THE PAIR INTERACTION FORCES AND THE FRICTION AND DIFFUSION COEFFICIENTS OF PARTICLES IN MOMENTUM SPACE

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Diffusion processes in momentum space in the systems containing a large number of particles are considered. The friction coefficient and diffusion tensor are derived directly on the bases of the dynamics of individual particles motion under the action of the pair interaction forces from each of them. The expression for the frictional force in the case of pre-Brownian motion of particles with Coulomb interaction is obtained.

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INTRODUCTION

The transport phenomena and relaxation processes in the systems consisting of a large number of particles are studied using the kinetic equations (see, e.g., [1-6]). The collision integrals occurring in these equations are derived for specific studied physical processes. Thus for a fully ionized plasma the integral of collisions has been obtain in [1] by transformation of Boltzmann's collision integral. The general method of construction of the kinetic equations from equations of motion of particles has been given in [2].

The kinetic equations describe the evolution of particles systems on times greater than some characteristic time of particles motion randomization. Therefore the friction and diffusion of the charged particles in momentum space are investigated theoretically by means of such equations at a kinetic stage of the system evolution, when motion of particles is completely random. For a smaller time intervals, in case of pre-Brownian motion of the particles, the expression for mean square spread in momenta of Coulomb interacting nonrelativistic charged particles was derived in [7] based on the dynamics of particles motion. The same method was used to investigate the diffusion in momenta space at collisions of the relativistic charged particles [8]. The change of the mean square spread in momenta of the charged particles under the influence of their electromagnetic radiation in external periodic fields was investigated in [9-12].

In the given work the change in mean value of momentum of nonrelativistic particles in the absence of external fields is considered. The expression for the friction force describing average change of particles momentum per a time unit, both at the kinetic stage of evolution of a system, and at the initial stage in the case of pre-Brownian motion of particles is derived. The relationship between mean square spread in momenta of particles and the frictional force at the initial stage of evolution of system is analyzed.

1. FRICTION FORCE AND DIFFUSION COEFFICIENTS

Let's consider the system consisting of N identical, nonrelativistic particles, occupying volume V, whose motion complies with laws of classical mechanics with arbitrary interaction between particles. The equations of motion of the individual (test) particle we will write in the form

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}[x(t), t] = \sum_{s=1}^{N} \mathbf{F}^{(s)}[x(t), t, x_s(t, x_{0s})] \,, \qquad (1)$$

$$\frac{d\mathbf{r}}{dt} = \mathbf{v} \,, \qquad (2)$$

where $\mathbf{r}$ and $\mathbf{p}$ are the coordinates and momentum of particle, $\mathbf{p} = m\mathbf{v}$, m is a particle mass, $\mathbf{F}$ is microscopic force, $\mathbf{F}^{(s)}$ is the pair interaction force of particles, $x = \{\mathbf{r}, \mathbf{p}\}$ is the set of coordinates and momentum of particles, $x_{0s} = \{\mathbf{r}_{0s}, \mathbf{p}_{0s}\}$ is the coordinates and momentum of s-th particle at the initial instant t_0.

We will assume that pair interaction force between particles is known. Integration in the Eqs. (1) and (2) yields the expressions for the coordinates and momentum of particle

$$\mathbf{p}(t) = \mathbf{p}_0 + \int_{t_0}^{t} dt' \mathbf{F}[x(t'), t'] \,, \qquad (3)$$

$$\mathbf{r}(t) = \mathbf{r}^{(0)}(t) + \int_{t_0}^{t} dt'(t - t')\mathbf{F}[x(t'), t'] \,, \qquad (4)$$

where $\mathbf{r}^{(0)} = r_0 + \mathbf{v}_0(t - t_0)$.

Neglecting influence of average forces on motion of particles, we will consider small deviations of the coordinates and the momentum ($\Delta\mathbf{r}$, $\Delta\mathbf{p}$) from equilibrium values, where $\Delta\mathbf{r} = \mathbf{r} - \mathbf{r}^{(0)}$, $\Delta\mathbf{p} = \mathbf{p} - \mathbf{p}_0$. Expanding the expression for microscopic force into series on small deviations from equilibrium values in the right-hand side of Eq. (1), taking into account the Eqs. (3) and (4), we obtain the following equation for average change of the momentum of the test particle per unit time:

$$\left\langle \frac{d}{dt} p_i \right\rangle = \left\langle F_i\left[x^{(0)}(t), t\right] \right\rangle + \frac{1}{m} \int_{t_0}^{t} dt' L_j(x, t - t') \left\langle F_i^{(1)}(x, t) F_j^{(1)}\left[x^{(0)}(t'), t'\right] \right\rangle \Big|_{x = x^{(0)}(t)} \,, \qquad (5)$$

where $L_j(x,t) = t\frac{\partial}{\partial y_j} - \frac{\partial}{\partial \mathrm{v}_j}$, $x^{(0)} = \{\mathbf{r}^{(0)}, \mathbf{p}_0\}$, angular brackets mean the ensemble average, y_j is the Cartesian coordinates of the vector $\mathbf{r}$, j=1,2,3.

