НОВІ МЕТОДИ В СИСТЕМНОМУ АНАЛІЗІ,
ІНФОРМАТИЦІ ТА ТЕОРІЇ ПРИЙНЯТТЯ РІШЕНЬ

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INITIAL TIME VALUE PROBLEM SOLUTIONS FOR EVOLUTION INCLUSIONS WITH S_k TYPE OPERATORS

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For a large class of operator inclusions, including those generated by maps of S_k type, we obtain a general theorem on existence of solutions. We apply this result to some particular examples. This theorem is proved using the method of Faedo-Galerkin.

INTRODUCTION

One of the most effective approach to investigate nonlinear problems, represented by partial differential equations, inclusions and inequalities with boundary values, consists in the reduction of them into differential-operator inclusions in infinite-dimensional spaces governed by nonlinear operators. In order to study these objects the modern methods of nonlinear analysis have been used [7, 8, 17, 28]. Convergence of approximate solutions to an exact solution of the differential-operator equation or inclusion is frequently proved on the basis of a monotony or a pseudomonotony of corresponding operator. In applications, as a pseudomonotone operator the sum of radially continuous monotone bounded operator and strongly continuous operator was considered [8]. Concrete examples of pseudomonotone operators were obtained by extension of elliptic differential operators when only their summands complying with highest derivatives satisfied the monotony property [17]. The papers of F. Browder and P. Hess [3, 4] became classical in the given direction of investigations. In particular in F. Browder and P. Hess work [4] the class of generalized pseudomonotone operators was introduced. Let W be real Banach space continuously embedded in real reflexive Banach space Y with dual space Y^*, $\langle\cdot,\cdot\rangle_Y : Y^* \times Y \to \mathbb{R}$ be the pairing. Further, as $C_v(Y^*)$ we consider the family of all nonempty closed convex bounded subsets of the space Y^*. Multi-valued map $A: Y \to C_v(Y^*)$ refers to be *generalized pseudomonotone on* W if for each pair of sequences $\{y_n\}_{n\ge 1} \subset W$ and $\{d_n\}_{n\ge 1} \subset Y^*$ such that $d_n \in A(y_n)$, $y_n \to y$ weakly in W, $d_n \to d$ weakly in Y^*, from the inequality

$$\overline{\lim_{n\to\infty}} \langle d_n, y_n\rangle_Y \le \langle d, y\rangle_Y$$

it follows that $d \in A(y)$ and $\langle d_n, y_n \rangle_Y \to \langle d, y \rangle_Y$. I.V. Skrypnik's idea of passing to subsequences in classical definitions [26], realized for stationary and evolution inclusions in M.Z. Zgurovsky, P.O. Kasyanov, V.S. Mel'nik and J. Valero papers (see [12–16], [18–21] and citations there) enabled to consider the class of w_{λ_0}-pseudomonotone maps which includes, in particular, a class of generalized pseudomonotone on W multi-valued operators and it is *closed within summing*. Let us remark that any multi-valued map $A: Y \to C_v(Y^*)$ naturally generates *upper* and, accordingly, *lower form*:

$$[A(y), \omega]_+ = \sup_{d \in A(y)} \langle d, w \rangle_Y, \quad [A(y), \omega]_- = \inf_{d \in A(y)} \langle d, w \rangle_Y, \quad y, \omega \in X .$$

Properties of the given objects have been investigated by M.Z. Zgurovsky and V.S. Mel'nik (see [16, 18, 21]). Thus, together with the classical coercivity condition for singlevalued maps

$$\frac{\langle A(y), y \rangle_Y}{\|y\|_Y} \to +\infty \quad \text{as} \quad \|y\|_Y \to +\infty$$

which ensures the important a priori estimations, arises +-coercivity (and, accordingly, –-coercivity) for multivalued maps

$$\frac{[A(y), y]_{+(-)}}{\|y\|_Y} \to +\infty, \quad \text{as} \quad \|y\|_Y \to +\infty.$$

+-coercivity is weaker condition than –-coercivity.

Recent development of the monotony method in the theory of differential-operator inclusions and evolutionary variational inequalities ensures resolvability of the given objects under the conditions of coercivity, quasiboundedness and the generalized pseudomonotony (see for example [5–6, 9–10, 23–25, 27] and citations there). V.S. Mel'nik's results [22] allows to consider evolution inclusions with $+$-coercive w_{λ_0}-pseudomonotone quasibounded multimappings (see [12]–[16], [31] and citations there).

In this paper we introduce the differential-operator scheme for investigation nonlinear boundary-value problems with summands complying with highest derivatives are not satisfied monotone condition. A multi-valued map $A: Y \to$ $\to C_v(Y^*)$ satisfies the *property* S_k *on* W, if for any sequence $\{y_n\}_{n \geq 0} \subset W$ such that $y_n \to y_0$ weakly in W, $d_n \to d_0$ weakly in Y^* as $n \to +\infty$, where $d_n \in A(y_n)$ $\forall n \geq 1$, from

$$\lim_{n \to \infty} \langle d_n, y_n - y_0 \rangle_Y = 0,$$

it follows that $d_0 \in A(y_0)$. Now we consider the simple example of S_k type operator. Let $\Omega = (0,1)$, $Y = H_0^1(\Omega)$ be the real Sobolev space with dual space $Y^* = H^{-1}(\Omega)$ (see for details [8]). Let $A: Y \times [-1,1] \to Y^*$ defined by the rule

$$A(y, \alpha) = -\frac{d}{dx}\left(\alpha \frac{d}{dx} y\right).$$

Then the multivalued map

$$\mathcal{A}(y)=\{A(y,\alpha)|\alpha\in[-1,1]\},\ y\in Y$$

satisfies the property S_k, it is +-coercive, but it is not –-coercive, it is not generalized pseudomonotone and $(-\mathcal{A})$ is not generalized pseudomonotone too (see [11] for detailes). We remark that stationary inclusions for multimaps with S_k properties were considered by V.O. Kapustyan, P.O. Kasyanov, O.P. Kogut [11], the evolution inclusions for $+$-coercive w_{λ_0}-pseudomonotone quasibounded maps by V.S. Mel'nik, P.O. Kasyanov, J. Valero (see [12]–[16], [31] and citations there). The obtained in this paper results are new results for evolution equations too.

