UDK 517.5

©2010. **D.A. Kovtonyuk, I.V. Petkov, V.I. Ryazanov**

THE BELTRAMI EQUATIONS AND LOWER Q-HOMEOMORPHISMS

In this article it is shown that each homeomorphic $W^{1,1}_{\rm loc}$ solution to the Beltrami equation $\overline{\partial}f = \mu\,\partial f$ is the so-called lower Q-homeomorphism with $Q(z) = K_\mu(z)$ where $K_\mu(z)$ is dilatation quotient of this equation. It is developed on this base the theory of the boundary behavior and the removability of singularities of such solutions.

Key words: *Beltrami equations, lower Q-homeomorphism*

1. Introduction. In this paper we present applications of our results on the so-called lower Q-homeomorphisms in the monograph [9] to the study of the boundary behavior of solutions for the Beltrami equations with degeneration.

Let D be a domain in the complex plane $\mathbb{C}$, i.e., a connected and open subset of $\mathbb{C}$, and let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. (almost everywhere) in D. The **Beltrami equation** is the equation of the form

$$f_{\overline{z}} \;=\; \mu(z) f_z \tag{1}$$

where $f_{\overline{z}} = \overline{\partial}f = (f_x + if_y)/2$, $f_z = \partial f = (f_x - if_y)/2$, $z = x + iy$, and f_x and f_y are partial derivatives of f in x and y, correspondingly. The function μ is called the **complex coefficient** and

$$K_\mu(z) \;=\; \frac{1 + |\mu(z)|}{1 - |\mu(z)|} \tag{2}$$

the **dilatation quotient** for the equation (1). The Beltrami equation (1) is said to be **degenerate** if $\operatorname{ess\,sup} K_\mu(z) = \infty$.

The existence theorem for homeomorphic $W^{1,1}_{\rm loc}$ solutions was established to many degenerate Beltrami equations, see, e.g., the recent monographs [1] and [9] and the surveys [6] and [13].

A continuous mapping γ of an open subset Δ of the real axis $\mathbb{R}$ or a circle into D is called a **dashed line**, see, e.g., Section 6.3 in [9]. Recall that every open set Δ in $\mathbb{R}$ consists of a countable collection of mutually disjoint intervals. This is the motivation for the term.

Given a family Γ of dashed lines γ in complex plane $\mathbb{C}$, a Borel function $\varrho : \mathbb{C} \to [0, \infty]$ is called **admissible** for Γ, write $\varrho \in \operatorname{adm}\Gamma$, if

$$\int\limits_{\gamma} \varrho\, ds \;\geqslant\; 1 \tag{3}$$

for every $\gamma \in \Gamma$. The **(conformal) modulus** of Γ is the quantity

$$M(\Gamma) \;=\; \inf_{\varrho \in \operatorname{adm}\Gamma} \int\limits_{\mathbb{C}} \varrho^2(z)\, dm(z) \tag{4}$$

where $dm(z)$ corresponds to the Lebesgue measure in $\mathbb{C}$. We say that a property P holds for **a.e.** (almost every) $\gamma \in \Gamma$ if a subfamily of all lines in Γ for which P fails has the modulus zero, cf. [3]. Later on, we also say that a Lebesgue measurable function $\varrho : \mathbb{C} \to [0, \infty]$ is **extensively admissible** for Γ, write $\varrho \in \operatorname{ext} \operatorname{adm} \Gamma$, if (3) holds for a.e. $\gamma \in \Gamma$, see, e.g., Section 9.2 in [9].

The following concept was motivated by Gehring's ring definition of quasiconformality in [4]. Given domains D and D' in $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, $z_0 \in \overline{D} \setminus \{\infty\}$, and a measurable function $Q : D \to (0, \infty)$, we say that a homeomorphism $f : D \to D'$ is a **lower Q-homeomorphism at the point** z_0 if

$$M(f\Sigma_\varepsilon) \geqslant \inf_{\varrho \in \operatorname{ext} \operatorname{adm} \Sigma_\varepsilon} \int\limits_{D \cap R_\varepsilon} \frac{\varrho^2(x)}{Q(x)}\, dm(x) \tag{5}$$

for every ring

$$R_\varepsilon = \{z \in \overline{\mathbb{C}} : \varepsilon < |z - z_0| < \varepsilon_0\}, \quad \varepsilon \in (0, \varepsilon_0),\ \varepsilon_0 \in (0, d_0),$$

where

$$d_0 = \sup_{z \in D} |z - z_0|,$$

and Σ_ε denotes the family of all intersections of the circles

$$S(r) = S(z_0, r) = \{z \in \mathbb{C} : |z - z_0| = r\}, \quad r \in (\varepsilon, \varepsilon_0),$$

with the domain D.

