), (33), (34), (42) and (45), we establish that condition (5) is satisfied. Thus, assertion (a) holds. The proposition is proved.

Remark 3. From the statements of Propositions 1 and 3 it follows that in the nonweighted case and in the case of power weights with positive exponents, for the existence of $m, \sigma \in \mathbb{R}^n$ satisfying conditions (4) and (5), the necessary requirement on α and β is the same, exactly $\alpha > \beta n/(2\beta + n - 2)$.

Proposition 4. *Let $\theta \in \Omega$, $\gamma, \tau > 0$, and suppose that assertions* (33) *and* (34) *hold. Then the following assertions are equivalent*:

(a) *there exist $m, \sigma \in \mathbb{R}^n$ with positive coordinates such that conditions* (6) *and* (7) *are satisfied*;

(b) *the inequalities*

$$(2\beta - 1)\gamma + \tau < \alpha n(2\beta - 1) - (3n - 2)\beta, \tag{46}$$

$$\gamma + (2\alpha - 1)\tau < \alpha(2\beta n - 3n + 2) - \beta n, \tag{47}$$

$$\frac{\gamma}{\alpha} + \frac{\tau}{\beta} < \frac{\alpha(2\beta + n - 2) - \beta n}{\alpha(\beta - 1)} \tag{48}$$

are valid.

Proof. Suppose that assertion (a) holds. Let $i \in \{1, \dots, l\}$. By condition (6) and assumption (9), we have

$$\frac{1}{m_i} < \alpha - 1 - \frac{\overline{q}}{p_m(\overline{q} - 1)}\,. \tag{49}$$

Moreover, since, by condition (6), $1/\nu_i \in L^{m_i}(\Omega)$, from this inclusion and (33) we deduce that $\gamma m_i < n$. This and (49) imply that

$$\frac{\gamma}{n} < \alpha - 1 - \frac{\overline{q}}{p_m(\overline{q} - 1)}\,. \tag{50}$$

Analogously, taking an arbitrary $i \in \{l+1, \dots, n\}$, by virtue of condition (6) and assumption (10), we have

$$\frac{1}{m_i} < \beta - 1 - \frac{\overline{q}}{p_m(\overline{q} - 1)}\,. \tag{51}$$

Furthermore, since, by condition (6), $1/\nu_i \in L^{m_i}(\Omega)$, from this inclusion and (34) we derive that $\tau m_i < n$. This and (51) imply that

$$\frac{\tau}{n} < \beta - 1 - \frac{\overline{q}}{p_m(\overline{q} - 1)}\,. \tag{52}$$

From the above considerations it follows that assertions (37) and (38) holds, and then, in view of (9) and (10), inequality (39) is valid. Using (12), (39) and (50), we establish that inequality (46) is valid, and, by means of (12), (39) and (52), we find that inequality (47) is valid. Finally, from condition (7) and assumption (10) it follows that $(\beta-1)\overline{q} < p_m(\overline{q}-1)$. Hence, using (12) and (39), we establish that inequality (48) is valid. Thus, assertion (b) holds.

Conversely, suppose that assertion (b) holds, Then, by virtue of inequalities (46)–(48), we have

$$[(2\beta-1)\gamma+\tau][\alpha n(2\beta-1)-(3n-2)\beta]^{-1} < 1,$$

$$[\gamma+(2\alpha-1)\tau][\alpha(2\beta n-3n+2)-\beta n]^{-1} < 1,$$

$$\alpha(\beta-1)\left(\frac{\gamma}{\alpha}+\frac{\tau}{\beta}\right)[\alpha(2\beta+n-2)-\beta n]^{-1} < 1.$$

Taking these inequalities into account, we fix a number $\varepsilon < 1$ such that

$$[(2\beta-1)\gamma+\tau][\alpha n(2\beta-1)-(3n-2)\beta]^{-1} < \varepsilon, \tag{53}$$

$$[\gamma+(2\alpha-1)\tau][\alpha(2\beta n-3n+2)-\beta n]^{-1} < \varepsilon, \tag{54}$$

$$\alpha(\beta-1)\left(\frac{\gamma}{\alpha}+\frac{\tau}{\beta}\right)[\alpha(2\beta+n-2)-\beta n]^{-1} < \varepsilon, \tag{55}$$

and set

$$\mu = \frac{1}{\overline{q}} - \frac{1}{n} + \frac{1}{2n\varepsilon}\left(\frac{\gamma}{\alpha}+\frac{\tau}{\beta}\right). \tag{56}$$