PROBLEM DEFINITION

Let $(V_1,\|\cdot\|_{V_1})$ and $(V_2,\|\cdot\|_{V_2})$ be some reflexive separable Banach spaces, continuously embedded in the Hilbert space $(H,(\cdot,\cdot))$ such that

$$V:=V_1\cap V_2 \text{ is dense in spaces } V_1, V_2 \text{ and } H \tag{1}$$

After the identification $H\equiv H^*$ we get

$$V_1\subset H\subset V_1^*,\qquad V_2\subset H\subset V_2^*, \tag{2}$$

with continuous and dense embeddings [8], where $(V_i^*,\|\cdot\|_{V_i^*})$ is the topologically conjugate of V_i space with respect to the canonical bilinear form

$$\langle\cdot,\cdot\rangle_{V_i}:V_i^*\times V_i\to\mathbb{R}\ \ (i=1,2)$$

which coincides on $H\times V$ with the inner product $(\cdot,\cdot)$ on H. Let us consider the functional spaces

$$X_i=L_{r_i}(S;H)\cap L_{p_i}(S;V_i),$$

where $S=[0,T]$, $0<T<+\infty$, $1<p_i\le r_i<+\infty$ $(i=1,2)$. The spaces X_i are Banach spaces with the norms $\|y\|_{X_i}=\|y\|_{L_{p_i}(S;V_i)}+\|y\|_{L_{r_i}(S;H)}$. Moreover, X_i is a reflexive space.

Let us also consider the Banach space $X=X_1\cap X_2$ with the norm $\|y\|_X=\|y\|_{X_1}+\|y\|_{X_2}$. Since the spaces $L_{q_i}(S;V_i^*)+L_{r_i'}(S;H)$ and X_i^* are isometrically isomorphic, we identify them. Analogously,

$$X^*=X_1^*+X_2^*=L_{q_1}(S;V_1^*)+L_{q_2}(S;V_2^*)+L_{r_1'}(S;H)+L_{r_2'}(S;H),$$

where $r_i^{-1}+r_{i'}^{-1}=p_i^{-1}+q_i^{-1}=1$.

Let us define the duality form on $X^*\times X$

$$\langle f,y\rangle=\int_S(f_{11}(\tau),y(\tau))_H\,d\tau+\int_S(f_{12}(\tau),y(\tau))_H\,d\tau+\int_S\langle f_{21}(\tau),y(\tau)\rangle_{V_1}d\tau+$$

$$+\int_S\langle f_{22}(\tau),y(\tau)\rangle_{V_2}d\tau=\int_S(f(\tau),y(\tau))d\tau,$$

where $f = f_{11} + f_{12} + f_{21} + f_{22}$, $f_{1i} \in L_{r_i'}(S;H)$, $f_{2i} \in L_{q_i}(S;V_i^*)$. Remark, that for each $f \in X^*$

$$\|f\|_{X^*} = \inf_{\substack{f=f_{11}+f_{12}+f_{21}+f_{22}: \\ f_{1i}\in L_{r_i'}(S;H),\ f_{2i}\in L_{q_i}(S;V_i^*)(i=1,2)}} \max\Big\{ \|f_{11}\|_{L_{r_1}'(S;H)};$$

$$\|f_{12}\|_{L_{r_2}'(S;H)}; \|f_{21}\|_{L_{q_1}(S;V_1^*)}; \|f_{22}\|_{L_{q_2}(S;V_2^*)} \Big\}.$$

Following by [17], we may assume that there is a separable Hilbert space V_σ such that $V_\sigma \subset V_1$, $V_\sigma \subset V_2$ with continuous and dense embedding, $V_\sigma \subset H$ with compact and dense embedding. Then

$$V_\sigma \subset V_1 \subset H \subset V_1^* \subset V_\sigma^*, \quad V_\sigma \subset V_2 \subset H \subset V_2^* \subset V_\sigma^*$$

with continuous and dense embedding. For $i = 1,2$ let us set

$$X_{i,\sigma} = L_{r_i}(S;H) \cap L_{p_i}(S;V_\sigma), \quad X_\sigma = X_{1,\sigma} \cap X_{2,\sigma},$$

$$X_{i,\sigma}^* = L_{r_i'}(S;H) + L_{q_i}(S;V_\sigma^*), \quad X_\sigma^* = X_{1,\sigma}^* + X_{2,\sigma}^*,$$

$$W_{i,\sigma} = \{y \in X_i \mid y' \in X_{i,\sigma}^*\}, \qquad W_\sigma = W_{1,\sigma} \cap W_{2,\sigma}.$$

For multivalued (in general) map $A : X \rightrightarrows X^*$ let us consider such problem:

$$\begin{cases} u' + A(u) \ni f, \\ u(0) = a, u \in W \subset C(S;H), \end{cases} \tag{3}$$

where $a \in H$ and $f \in X^*$ are arbitrary fixed elements. The goal of this work is to prove the solvability for the given problem by the Faedo-Galerkin method.