The notion can be extended to the case $z_0 = \infty \in \overline{D}$ in the standard way by applying the inversion T with respect to the unit circle in $\overline{\mathbb{C}}$, $T(x) = z/|z|^2$, $T(\infty) = 0$, $T(0) = \infty$. Namely, a homeomorphism $f : D \to D'$ is a **lower Q-homeomorphism at** $\infty \in \overline{D}$ if $F = f \circ T$ is a lower Q_*-homeomorphism with $Q_* = Q \circ T$ at 0. We also say that a homeomorphism $f : D \to \overline{\mathbb{C}}$ is a **lower Q-homeomorphism in** ∂D if f is a lower Q-homeomorphism at every point $z_0 \in \partial D$.

Further we show that each homeomorphic $W^{1,1}_{\mathrm{loc}}$ solution of the Beltrami equation (1) is a lower Q-homeomorphism with $Q(z) = K_\mu(z)$ and, thus, the whole theory of the boundary behavior in [7], see also Chapter 9 in [9], can be applied to such solutions. In other words, in the plane this holds for homeomorphisms with finite distortion by Iwaniec, see, e.g., related references in the monographs [1] and [9].

2. The main result.

Theorem. *Let f be a homeomorphic $W^{1,1}_{\mathrm{loc}}$ solution of the Beltrami equation (1). Then f is a lower Q-homeomorphism at each point $z_0 \in \overline{D}$ with $Q(z) = K_\mu(z)$.*

Proof. Let B be a (Borel) set of all points z in D where f has a total differential with $J_f(z) \neq 0$ a.e. It is known that B is the union of a countable collection of Borel sets B_l, $l = 1, 2, \ldots$, such that $f_l = f|_{B_l}$ is a bi-Lipschitz homeomorphism, see e.g. Lemma 3.2.2 in [2]. With no loss of generality, we may assume that the B_l are mutually disjoint. Denote also by B_* the set of all points $z \in D$ where f has a total differential with $f'(z) = 0$.

Note that the set $B_0 = D \setminus (B \cup B_*)$ has the Lebesgue measure zero in $\mathbb{C}$ by Gehring-Lehto-Menchoff theorem, see [5] and [11]. Hence by Theorem 2.11 in [8], see also Lemma 9.1 in [9], length$(\gamma \cap B_0) = 0$ for a.e. paths γ in D. Let us show that length$(f(\gamma) \cap f(B_0)) = 0$ for a.e. circle γ centered at z_0.

The latter follows from absolute continuity of f on closed subarcs of $\gamma \cap D$ for a.e. such circle γ. Indeed, the class $W^{1,1}_{\rm loc}$ is invariant with respect to local quasi-isometries, see e.g. Theorem 1.1.7 in [10], and the functions in $W^{1,1}_{\rm loc}$ is absolutely continuous on lines, see e.g. Theorem 1.1.3 in [10]. Applying say the transformation of coordinates $\log(z - z_0)$, we come to the absolute continuity on a.e. such circle γ.

Thus, length$(\gamma_* \cap f(B_0)) = 0$ where $\gamma_* = f(\gamma)$ for a.e. circle γ centered at z_0. Now, let $\varrho_* \in {\rm adm}\, f(\Gamma)$ where Γ is the collection of all dashed lines $\gamma \cap D$ for such circles γ and $\varrho_* \equiv 0$ outside $f(D)$. Set $\varrho \equiv 0$ outside D and

$$\varrho(z) \ : \ = \ \varrho_*(f(z))\,(|f_z| + |f_{\bar z}|) \qquad \text{for a.e.} \ \ z \in D$$

Arguing piecewise on B_l, we have by Theorem 3.2.5 under $m = 1$ in [2] that

$$\int\limits_{\gamma} \varrho\, ds \ \geqslant \ \int\limits_{\gamma_*} \varrho_*\, ds_* \ \geqslant \ 1 \qquad \text{for a.e.} \ \ \gamma \in \Gamma$$

because length$(f(\gamma) \cap f(B_0)) = 0$ and length$(f(\gamma) \cap f(B_*)) = 0$ for a.e. $\gamma \in \Gamma$, consequently, $\varrho \in {\rm ext\,adm}\,\Gamma$.