Using (12), (53), (54) and (56), we obtain the inequalities

$$\frac{\mu\overline{q}}{\overline{q}-1} < \alpha - 1 - \frac{\gamma}{n\varepsilon}, \qquad \frac{\mu\overline{q}}{\overline{q}-1} < \beta - 1 - \frac{\tau}{n\varepsilon}, \tag{57}$$

and using (12), (55) and (56), we get $\mu(\beta-1)\overline{q} < \overline{q}-1$. Taking into account the latter inequality, we fix $s > 0$ such that

$$\frac{1}{s} < 1 - \frac{\mu(\beta-1)\overline{q}}{\overline{q}-1}. \tag{58}$$

Now, let m, $\sigma \in \mathbb{R}^n$ be such that

$$m_i = \frac{n\varepsilon}{\gamma}, \quad i = 1, \ldots, l, \tag{59}$$

$$m_i = \frac{n\varepsilon}{\tau}, \quad i = l+1, \ldots, n, \tag{60}$$

$$\sigma_i = s, \qquad i = 1, \ldots, n. \tag{61}$$

From (9), (10), (56), (59) and (60) it follows that

$$\mu p_m = 1. \tag{62}$$

Using (9), (10), (57), (59), (60) and (62), we establish that

$$\forall\, i \in \{1, \dots, n\}, \qquad \frac{\overline{q}}{p_m(\overline{q}-1)} < q_i - 1 - \frac{1}{m_i}\,. \tag{63}$$

Moreover, using (33), (34), (59), (60) and the inequality $\varepsilon < 1$, we obtain that for every $i \in \{1, \dots, n\}$, $1/\nu_i \in L^{m_i}(\Omega)$. This and (63) imply that condition (6) is satisfied. Finally, using (11), (33), (34), (58), (61) and (62), we establish that condition (7) is satisfied. Thus, assertion (a) holds. The proposition is proved.

Remark 4. From the statements of Propositions 2 and 4 it follows that in the nonweighted case and in the case of power weights with positive exponents, for the existence of m, $\sigma \in \mathbb{R}^n$ with positive coordinates satisfying conditions (6) and (7), the principal necessary requirement on α and β is the same, exactly is inequality (23). To see this, it is sufficient to note that by virtue of (46) and the initial inequality $\alpha \leqslant \beta$, we have $\beta > 2 - 1/n$, and then, in view of (46), $\alpha(2\beta n - 3n + 2) - \beta n > 0$.

Proposition 5. *Let $\theta \in \Omega$, $\gamma, \tau > 0$, and suppose that the following assertions hold*:

$$\text{if } i \in \{1, \dots, l\}, \ \text{ then for every } x \in \Omega \setminus \{\theta\},\ \nu_i(x) = |x|^{-\gamma}\,, \tag{64}$$

$$\text{if } i \in \{l+1, \dots, n\}, \ \text{ then for every } x \in \Omega \setminus \{\theta\},\ \nu_i(x) = |x|^{-\tau}\,. \tag{65}$$

Then the following assertions are equivalent:

(a) *there exist $m, \sigma \in \mathbb{R}^n$ such that conditions* (4) *and* (5) *are satisfied*;

(b) *the inequalities*

$$\gamma < \frac{\alpha[\beta(n-2) - \alpha(n-2\beta)]}{2\alpha\beta - \alpha - \beta}\,, \qquad \tau < \frac{\beta[\alpha(2\beta + n - 2) - \beta n]}{2\alpha\beta - \alpha - \beta} \tag{66}$$

are valid.