THE CLASS $\mathcal{H}(X^*)$

Let us note that $B \in \mathcal{H}(X^*)$, if for an arbitrary measurable set $E \subset S$ and for arbitrary $u, v \in B$ the inclusion $u + (v - u)\chi_E \in B$ is true. Here and further for $d \in X^*$

$$(d\chi_E)(\tau) = d(\tau)\chi_E(\tau) \text{ for a.e. } \tau \in S, \quad \chi_E(\tau) = \begin{cases} 1, & \tau \in E, \\ 0, & \text{else.} \end{cases}$$

Lemma 1 [30]. $B \in \mathcal{H}(X^*)$ if and only if $\forall n \ge 1$, $\forall \{d_i\}_{i=1}^n \subset B$ and for arbitrary measurable pairwise disjoint subsets $\{E_j\}_{j=1}^n$ of the set S: $\cup_{j=1}^n E_j = S$ the following $\sum_{j=1}^n d_j \chi_{E_j} \in B$ is true.

Let us remark, that $\varnothing, X^* \in \mathcal{H}(X^*)$; $\forall f \in X^*$ $\{f\} \in \mathcal{H}(X^*)$; if $K : S \rightrightarrows V^*$ is an arbitrary multi-valued map, then

$$\{f \in X^* \mid f(t) \in K(t) \text{ for a.e. } t \in S\} \in \mathcal{H}(X^*).$$

At the same time for an arbitrary $v \in V^* \setminus \overline{0}$ that is not equal to 0 the closed convex set $B = \{f \in X^* \mid f \equiv \alpha v, \alpha \in [0,1]\} \notin \mathcal{H}(X^*)$, as $g(\cdot) = v \cdot \chi_{[0;T/2]}(\cdot) \notin B$.

CLASSES OF MULTI-VALUED MAPS

Let us consider now the main classes of multi-valued maps. Let Y be some reflexive Banach space, Y^* be its topologically adjoint, $\langle \cdot,\cdot \rangle_Y : Y^* \times Y \to \mathbb{R}$ be the pairing, $A : Y \rightrightarrows Y^*$ be the strict multi-valued map, i.e. $A(y) \neq \varnothing \quad \forall y \in X$. For this map let us define the upper $\|A(y)\|_+ = \sup_{d \in A(y)} \|d\|_{X^*}$ and the lower $\|A(y)\|_- = \inf_{d \in A(y)} \|d\|_{X^*}$ norms, where $y \in Y$. Let us consider the next maps which are connected with A: $\operatorname{co} A : Y \rightrightarrows Y^*$ and $\overline{\operatorname{co}}\, A : Y \rightrightarrows Y^*$, which are defined by the next relations $(\operatorname{co} A)(y) = \operatorname{co}(A(y))$ and $(\overline{\operatorname{co}}\, A(y)) = \overline{\operatorname{co}(A(y))}$ respectively, where $\overline{\operatorname{co}(A(y))}$ is the weak closeness of the convex hull of the set $A(y)$ in the space Y^*. It is known that strict multi-valued maps $A, B : Y \rightrightarrows Y^*$ have such properties [16, 18, 20]:

1) $[A(y), v_1 + v_2]_+ \le [A(y), v_1]_+ + [A(y), v_2]_+$,
$[A(y), v_1 + v_2]_- \ge [A(y), v_1]_- + [A(y), v_2]_- \quad \forall y,\ v_1,\ v_2 \in Y$;
2) $[A(y), v]_+ = -[A(y), -v]_-$,
$[A(y) + B(y), v]_{+(-)} = [A(y), v]_{+(-)} + [B(y), v]_{+(-)} \quad \forall y, v \in Y$;
3) $[A(y), v]_{+(-)} = [\overline{\operatorname{co}} A(y), v]_{+(-)} \quad \forall y, v \in Y$;
4) $[A(y), v]_{+(-)} \le \|A(y)\|_{+(-)} \|v\|_Y$, $\|A(y) + B(y)\|_+ \le \|A(y)\|_+ + \|B(y)\|_+$,

partially the inclusions $d \in \overline{\operatorname{co}}\, A(y)$ is true if and only if

$$[A(y), v]_+ \ge \langle d, v \rangle_Y \quad \forall v \in Y.$$

Let $D \subset Y$. If $a(\cdot,\cdot) : D \times Y \to \mathbb{R}$, then for every $y \in D$ the functional $Y \ni w \mapsto a(y, w)$ is positively homogeneous convex and lower semi-continuous if and only if there exists the multi-valued map $A : Y \rightrightarrows Y^*$ with the definition domain $D(A) = D$ such, that

$$a(y, w) = [A(y), w]_+ \quad \forall y \in D(A), \forall w \in Y.$$

Further, $y_n \rightharpoonup y$ in Y will mean, that y_n converges weakly to y in Y.

Let W be some normalized space that continuously embedded into Y. Let us consider multi-valued map $A : Y \rightrightarrows Y^*$.