On the other hand, again arguing piecewise on B_l, we have the inequality

$$\int\limits_{D} \frac{\varrho^2(x)}{K_\mu(z)}\ dm(z) \ \leqslant \ \int\limits_{f(D)} \varrho_*^2(w)\ dm(w)$$

because $\varrho(z) = 0$ on B_*. Consequently, we obtain that

$$M(f\Gamma) \ \geqslant \ \inf_{\varrho \in {\rm ext\,adm}\,\Gamma} \int\limits_{D} \frac{\varrho^2(z)}{K_\mu(z)}\ dm(z)\,,$$

i.e., f is really a lower Q-homeomorphism with $Q(z) = K_\mu(z)$.

1. *Astala K., Iwaniec T. and Martin G.J.* Elliptic differential equations and quasiconformal mappings in the plane. – Princeton Math. Ser., v. 48, Princeton Univ. Press, Princeton, 2009.
2. *Federer H.* Geometric Measure Theory. – Springer-Verlag, Berlin 1969.
3. *Fuglede B.* Extremal length and functional completion // Acta Math. – 1957. – V. 98. – P. 171-219.
4. *Gehring F.W.* Rings and quasiconformal mappings in space // Trans. Amer. Math. Soc. – 1962. – V. 103. – P. 353-393.
5. *Gehring F.W., Lehto O.* On the total differentiability of functions of a complex variable // Ann. Acad. Sci. Fenn. A1. Math. – 1959. – V. 272. – P. 1-9.
6. *Gutlyanskii V., Ryazanov V., Srebro U. and Yakubov E.* On recent advances in the degenerate Beltrami equations // Ukraininan Math. Bull. – 2010. – V. 7, no. 4. – P. 467-515.
7. *Kovtonyuk D., Ryazanov V.* On the theory of lower Q-homeomorphisms // Ukrainian Math. Bull. – 2008. – V. 5, no. 2. – P. 157-181.

8. *Kovtonyuk D., Ryazanov V.* On the theory of mappings with finite area distortion // J. Anal. Math. – 2008. – V. 104. – P. 291-306.
9. *Martio O., Ryazanov V., Srebro U., Yakubov E.* Moduli in Modern Mapping Theory. – Springer, New York, 2009.
10. *Maz'ya V.* Sobolev Classes. – Springer-Verlag, Berlin, 1985.
11. *Menchoff D.* Sur les differentielles totales des fonctions univalentes // Math. Ann. – 1931. – V. 105. – P. 75-85.
12. *Ryazanov V., Srebro U. and Yakubov E.* BMO-quasiconformal mappings // J. d'Anal. Math. – 2001. – V. 83. – P. 1-20.
13. *Srebro U. and Yakubov E.* The Beltrami equation // Handbook in Complex Analysis. Geometric function theory. – 2005. – V. 2. – P. 555-597.

Д.А. Ковтонюк, И.В. Петков, В.И. Рязанов

Уравнения Бельтрами и нижние Q-гомеоморфизмы.

В работе показано, что любое гомеоморфное $W^{1,1}_{\rm loc}$ решение уравнения Бельтрами $\overline{\partial}f = \mu\,\partial f$ является так называемым нижним Q-гомеоморфизмом с $Q(z) = K_\mu(z)$, где $K_\mu(z)$ – коэффициент дилатации этого уравнения. На этой основе развита теория граничного поведения и устранимость сингулярностей таких решений.

Ключевые слова: *Уравнения Бельтрами, нижние Q-гомеоморфизмы*

Д.О. Ковтонюк, І.В. Пєтков, В.І. Рязанов

Рівняння Бельтрамі та нижні Q-гомеоморфізми.

У роботі показано, що будь-який гомеоморфний $W^{1,1}_{\rm loc}$ розв'язок рівняння Бельтрамі $\overline{\partial}f = \mu\,\partial f$ є так званим нижнім Q-гомеоморфізмом з $Q(z) = K_\mu(z)$, де $K_\mu(z)$ – коефіцієнт дилатації цього рівняння. На цій основі розвинуто теорію граничної поведінки і усунення сингулярностей таких розвязків.

Ключові слова: *Рівняння Бельтрамі, нижні Q-гомеоморфізми*

Ин-т прикл. математики и механики НАН Украины, Донецк
denis_kovtonyuk@bk.ru, igorpetkov@list.ru,
vlryazanov1@rambler.ru

Received 10.12.10