Proof. First of all we observe that, in view of (12),

$$\frac{\overline{q}}{\overline{q}-1}\left(\frac{1}{\overline{q}} - \frac{1}{n}\right) = \frac{n(\alpha+\beta) - 2\alpha\beta}{n(2\alpha\beta - \alpha - \beta)}\,. \tag{67}$$

Suppose that assertion (a) holds. Let $i \in \{1, \dots, l\}$. Then, by virtue of condition (5) and assumption (9), we have $\sigma_i > 0$,

$$\frac{1}{\sigma_i} < 1 - \frac{(\alpha-1)\overline{q}}{p_m(\overline{q}-1)} \tag{68}$$

and $\nu_i \in L^{\sigma_i}(\Omega)$. From the latter inclusion and (64), taking into account that $\theta \in \Omega$, we deduce that $\gamma\sigma_i < n$. Then, by (68), $\frac{\gamma}{n} < 1 - \frac{(\alpha-1)\overline{q}}{p_m(\overline{q}-1)}$. This along with (13) and (67) implies that the first inequality of (66) is valid. Analogously, let $i \in \{l+1, \dots, n\}$. Then, by virtue of condition (5) and assumption (10), we have $\sigma_i > 0$,

$$\frac{1}{\sigma_i} < 1 - \frac{(\beta - 1)\overline{q}}{p_m(\overline{q} - 1)} \tag{69}$$

and $\nu_i \in L^{\sigma_i}(\Omega)$. From the latter inclusion and (65), taking into account that $\theta \in \Omega$, we derive that $\tau\sigma_i < n$. Then, by (69), $\frac{\tau}{n} < 1 - \frac{(\beta-1)\overline{q}}{p_m(\overline{q}-1)}$. This along with (13) and (67) implies that the second inequality of (66) is valid. Thus, assertion (b) holds.

Conversely, suppose that assertion (b) holds. Then, by virtue of (66) and (67), we have

$$\frac{\gamma}{n} < 1 - \frac{(\alpha-1)\overline{q}}{\overline{q}-1}\left(\frac{1}{\overline{q}} - \frac{1}{n}\right), \qquad \frac{\tau}{n} < 1 - \frac{(\beta-1)\overline{q}}{\overline{q}-1}\left(\frac{1}{\overline{q}} - \frac{1}{n}\right).$$

Taking these inequalities into account, we fix $s_1, s_2 > 0$ such that

$$\frac{\gamma}{n} < \frac{1}{s_1} < 1 - \frac{(\alpha-1)\overline{q}}{\overline{q}-1}\left(\frac{1}{\overline{q}} - \frac{1}{n}\right), \tag{70}$$

$$\frac{\tau}{n} < \frac{1}{s_2} < 1 - \frac{(\beta-1)\overline{q}}{\overline{q}-1}\left(\frac{1}{\overline{q}} - \frac{1}{n}\right), \tag{71}$$

and then we fix a number t such that

$$t \geqslant \frac{1}{\alpha - 1}, \tag{72}$$

$$\frac{\alpha-1}{(\overline{q}-1)t} < 1 - \frac{(\alpha-1)\overline{q}}{\overline{q}-1}\left(\frac{1}{\overline{q}} - \frac{1}{n}\right) - \frac{1}{s_1}, \tag{73}$$

$$\frac{\beta-1}{(\overline{q}-1)t} < 1 - \frac{(\beta-1)\overline{q}}{\overline{q}-1}\left(\frac{1}{\overline{q}} - \frac{1}{n}\right) - \frac{1}{s_2}. \tag{74}$$

Now, let $m, \sigma \in \mathbb{R}^n$ be such that

$$m_i = t, \qquad i = 1, \dots, n, \tag{75}$$

$$\sigma_i = s_1, \qquad i = 1, \dots, l, \tag{76}$$

$$\sigma_i = s_2, \qquad i = l+1, \dots, n. \tag{77}$$

Using (11), (64), (65), (72) and (75), we establish that condition (4) is satisfied. At the same time, in view of (75), we have

$$\frac{1}{p_m} = \frac{1}{\overline{q}} - \frac{1}{n} + \frac{1}{\overline{q}t}. \tag{78}$$

From (9), (10), (73), (74) and (76)–(78) it follows that

$$\forall\, i \in \{1, \dots, n\}, \qquad \sigma_i > 0, \quad \frac{1}{\sigma_i} < 1 - \frac{(q_i - 1)\overline{q}}{p_m(\overline{q}-1)}. \tag{79}$$

Moreover, using (64), (65), (76), (77) along with the inclusion $\theta \in \Omega$ and taking into account that, in view of (70) and (71), $\gamma s_1 < n$ and $\tau s_2 < n$, we establish that for every $i \in \{1, \ldots, n\}$, $\nu_i \in L^{\sigma_i}(\Omega)$. This and (79) imply that condition (5) is satisfied. Thus, assertion (a) holds. The proposition is proved.