Definition 1. The strict multi-valued map $A : Y \rightrightarrows Y^*$ is called:

- λ_0*-pseudomonotone on* W, if for any sequence $\{y_n\}_{n\ge 0}\subset W$ such, that $y_n \rightharpoonup y_0$ in W, $d_n \rightharpoonup d_0$ in Y^* as $n\to+\infty$, where $d_n \in \overline{\mathrm{co}}\, A(y_n)$ $\forall n\ge 1$, from the inequality

$$\overline{\lim_{n\to\infty}} \langle d_n, y_n - y_0\rangle_Y \le 0 \quad (4)$$

it follows the existence of subsequence $\{y_{n_k}, d_{n_k}\}_{k\ge 1}$ from $\{y_n, d_n\}_{n\ge 1}$, for that

$$\underline{\lim_{k\to\infty}} \langle d_{n_k}, y_{n_k} - w\rangle_Y \ge [A(y_0), y_0 - w]_- \quad \forall w\in Y \quad (5)$$

is fulfilled;

- *bounded*, if for every $L>0$ there exists such $l>0$, that

$$\forall\, y\in Y : \|y\|_Y \le L, \text{ it follows that } \|A(y)\|_+ \le l.$$

Definition 2. The strict multi-valued map $A: X \rightrightarrows X^*$ is called:

- *the operator of the Volterra type* , if for arbitrary $u, v\in X$, $t\in S$ from the equality $u(s)=v(s)$ for a.e. $s\in[0,t]$, it follows, that $[A(u),\xi_t]_+ = [A(v),\xi_t]_+$

$$\forall \xi_t \in X:\ \xi_t(s)=0 \text{ for a.e. } s\in S\setminus[0,t];$$

- +(-)-*coercive*, if there exists the real function $\gamma:\mathbb{R}_+\to\mathbb{R}$ such, that $\gamma(s)\to+\infty$ as $s\to+\infty$ and

$$[A(y),y]_{+(-)} \ge \gamma(\|y\|_Y)\|y\|_Y \quad \forall y\in Y;$$

- *demi-closed*, if from that fact, that $y_n\to y$ in Y, $d_n\rightharpoonup d$ in Y^*, where

$$d_n\in A(y_n),\ n\ge 1, \text{ it follows, that } d\in A(y).$$

Let us consider multi-valued maps, that act from X_m into X_m^*, $m\ge 1$. Let us remark, that embeddings $X_m\subset Y_m\subset X_m^*$ are continuous, and the embedding W_m into X_m is compact [17].

Definition 3. The multi-valued map $\mathcal{A}: X_m\to C_v(X_m^*)$ is called (W_m, X_m^*)-weakly closed, if from that fact, that $y_n\rightharpoonup y$ in W_m, $d_n\rightharpoonup d$ in X_m^*, $d_n\in\mathcal{A}(y_n)$ $\forall n\ge 1$ it follows, that $d\in\mathcal{A}(y)$.

Lemma 2. The multi-valued map $\mathcal{A}: X_m\to C_v(X_m^*)$ satisfies the property S_k on W_m if and only if $\mathcal{A}: X_m\to C_v(X_m^*)$ is (W_m, X_m^*)-weakly closed.

Proof. Let us prove the necessity. Let $y_n\rightharpoonup y$ in W_m, $d_n\rightharpoonup d$ in X_m^*, where $d_n\in\mathcal{A}(y_n)$ $\forall n\ge 1$. Then $y_n\to y$ in X_m and $\langle d_n, y_n - y\rangle_{X_m}\to 0$ as $n\to+\infty$. Therefore, in virtue of $\mathcal{A}$ satisfies the S_k property on W_m, we obtain, that $d\in\mathcal{A}(y)$.

Let us prove sufficiency. Let $y_n\rightharpoonup y$ in W_m, $d_n\rightharpoonup d$ in X_m^*, $\langle d_n, y_n - y\rangle_{X_m}\le 0$ as $n\to+\infty$, where $d_n\in\mathcal{A}(y_n)$ $\forall n\ge 1$. Then $y_n\to y$ in X_m and $d\in\mathcal{A}(y)$.

The lemma is proved.

Corollary 1. If the multi-valued map $\mathcal{A}: X_m \to C_v(X_m^*)$ satisfies the property S_k on W_m, then $\mathcal{A}$ is λ_0-pseudomonotone on W_m.

THE MAIN RESULTS

In the next theorem we will prove the solvability and justify the Faedo-Galerkin method for the problem (3).

Theorem 1. Let $a = \overline{0}$, $A: X \to C_v(X^*) \cap \mathcal{H}(X^*)$ be $+$-coercive bounded map of the Volterra type, that satisfies the property S_k on W_σ. Then for arbitrary $f \in X^*$ there exists at least one solution of the problem (3), that can be obtained by the Faedo-Galerkin method.

Proof. From +-coercivity for $A: X \rightrightarrows X^*$ it follows, that $\forall\, y \in X$

$$[A(y), y]_+ \geq \gamma(\|y\|_X)\|y\|_X .$$

So, $\exists\, r_0 > 0:\ \gamma(r_0) > \|f\|_{X^*} \geq 0.$ Therefore,

$$\forall y \in X: \|y\|_X = r_0 \quad [A(y) - f, y]_+ \geq 0. \tag{6}$$

The solvability of approximate problems.

Let us consider the complete vectors system $\{h_i\}_{i \geq 1} \subset V$ such that

α_1) $\{h_i\}_{i\geq 1}$ orthonormal in H;

α_2) $\{h_i\}_{i\geq 1}$ orthogonal in V;

α_3) $\forall\, i \geq 1\ (h_i, v)_V = \lambda_i (h_i, v) \quad \forall v \in V$,

where $0 \leq \lambda_1 \leq \lambda_2, ..., \lambda_j \to \infty$ as $j \to \infty$, $(\cdot,\cdot)_V$ is the natural inner product in V, i.e. $\{h_i\}_{i\geq 1}$ is a special basis [29]. Let for each $m \geq 1$ $H_m = \operatorname{span}\{h_i\}_{i=1}^m$, on which we consider the inner product induced from H that we again denote by $(\cdot,\cdot)$. Due to the equivalence of H^* and H it follows that $H_m^* \equiv H_m$; $X_m = L_{p_0}(S; H_m)$, $X_m^* = L_{q_0}(S; H_m)$, $p_0 = \max\{r_1, r_2\}$, $q_0 > 1$: $1/p_0 + 1/q_0 = 1$, $\langle\cdot,\cdot\rangle_{X_m} = \langle\cdot,\cdot\rangle_X |_{X_m^* \times X_m}$, $W_m := \{y \in X_m \mid y' \in X_m^*\}$, where y' is the derivative of an element $y \in X_m$ is considered in the sense of $\mathcal{D}^*(S, H_m)$. For any $m \geq 1$ let $I_m \in \mathcal{L}(X_m; X)$ be the canonical embedding of X_m in X, I_m^* be the adjoint operator to I_m. Then

$$\forall\, m \geq 1 \quad \|I_m^*\|_{\mathcal{L}(X_\sigma^*; X_\sigma^*)} = 1. \tag{7}$$

Let us consider such maps [12]:

$$A_m := I_m^* \circ A \circ I_m : X_m \to C_v(X^*), \ \ f_m := I_m^* f.$$

So, from (6) and corollary 1, applying analogical thoughts with [12], [14] we will obtain, that

j_1) A_m is λ_0-pseudomonotone on W_m;

j_2) A_m is bounded;

j_3) $[A_m(y)-f_m, y]_+ \ge 0 \quad \forall y \in X_m : \|y\|_X = r_0$.

Let us consider the operator $L_m : D(L_m) \subset X_m \to X_m^*$ with the definition domain

$$D(L_m) = \{y \in W_m \mid y(0) = \bar{0}\} = W_m^0,$$

that acts by the rule:

$$\forall y \in W_m^0 \quad L_m y = y',$$

where the derivative y' we consider in the sense of the distributions space $\mathcal{D}^*(S; H_m)$. From [12] for the operator L_m the next properties are true:

j_4) L_m is linear;

j_5) $\forall\, y \in W_m^0\ \langle L_m y, y\rangle \ge 0$;

j_6) L_m is maximal monotone.

Therefore, conditions j_1)–j_6) and the theorem 3.1 from [13] guarantees the existence at least one solution $y_m \in D(L_m)$ of the problem:

$$L_m(y_m) + A_m(y_m) \ni f_m, \quad \|y_m\|_X \le r_0,$$

that can be obtained by the method of singular perturbations. This means, that y_m is the solution of such problem:

$$\begin{cases} y_m' + A_m(y_m) \ni f_m \\ y_m(0) = \bar{0},\ y_m \in W_m, \|y_m\|_X \le R, \end{cases} \tag{8}$$

where $R = r_0$.

P a s s i n g t o t h e l i m i t.

From the inclusion from (8) it follows, that $\forall m \ge 1\ \exists d_m \in A(y_m)$:

$$I_m^* d_m = f_m - y_m' \in A_m(y_m) = I_m^* A(y_m). \tag{9}$$

1°. The boundedness of $\{d_m\}_{m\ge1}$ in X^* follows from the boundedness of A and from (8). Therefore,

$$\exists c_1 > 0 : \quad \forall m \ge 1 \quad \|d_m\|_{X^*} \le c_1. \tag{10}$$

2°. Let us prove the boundedness $\{y_m'\}_{m\ge1}$ in X_σ^*. From (9) it follows, that $\forall m \ge 1\ \ y_m' = I_m^*(f - d_m)$, and, taking into account (7), (8) and (10) we have:

$$\|y_m'\|_{X_\sigma^*} \le \|y_m\|_{W_\sigma} \le c_2 < +\infty. \tag{11}$$

In virtue of (8) and the continuous embedding $W_m \subset C(S; H_m)$ we obtain (see [24]) that $\exists c_3 > 0$ such, that

$$\forall m \ge 1, \forall t \in S \quad \|y_m(t)\|_H \le c_3. \tag{12}$$

3°. In virtue of estimations from (10)–(12), due to the Banach-Alaoglu theorem, taking into account the compact embedding $W \subset Y$, it follows the existence of subsequences

$$\{y_{m_k}\}_{k\ge1} \subset \{y_m\}_{m\ge1}, \quad \{d_{m_k}\}_{k\ge1} \subset \{d_m\}_{m\ge1}$$

and elements $y \in W$, $d \in X^*$, for which the next converges take place:

$$y_{m_k} \rightharpoonup y \text{ in } W, \quad d_{m_k} \rightharpoonup d \text{ in } X^*$$

$$y_{m_k}(t) \rightharpoonup y(t) \text{ in } H \text{ for each } t \in S, \tag{13}$$

$$y_{m_k}(t) \to y(t) \text{ in } H \text{ for a.e. } t \in S, \text{ as } k \to \infty.$$

From here, as $\forall k \ge 1$ $y_{m_k}(0) = \bar{0}$, then $y(0) = \bar{0}$.