Remark 5. From the statements of Propositions 3 and 5 it follows that, for the existence of $m, \sigma \in \mathbb{R}^n$ satisfying conditions (4) and (5), the necessary requirements on α and β in the case of power weights with negative exponents in general are stronger than that in the case of power weights with positive exponents.

Proposition 6. *Let* $\theta \in \Omega$, $\gamma, \tau > 0$, *and suppose that assertions* (64) *and* (65) *hold. Then the following assertions are equivalent*:

(a) *there exist* $m, \sigma \in \mathbb{R}^n$ *with positive coordinates such that conditions* (6) *and* (7) *are satisfied*;

(b) *the inequality*

$$\frac{(3n-2)\beta}{(2\beta-1)n} < \alpha \tag{80}$$

and inequalities (66) *are valid.*

Proof. Suppose that assertion (a) holds. Then, by virtue of condition (6) and assumption (9), we have $1/p_m < (\alpha-1)(\overline{q}-1)/\overline{q}$. Hence, using (13), we get $1/\overline{q} < 1 - \frac{n-1}{n\alpha}$. This and (12) imply that inequality (80) is valid. The validity of inequalities (66) is established in the same way as in the proof of Proposition 5. Thus, assertion (b) holds.

Conversely, suppose that assertion (b) holds. Then, taking into account (66) and (67), we fix $s_1, s_2 > 0$ such that inequalities (70) and (71) are valid. Observe that, by (12) and (80), $\frac{\overline{q}}{\overline{q}-1}\big(\frac{1}{\overline{q}} - \frac{1}{n}\big) < \alpha - 1$. Taking into account this inequality and (70), (71), we fix a number $t > 0$ such that

$$\frac{\overline{q}}{(\overline{q}-1)t} < \alpha - 1 - \frac{\overline{q}}{\overline{q}-1}\left(\frac{1}{\overline{q}} - \frac{1}{n}\right) \tag{81}$$

and inequalities (73) and (74) are valid. Now, let m, $\sigma \in \mathbb{R}^n$ be such that $m_i = t$, $i = 1, \ldots, n$; $\sigma_i = s_1$, $i = 1, \ldots, l$, and $\sigma_i = s_2$, $i = l+1, \ldots, n$. It is easy to see that equality (78) is valid. Then, using (11), (64), (65) and (81), we establish that condition (6) is satisfied. Moreover, using (9), (10), (64), (65), (73), (74), (78), the inclusion $\theta \in \Omega$ and the inequalities $\gamma s_1 < n$ and $\tau s_2 < n$, we obtain that condition (7) is satisfied. Thus, assertion (a) holds. The proposition is proved.

Remark 6. From the statements of Propositions 4 and 6 it follows that, for the existence of m, $\sigma \in \mathbb{R}^n$ with positive coordinates satisfying conditions (6) and (7), the necessary requirements on α and β in the case of power weights with negative exponents in general are stronger than that in the case of power weights with positive exponents.

Proposition 7. *Let* $\theta \in \Omega$, $\gamma, \tau > 0$, *and suppose that the following assertions hold*:

$$\text{if } i \in \{1, \ldots, l\}, \text{ then for every } x \in \Omega,\ \nu_i(x) = |x|^{\gamma}, \tag{82}$$

$$\text{if } i \in \{l+1, \ldots, n\}, \text{ then for every } x \in \Omega \setminus \{\theta\},\ \nu_i(x) = |x|^{-\tau}. \tag{83}$$

Then the following assertions are equivalent:

(a) *there exist* $m, \sigma \in \mathbb{R}^n$ *such that conditions* (4) *and* (5) *are satisfied*;
(b) *the inequalities*

$$\gamma < n(\alpha - 1), \tag{84}$$

$$\gamma(\beta - 1) + \tau(2\alpha - 1 - \alpha/\beta) < \alpha(2\beta + n - 2) - \beta n \tag{85}$$

are valid.