4°. Let us prove, that

$$y' = f - d. \tag{14}$$

Let $\varphi \in D(S), n \in \mathbb{N}$ and $h \in H_n$. Then $\forall k \ge 1$: $m_k \ge n$ we have:

$$(\int_S \varphi(\tau)(y_{m_k}{}'(\tau) + d_{m_k}(\tau))d\tau, h) = \langle y_{m_k}{}' + d_{m_k}, \psi \rangle,$$

where $\psi(\tau) = h \cdot \varphi(\tau) \in X_n \subset X$. Let us remark, that here we use the property of Bochner integral [8](theorem IV.1.8, c.153). Since for $m_k \ge n$ $H_{m_k} \supset H_n$, then $\langle y_{m_k}{}' + d_{m_k}, \psi \rangle = \langle f_{m_k}, \psi \rangle$. Therefore, $\forall k \ge 1$: $m_k \ge n$

$$\langle f_{m_k}, \psi \rangle = \left(\int_S \varphi(\tau) f(\tau) d\tau, h \right).$$

Hence, for all $k \ge 1$: $m_k \ge n$

$$\left(\int_S \varphi(\tau) y'_{m_k}(\tau) d\tau, h \right) = \left\langle f - d_{m_k}, \psi \right\rangle \to$$

$$\to \left(\int_S \varphi(\tau)(f(\tau) - d(\tau) d\tau, h \right) \text{ as } k \to \infty. \tag{15}$$

The last follows from the weak convergence d_{m_k} to d in X^*.

From the convergence (13) we have:

$$\left(\int_S \varphi(\tau) y'_{m_k}(\tau) d\tau, h \right) \to \left(y'(\varphi), h \right) \text{ as } k \to \infty, \tag{16}$$

where

$$\forall \varphi \in \mathcal{D}(S) \;\; y'(\varphi) = -y(\varphi') = -\int_S y(\tau)\varphi'(\tau) d\tau.$$

Therefore, from (15) and (16) it follows, that

$$\forall \varphi \in \mathcal{D}(S)\ \forall h \in \bigcup_{m\ge 1} H_m \quad (y'(\varphi), h) = \left(\int_S \varphi(\tau)(f(\tau) - d(\tau))d\tau, h \right).$$

Since $\bigcup_{m\ge 1} H_m$ is dense in V we have, that

$$\forall \varphi \in \mathcal{D}(S) \quad y'(\varphi) = \int_S \varphi(\tau)(f(\tau) - d(\tau))d\tau.$$

Therefore, $y' = f - d \in X^*$.

5°. In order to prove, that y is the solution of the problem (3) it remains to show, that y satisfies the inclusion $y' + A(y) \ni f$. In virtue if identity (14), it is enough to prove, that $d \in A(y)$.

From (13) it follows the existence of $\{\tau_l\}_{l\ge 1} \subset S$ such that $\tau_l \nearrow T$ as $l \to +\infty$ and

$$\forall l \ge 1 \quad y_{m_k}(\tau_l) \to y(\tau_l) \ \text{ in } H \ \text{ as } k \to +\infty \tag{17}$$

Let us show that for any $l \ge 1$

$$\langle d, w\rangle \le [A(y), w]_+ \quad \forall w \in X : w(t) = 0 \ \text{ for a e. } t \in [\tau_l, T]. \tag{18}$$

Let us fix an arbitrary $\tau \in \{\tau_l\}_{l\ge 1}$. For $i = 1,2$ let us set

$$X_{i,\sigma}(\tau) = L_{r_i}(\tau, T; H) \cap L_{p_i}(\tau, T; V_\sigma), \quad X_\sigma(\tau) = X_{1,\sigma}(\tau) \cap X_{2,\sigma}(\tau),$$

$$X^*_{i,\sigma}(\tau) = L_{r_i'}(\tau, T; H) + L_{q_i}(\tau, T; V^*_\sigma), \quad X^*_\sigma(\tau) = X^*_{1,\sigma}(\tau) + X^*_{2,\sigma}(\tau),$$

$$W_{i,\sigma}(\tau) = \{y \in X_i(\tau) \mid y' \in X^*_{i,\sigma}(\tau)\}, \quad W_\sigma(\tau) = W_{1,\sigma}(\tau) \cap W_{2,\sigma}(\tau).$$

$$a_0 = y(\tau), \quad a_k = y_{m_k}(\tau), \quad k \ge 1.$$

Similarly we introduce $X(\tau)$, $X^*(\tau)$, $W(\tau)$. From (17) it follows that

$$a_k \to a_0 \ \text{ in } H \ \text{ as } k \to +\infty. \tag{19}$$

For any $k \ge 1$ let $z_k \in W(\tau)$ be such that

$$\begin{cases} z_k' + J(z_k) \ni \bar{0}, \\ z_k(\tau) = a_k, \end{cases} \tag{20}$$

where $J : X(\tau) \to C_v(X^*(\tau))$ be the duality (in general multivalued) mapping, i.e.

$$[J(u), u]_+ = [J(u), u]_- = \|u\|^2_{X(\tau)} = \|J(u)\|^2_+ = \|J(u)\|^2_-, \quad u \in X(\tau).$$

We remark that the problem (20) has a solution $z_k \in W(\tau)$ because J is monotone, coercive, bounded and demiclosed (see [1–2, 8, 13]). Let us also note that for any $k \ge 1$

$$\|z_k(T)\|_H^2 - \|a_k\|_H^2 = 2\langle z_k', z_k\rangle_{X(\tau)} + 2\|z_k\|_{X(\tau)}^2 = 0.$$

Hence,

$$\forall k \ge 1 \quad \|z_k'\|_{X^*(\tau)} = \|z_k\|_{X(\tau)} \le \frac{1}{\sqrt{2}}\|a_k\|_H \le c_3.$$

Due to (19), similarly to [8, 13], as $k \to +\infty$, z_k weakly converges in W to the unique solution $z_0 \in W$ of the problem (20) with initial time value condition $z(0) = a_0$. Moreover,

$$z_k \to z_0 \quad \text{in} \quad X(\tau) \quad \text{as} \quad k \to +\infty \tag{21}$$

because $\overline{\lim_{k\to+\infty}} \|z_k\|_{X(\tau)}^2 \le \|z_0\|_{X(\tau)}^2$, $z_k \rightharpoonup z_0$ in $X(\tau)$ and $X(\tau)$ is a Hilbert space.

For any $k \ge 1$ let us set

$$u_k(t) = \begin{cases} y_{m_k}(t), & \text{if } t \in [0,\tau], \\ z_k(t), & \text{elsewhere}, \end{cases} \qquad g_k(t) = \begin{cases} d_{m_k}(t), & \text{if } t \in [0,\tau], \\ \hat{d}_k(t), & \text{elsewhere}, \end{cases}$$

where $\hat{d}_k \in A(u_k)$ is an arbitrary. As $\{u_k\}_{k\ge1}$ is bounded, $A: X \rightrightarrows X^*$ is bounded, then $\{\hat{d}_k\}_{k\ge1}$ is bounded in X^*. In virtue of (21), (13), (17)

$$\lim_{k\to+\infty} \langle g_k, u_k - u\rangle = \lim_{k\to+\infty} \int_0^\tau \big(d_k(t), y_k(t) - y(t)\big)dt =$$

$$= \lim_{k\to+\infty} \int_0^\tau \big(f(t) - y_k'(t), y_k(t) - y(t)\big)dt = \lim_{k\to+\infty} \int_0^\tau \big(y_k'(t), y(t) - y_k(t)\big)dt =$$

$$= \lim_{k\to+\infty} \frac{1}{2}\Big(\|y_k(0)\|_H^2 - \|y_k(\tau)\|_H^2\Big) + \lim_{k\to+\infty} \int_0^\tau \big(y_k'(t), y(t)\big)dt =$$

$$= \frac{1}{2}\Big(\|y(0)\|_H^2 - \|y(\tau)\|_H^2\Big) + \int_0^\tau \big(y'(t), y(t)\big)dt = 0.$$

So,

$$\lim_{k\to+\infty} \langle g_k, u_k - u\rangle = 0. \tag{22}$$

Let us show that $g_k \in A(u_k)$ $\forall k \ge 1$. For any $w \in X$ let us set

$$w_\tau(t) = \begin{cases} w(t), & \text{if } t \in [0,\tau], \\ \bar{0}, & \text{elsewhere}, \end{cases} \qquad w^\tau(t) = \begin{cases} \bar{0}, & \text{if } t \in [0,\tau], \\ w(t), & \text{elsewhere}. \end{cases}$$

In virtue of A is the Volterra type operator we obtain that

$$\langle g_k, w\rangle = \langle d_{m_k}, w_\tau\rangle + \langle \hat{d}_k, w^\tau\rangle \le$$

$$\le [A(y_{m_k}), w_\tau]_+ + \langle \hat{d}_k, w^\tau\rangle =$$

$$= [A(u_k), w_\tau]_+ + \langle \hat{d}_k, w^\tau\rangle \le$$

$$\le [A(u_k), w_\tau]_+ + [A(u_k), w^\tau]_+ .$$

Due to $A(u_k) \in \mathcal{H}(X^*)$, similarly to [30], we obtain that

$$[A(u_k), w_\tau]_+ + [A(u_k), w^\tau]_+ = [A(u_k), w]_+ .$$

As $w \in X$ is an arbitrary, then $g_k \in A(u_k)$ $\forall k \ge 1$. Due to $\{u_k\}_{k\ge 1}$ is bounded in X, then $\{g_k\}_{k\ge 1}$ is bounded in X^*. Thus, up to a subsequence $\{u_{k_j}, g_{k_j}\}_{j\ge 1} \subset \{u_k, g_k\}_{k\ge 1}$, for some $u \in W$, $g \in X^*$ the next convergence takes place

$$u_{k_j} \rightharpoonup u \text{ in } W_\sigma, \quad g_{k_j} \rightharpoonup g \text{ in } X^* \text{ as } j \to \infty . \tag{23}$$

We remark that

$$u(t) = y(t), \quad g(t) = d(t) \quad \text{for a.e. } .t \in [0, \tau] . \tag{24}$$

In virtue of (22), (23), as A satisfies the property S_k on W_σ, we obtain that $g \in A(u)$. Hence, due to (24), as A is the Volterra type operator, for any $w \in X$ such that $w(t) = 0$ for a.e. $t \in [\tau, T]$ we have

$$\langle d, w\rangle = \langle g, w\rangle \le [A(u), w]_+ = [A(y), w]_+ ..$$

As $\tau \in \{\tau_l\}_{l\ge 1}$ is an arbitrary, we obtain (18).

From (18), due to the functional $w \to [A(y), w]_+$ is convex and lower semicontinuous on X (hence it is continuous on X) we obtain that for any $w \in X$ $\langle d, w\rangle \le [A(y), w]_+$. So, $d \in A(y)$.

The theorem is proved.

In a standard way (see [17]), by using the results of the theorem 1, we can obtain such proposition.