Proof. Suppose that assertion (a) holds. Let $i \in \{1, \dots, l\}$. Then, by condition (4) and assumption (9), $m_i \geqslant 1/(\alpha - 1)$ and $1/\nu_i \in L^{m_i}(\Omega)$. The latter inclusion along with the inclusion $\theta \in \Omega$ and (82) implies that $\gamma m_i < n$. From the given inequalities for m_i we deduce that inequality (84) is valid. Moreover, we conclude that for every $i \in \{1, \dots, l\}$, $1/m_i > \gamma/n$. Using this fact and (9), we get

$$\frac{1}{p_m} > \frac{1}{\overline{q}} - \frac{1}{n} + \frac{\gamma}{2\alpha n}\,. \tag{86}$$

Next, let $i \in \{l + 1, \dots, n\}$. By condition (5) and assumption (10),

$$\frac{1}{\sigma_i} < 1 - \frac{(\beta - 1)\overline{q}}{p_m(\overline{q} - 1)} \tag{87}$$

and $\nu_i \in L^{\sigma_i}(\Omega)$. The latter inclusion along with the inclusion $\theta \in \Omega$ and (83) implies that $\tau\sigma_i < n$. From this and (87) it follows that $\frac{\tau}{n} < 1 - \frac{(\beta-1)\overline{q}}{p_m(\overline{q}-1)}$. Hence, using (12) and (86), we establish that inequality (85) is valid. Thus, assertion (b) holds.

Conversely, suppose that assertion (b) holds. Then, by virtue of inequalities (84) and (85), $1 < n(\alpha - 1)/\gamma$, $1 < \frac{1}{\gamma(\beta-1)}\,[\alpha(2\beta + n - 2) - \beta n - \tau(2\alpha - 1 - \alpha/\beta)]$. Taking these inequalities into account, we fix a number $\varepsilon > 0$ such that

$$1 < \frac{1}{\varepsilon} < \frac{n(\alpha - 1)}{\gamma}\,, \tag{88}$$

$$\frac{1}{\varepsilon} < \frac{1}{\gamma(\beta - 1)}\,[\alpha(2\beta + n - 2) - \beta n - \tau(2\alpha - 1 - \alpha/\beta)].$$

The latter inequality implies that $0 < \alpha(2\beta + n - 2) - \beta n - \tau(2\alpha - 1 - \alpha/\beta) - \gamma(\beta - 1)/\varepsilon$. Taking this inequality into account, we fix a number t such that

$$t \geqslant \frac{1}{\beta - 1}\,, \tag{89}$$

$$\frac{(\beta - 1)\alpha n}{\beta t} < \alpha(2\beta + n - 2) - \beta n - \tau(2\alpha - 1 - \alpha/\beta) - \frac{\gamma(\beta - 1)}{\varepsilon}\,, \tag{90}$$

and define

$$d = \frac{1}{\overline{q}} - \frac{1}{n} + \frac{\gamma}{2\alpha n\varepsilon} + \frac{1}{2\beta t}\,. \tag{91}$$

From (12), (90) and (91) it follows that $\tau/n < 1 - (\beta - 1)d\overline{q}/(\overline{q} - 1)$. Taking this inequality into account, we fix $s > 0$ such that

$$\frac{\tau}{n} < \frac{1}{s} < 1 - \frac{(\beta - 1)d\overline{q}}{\overline{q} - 1}\,. \tag{92}$$

Now, let m, $\sigma \in \mathbb{R}^n$ be such that

$$m_i = \frac{n\varepsilon}{\gamma}\,, \quad i = 1, \ldots, l, \tag{93}$$

$$m_i = t, \qquad i = l+1, \ldots, n, \tag{94}$$

$$\sigma_i = s, \qquad i = 1, \ldots, n. \tag{95}$$

Using (9), (10), (91), (93) and (94), we establish that

$$dp_m = 1, \tag{96}$$

and from (9), (10), (88), (89), (93) and (94) we derive that

$$\forall\, i \in \{1, \ldots, n\}, \qquad m_i \geqslant 1/(q_i - 1). \tag{97}$$

Moreover, using (82), (83), (93), (94), the inclusion $\theta \in \Omega$ and the inequality $\varepsilon < 1$, we establish that for every $i \in \{1, \ldots, n\}$, $1/\nu_i \in L^{m_i}(\Omega)$. This and (97) imply that condition (4) is satisfied. Finally, from (11), (92), (95) and (96) we deduce that