Corollary 2. Let $A : X \to C_v(X^*) \cap \mathcal{H}(X^*)$ be bounded map of the Volterra type, that satisfies the property S_k on W_σ. Moreover, let for some $c > 0$

$$\frac{[A(y), y]_+ - c\|A(y)\|_+}{\|y\|_X} \to +\infty \tag{25}$$

as $\|y\|_X \to +\infty$. Then for any $a \in H$, $f \in X^*$ there exists at least one solution of the problem (3), that can be obtained by the Faedo-Galerkin method.

Proof. Let us set $\varepsilon = \dfrac{\|a\|_H^2}{2c^2}$. We consider $w \in W$:

$$\begin{cases} w' + \varepsilon J(w) = \overline{0}, \\ w(0) = a, \end{cases}$$

where $J : X \to C_v(X^*)$ be the duality map. Hence $\|w\|_X \le c$. We define $\hat{A} : X \to C_v(X^*) \cap \mathcal{H}(X^*)$ by the rule: $\hat{A}(z) = A(z + w)$, $z \in X$. Let us set $\hat{f} = f - w' \in X^*$. If $z \in W$ is the solution of the problem

$$\begin{cases} z' + \hat{A}(z) \ni f, \\ z(0) = \overline{0}, \end{cases}$$

then $y = z + w$ is the solution of the problem (3). It is clear that $\hat{A}$ is a bounded map of the Volterra type, that satisfies the property S_k on W. So, due to the theorem 1, it is enough to prove the $+$-coercivity for the map $\hat{A}$. This property follows from such estimates:

$$[\hat{A}(z), z]_+ \geq [A(z+w), z+w]_+ - [A(z+w), w]_+ \geq$$

$$\geq [A(z+w), z+w]_+ - c\|A(z+w)\|_+,$$

$$\|z\|_X \geq \|z+w\|_X - c.$$

The corollary is proved.

Analyzing the proof of the theorem 1 we can obtain such result.

Corollary 3. *Let* $A : X \to C_v(X^*) \cap \mathcal{H}(X^*)$ be bounded map of the Volterra type, that satisfies the property S_k on W_σ, $\{a_n\}_{n\geq 0} \subset H$: $a_n \to a_0$ in H as $n \to +\infty$, $y_n \in W$, $n \geq 1$ be the corresponding to initial data a_n solution of the problem (3). If $y_n \rightharpoonup y_0$ in X, as $n \to +\infty$, then $y \in W$ is the solution of the problem (3) with initial data a_0. Moreover, up to a subsequence, $y_n \rightharpoonup y_0$ in $W_\sigma \cap C(S; H)$.

EXAMPLE

Let us consider the bounded domain $\Omega \subset \mathbb{R}^n$ with rather smooth boundary $\partial\Omega$, $S = [0, T]$, $Q = \Omega \times (0; T)$, $\Gamma_T = \partial\Omega \times (0; T)$. For $a, b \in \mathbb{R}$ we set $[a, b] = \{\alpha a + (1-\alpha) b | \alpha \in [0,1]\}$. Let $V = H_0^1(\Omega)$ be real Sobolev space, $V^* = H^{-1}(\Omega)$ be its dual space, $H = L_2(\Omega)$, $a \in H$, $f \in X^*$. We consider such problem:

$$\frac{\partial y(x,t)}{\partial t} + [-\Delta y(x,t), \Delta y(x,t)] \ni f(x,t) \quad \text{in } Q,$$

$$y(x,0) = a(x) \quad \text{in } \Omega,$$

$$y(x,t) = 0 \quad \text{in } \Gamma_T. \tag{26}$$

We consider $A : X \to C_v(X^*) \cap \mathcal{H}(X^*)$,

$$A(y) = \{\Delta y \cdot p | \, p \in L_\infty(S), |p(t)| \leq 1 \text{ a.e. in } S\}.$$

where Δ means the energetic extension in X of Laplacian (see [8] for details), $(\Delta y \cdot p)(x,t) = \Delta y(x,t) \cdot p(t)$ for a.e. $(x,t) \in Q$.

We remark that

$$\|A(y)\|_+ = \|y\|_X, [A(y), y]_+ = \|y\|_X^2. \tag{27}$$

We rewrite the problem (26) to the next one (see [8] for details):

$$y' + A(y) \ni f, \; y(0) = a. \qquad (28)$$

The solution of the problem (28) is called the generalized solution of (26). Due to the corollary 2 and (27), it is enough to check that A satisfies the property S_k on W. Indeed, let $y_n \rightharpoonup y$ in W, $d_n \rightharpoonup d$ in X^*, where $d_n = p_n \Delta y_n$, $p_n \in L_\infty(S)$, $|p_n(t)| \le 1$ for a.e. $t \in S$. Then $y_n \to y$ in Y and up to a subsequence $p_n \to p$ weakly star in $L_\infty(S)$, where $|p(t)| \le 1$ for a.e. $t \in S$. As $\|p_n \Delta y_n - p_n \Delta y\|_{L_2(S;H^{-2}(\Omega))} \le \|y_n - y\|_Y \to 0$, then $p_n \Delta y_n \to p \Delta y$ weakly in $L_2(S;H^{-2}(\Omega))$. Due to the continuous embedding $X^* \subset L_2(S;H^{-2}(\Omega))$ we obtain that $d = p \Delta y \in A(y)$. So, we obtain such statement.

Proposition 1. Under the listed above conditions the problem (26) has at least one generalized solution $y \in W$.

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