$$\forall\, i \in \{1, \ldots, n\}, \qquad \sigma_i > 0, \ \frac{1}{\sigma_i} < 1 - \frac{(q_i - 1)\overline{q}}{p_m(\overline{q} - 1)}\,,$$

and using (82), (83), (92), (95) and the inclusion $\theta \in \Omega$, we establish that for every $i \in \{1, \ldots, n\}$, $\nu_i \in L^{\sigma_i}(\Omega)$. Therefore, condition (5) is satisfied. Thus, assertion (a) holds. The proposition is proved.

Remark 7. From the statements of Propositions 1, 3 and 7 it follows that in the case of power weights defined by assertions (82) and (83), for the existence of m, $\sigma \in \mathbb{R}^n$ satisfying conditions (4) and (5), the necessary requirement on α and β is the same as in the nonweighted case and in the case of power weights with positive exponents, exactly $\alpha > \beta n/(2\beta + n - 2)$.

Proposition 8. *Let $\theta \in \Omega$, $\gamma, \tau > 0$, and suppose that assertions* (82) *and* (83) *hold. Then the following assertions are equivalent*:

(a) *there exist $m, \sigma \in \mathbb{R}^n$ with positive coordinates such that conditions* (6) *and* (7) *are satisfied*;

(b) *the inequality*

$$\gamma < \frac{\alpha(2\beta - 1)n - (3n - 2)\beta}{2\beta - 1} \tag{98}$$

and inequality (85) *are valid.*

Proof. Suppose that assertion (a) holds. Then assertion (a) of Proposition 7 holds. Therefore, by virtue of Proposition 7, inequality (85) is valid, and, according to the first part of the proof of Proposition 7, inequality (86) is also valid. Let $i \in \{1, \ldots, l\}$. By condition (6) and assumption (9), we have

$$\frac{1}{m_i} < \alpha - 1 - \frac{\overline{q}}{p_m(\overline{q} - 1)} \tag{99}$$

and $1/\nu_i \in L^{m_i}(\Omega)$. The latter inclusion along with (82) and the inclusion $\theta \in \Omega$ implies that $\gamma m_i < n$. From this and (99) we derive that $\frac{\gamma}{n} < \alpha - 1 - \frac{\overline{q}}{p_m(\overline{q}-1)}$. Hence, using (12) and (86), we infer that inequality (98) is valid. Thus, assertion (b) holds.

Conversely, suppose that assertion (b) holds. Then, by virtue of inequalities (98) and (85), we have

$$\gamma(2\beta-1)[\alpha(2\beta-1)n-(3n-2)\beta]^{-1} < 1,$$

$$\gamma(\beta-1)[\alpha(2\beta+n-2)-\beta n-\tau(2\alpha-1-\alpha/\beta)]^{-1} < 1.$$

Taking these inequalities into account, we fix a number $\varepsilon < 1$ such that

$$\gamma(2\beta-1)[\alpha(2\beta-1)n-(3n-2)\beta]^{-1} < \varepsilon,$$

$$\gamma(\beta-1)[\alpha(2\beta+n-2)-\beta n-\tau(2\alpha-1-\alpha/\beta)]^{-1} < \varepsilon.$$

In turn, taking into consideration the last two inequalities, we fix a number t such that

$$t > n\varepsilon/\gamma, \tag{100}$$

$$\frac{n}{t} < \alpha(2\beta-1)n-(3n-2)\beta-\frac{\gamma(2\beta-1)}{\varepsilon} \tag{101}$$

and inequality (90) is valid. Defining the number d by (91), from (12) and (90) we deduce that $\frac{\tau}{n} < 1 - \frac{(\beta-1)d\overline{q}}{\overline{q}-1}$. Taking this inequality into account, we fix $s > 0$ such that inequality (92) is valid. Now, let $m, \sigma \in \mathbb{R}^n$ be such that relations (93)–(95) hold. Using (9), (10), (91), (93) and (94), we establish that equality (96) is valid, and using (9), (11), (12), (91), (93), (94), (96), (100) and (101), we obtain that

$$\forall\, i \in \{1,\dots,n\}, \qquad \frac{\overline{q}}{p_m(\overline{q}-1)} < q_i - 1 - \frac{1}{m_i}. \tag{102}$$

Moreover, since $\theta \in \Omega$ and $\varepsilon < 1$, from (82), (83), (93) and (94) it follows that for every $i \in \{1,\dots,n\}$, $1/\nu_i \in L^{m_i}(\Omega)$. This and (102) imply that condition (6) is satisfied. Finally, using (11), (82), (83), (92), (95), (96) and the inclusion $\theta \in \Omega$, we establish that condition (7) is satisfied. Thus, assertion (a) holds. The proposition is proved.

Remark 8. From the statements of Propositions 2, 4 and 8 it follows that in the case of power weights defined by assertions (82) and (83), for the existence of $m, \sigma \in \mathbb{R}^n$ with positive coordinates satisfying conditions (6) and (7), the necessary requirement on α and β is the same as in the nonweighted case and in the case of power weights with positive exponents, and exactly is inequality (23).

Proposition 9. *Let $\theta \in \Omega$, $\gamma, \tau > 0$, and suppose that the following assertions hold*:

$$\textit{if } i \in \{1,\dots,l\}, \textit{ then for every } x \in \Omega \setminus \{\theta\},\ \nu_i(x) = |x|^{-\gamma}, \tag{103}$$

$$\textit{if } i \in \{l+1,\dots,n\}, \textit{ then for every } x \in \Omega,\ \nu_i(x) = |x|^{\tau}. \tag{104}$$

Then the following assertions are equivalent:

(a) *there exist $m, \sigma \in \mathbb{R}^n$ such that conditions* (4) *and* (5) *are satisfied*;

(b) *the inequalities*

$$\tau < \min\left\{n(\beta-1),\ \frac{\beta[\alpha(2\beta+n-2)-\beta n]}{\alpha(\beta-1)}\right\},$$

$$\gamma(2\beta-1-\beta/\alpha)+\tau(\alpha-1) < \beta(n-2)-\alpha(n-2\beta)$$

are valid.

The proof of this proposition is realized by analogy with the proof of Proposition 7.

Remark 9. From the statements of Propositions 5 and 9 it follows that in the case of power weights defined by assertions (64) and (65) and in the case of power weights defined by assertions (103) and (104), for the existence of $m,\sigma\in\mathbb{R}^n$ satisfying conditions (4) and (5), the necessary requirements on α and β are the same. At the same time, from the statements of Propositions 7 and 9 it follows that in the case of power weights defined by assertions (103) and (104), for the existence of $m,\sigma\in\mathbb{R}^n$ satisfying conditions (4) and (5), the necessary requirements on α and β in general are stronger than that in the case of power weights defined by assertions (82) and (83).

Proposition 10. *Let* $\theta\in\Omega$, $\gamma,\tau>0$, *and suppose that assertions* (103) *and* (104) *hold. Then the following assertions are equivalent*:

(a) *there exist* $m,\sigma\in\mathbb{R}^n$ *with positive coordinates such that conditions* (6) *and* (7) *are satisfied*;

(b) *the inequalities*

$$\tau<\alpha(2\beta-1)n-(3n-2)\beta,\qquad (2\alpha-1)\tau<\alpha(2\beta n-3n+2)-\beta n,$$

$$\frac{\tau}{\beta}<\frac{\alpha(2\beta+n-2)-\beta n}{\alpha(\beta-1)},$$

$$\gamma(2\beta-1-\beta/\alpha)+\tau(\alpha-1) < \beta(n-2)-\alpha(n-2\beta)$$

are valid.

The proof of this proposition is realized by analogy with the proof of Proposition 8.

Remark 10. From the statements of Propositions 6 and 10 it follows that in the case of power weights defined by assertions (64) and (65) and in the case of power weights defined by assertions (103) and (104), for the existence of $m,\sigma\in\mathbb{R}^n$ with positive coordinates satisfying conditions (6) and (7), the necessary requirements on α and β are the same (in this connection see also Remark 4). At the same time, from the statements of Propositions 8 and 10 it follows that in the case of power weights defined by assertions (103) and (104), for the existence of $m,\sigma\in\mathbb{R}^n$ with positive coordinates satisfying conditions (6) and (7), the necessary requirements on α and β in general are stronger than those in the case of power weights defined by assertions (82) and (83).

5. Theorems on solvability of the Dirichlet problem for some model cases. On the basis of Theorems 1 and 2 and Propositions 1–10 one can obtain a set of theorems on the existence of T-solutions and W-solutions of problem (2), (3) for model cases of the exponents q_i and the weighted functions ν_i considered in the previous section. For instance, the following results hold true.

Theorem 3. *Suppose that* $n = 2l$ *where* $l \in \mathbb{N}$, *and let* α *and* β *be numbers such that* $1 < \alpha \leqslant \beta < n$. *Assume that* $q_i = \alpha$ *if* $i \in \{1, \ldots, l\}$, *and* $q_i = \beta$ *if* $i \in \{l+1, \ldots, n\}$. *Let* $\theta \in \Omega$, $\gamma, \tau > 0$, *and suppose that assertions* (33) *and* (34) *hold and inequalities* (35) *and* (36) *are valid. Then there exists a* T*-solution of problem* (2), (3).

Theorem 4. *Suppose that* $n = 2l$ *where* $l \in \mathbb{N}$, *and let* α *and* β *be numbers such that* $1 < \alpha \leqslant \beta < n$. *Assume that* $q_i = \alpha$ *if* $i \in \{1, \ldots, l\}$, *and* $q_i = \beta$ *if* $i \in \{l+1, \ldots, n\}$. *Let* $\theta \in \Omega$, $\gamma, \tau > 0$, *and suppose that assertions* (33) *and* (34) *hold and inequalities* (46)–(48) *are valid. Then there exists a* W*-solution of problem* (2), (3).

Theorem 3 is a consequence of Proposition 3 and Theorem 1, and Theorem 4 follows from Proposition 4 and Theorem 2.

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Условия разрешимости задачи Дирихле для вырождающихся анизотропных эллиптических уравнений второго порядка с L^1-данными.

В статье рассмотрена задача Дирихле для класса вырождающихся анизотропных эллиптических уравнений второго порядка с L^1-правыми частями в ограниченном открытом множестве пространства $\mathbb{R}^n$ ($n \geqslant 2$). Этот класс характеризуется наличием набора показателей $q_1, \ldots, q_n$ и набора весовых функций $\nu_1, \ldots, \nu_n$ в условиях роста и коэрцитивности относительно коэффициентов уравнений. На основе общих теорем, полученных в одной из наших недавних работ, установлены условия существования T-решений и W-решений данной задачи в модельных случаях, где n – четное число, $q_1 = \ldots = q_{n/2} \leqslant q_{n/2+1} = \ldots = q_n$ и ν_i – некоторые степенные веса.

Ключевые слова: *вырождающиеся анизотропные эллиптические уравнения второго порядка,* L^1*-данные, задача Дирихле,* T*-решение,* W*-решение, условия существования решений.*

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Умови розв'язності задачі Діріхле для виродних анізотропних еліптичних рівнянь другого порядку з L^1-даними.

У статті розглянуто задачу Діріхле для класу виродних анізотропних еліптичних рівнянь другого порядку з L^1-правими частинами в обмеженій відкритій множині простору $\mathbb{R}^n$ ($n \geqslant 2$). Цей клас характеризується наявністю набору показників $q_1, \dots, q_n$ і набору вагових функцій $\nu_1, \dots, \nu_n$ в умовах зростання та коерцитивності відносно коефіцієнтів рівнянь. На основі загальних теорем, отриманих в одній з наших недавніх робіт, встановлено умови існування T-розв'язків і W-розв'язків даної задачі в модельних випадках, де n – парне число, $q_1 = \ldots = q_{n/2} \leqslant q_{n/2+1} = \ldots = q_n$ і ν_i – деякі степеневі ваги.

Ключові слова: *виродні анізотропні еліптичні рівняння другого порядку, L^1-дані, задача Діріхле, T-розв'язок, W-розв'язок, умови існування розв'язків.*

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