The first term in the Eq. (5) describes the change in momentum caused by influence of forces induced by the particle, as we neglect the influence of the average forces on particles motion. In particular this term corresponds to polarization losses by a charged particle passing through plasma. The second term in this equation describes the change in the momentum due to the fluctuation forces acting on the test particle from other particles. Below we will consider this frictional force.

The equation for diffusion coefficient in momenta space is

$$\frac{d}{dt}\left\langle \Delta p_i \Delta p_j \right\rangle = \int_{t_0}^{t} dt' \left\langle \delta F_i[\mathbf{r}(t),t] \cdot \delta F_j[\mathbf{r}(t'),t'] + \right.$$
$$\left. + \delta F_j[\mathbf{r}(t),t] \cdot \delta F_i[\mathbf{r}(t'),t'] \right\rangle ,$$

where $\delta \mathbf{F} = F - \langle \mathbf{F} \rangle$. (6)

The average values of product of microscopic forces and the space-time correlation function of fluctuations of the forces appear in the integrand on the right-hand side of the Eqs. (5) and (6), calculated by means of the function of dynamic state of considered system in 6N-dimensional phase space of coordinates and momenta of particles at the initial instant t_0 [7, 8], is

$$\left\langle F_i(x,t) F_j(x',t') \right\rangle = K_{ij}(t,t') +$$
$$+ (1 - 1/N) \int \mathrm{F}_i^{(1)}[x,t;x_1(t,x_{01})] \times$$
$$\mathrm{F}_j^{(1)}[x',t';x_2(t',x_{02})] f_2(x_{01},x_{02};t_0) dx_{01} dx_{02} , \quad (7)$$
$$\left\langle \delta F_i(t) \delta F_j(t') \right\rangle \equiv \left\langle F_i(t) F_j(t') \right\rangle - \left\langle F_i(t) \right\rangle \left\langle F_j(t') \right\rangle ,$$
$$K_{ij}(t,t') =$$
$$= \int \mathrm{F}_i^{(1)}[x,t;x_1(t,x_0)] \mathrm{F}_j^{(1)}[x',t';x_1(t',x_0)] f_1(x_0,t_0) dx_0 , \quad (8)$$

where $f_1(x_0,t_0)$ is the single-particle distribution function, $f_2(x_0,x_0',t_0)$ is the two-particle distribution function.

Using the principle of the correlations reduction at the initial instant t_0, we write

$$f_2(x_0,x_0',t_0) = f_1(x_0,t_0)\, f_1(x_0',t_0) .$$

Then the second term in the right-hand side of the formula (7) can be presented in the form of product of average values of the microscopic forces. The contribution of this term in the integrand on the right-hand side of the Eq. (5) can be neglected, as on the considered time interval the change of the average value of the force on *x* is small, if these forces are not equal to zero. Using the above mentioned assumptions, it is possible to present the Eqs. (5) and (6) in the following form:

$$A_i \equiv \left\langle \frac{d}{dt} p_i \right\rangle = \frac{1}{m} \int_{t_0}^{t} dt' L_j(x, t-t') K_{ij}(t,t') , \quad (9)$$

$$\frac{d}{dt}\left\langle \Delta p_i \Delta p_j \right\rangle = \int_{t_0}^{t} dt' \left[K_{ij}(t,t') + K_{ji}(t,t') \right] . \quad (10)$$

These formulae can be used to calculate the friction force and mean square spread in the momenta of particles at the specific nature of their interaction in the system.

2. COLLISIONS OF PARTICLES WITH COULOMB INTERACTION

We will consider the system of the charged particles with Coulomb interaction. The pair interaction force acting on the particle with the charge q in the coordinate $\mathbf{r}$ at the time t from the particle, moving on the trajectory $\mathbf{r}_s(t, x_{0s})$, we will present in the form of [13]

$$\mathbf{F}^{(s)}(\mathbf{r}, t, x_{0s}) = -q^2 \frac{\partial}{\partial \mathbf{r}} \frac{1}{|\mathbf{r} - \mathbf{r}_s(t, x_{0s})|} . \quad (11)$$

Let's consider the spatially homogeneous system on time intervals during which motion of particles does not change essentially. Then the expression for the trajectory of particle may be written as $\mathbf{r}_s = \mathbf{r}_{0s} + \mathbf{v}_{0s}(t - t_0)$. We will substitute the pair interaction force (11) in (8) and we will integrate on initial coordinates $d\mathbf{r}_0$, assuming that $N \to \infty$ and volume $V \to \infty$, so the density of particles $n = N/V$ is constant. Substituting the coordinates of test particle for its unperturbed trajectories, passing thus to differentiation on momentum [7, 8], we obtain the following expression for the friction force due to the fluctuating electric fields

$$A_i = \frac{\partial}{\partial p_j} \int_{t_0}^{t} dt' (t - t') K_{ij}(t,t') , \quad (12)$$

where

$$K_{ij} = q^4 \int J_{ij}(\xi) f_1(\mathbf{p}_0) d\mathbf{p}_0 ,$$

$$J_{ij} = \left(\delta_{ij} - 3\frac{u_i u_j}{u^2} \right) \left(\frac{\sqrt{\xi^2 + r_m^2}}{2\xi^2} - \frac{r_m^2}{2\xi^3} \ln \frac{\xi + \sqrt{\xi^2 + r_m^2}}{r_m} \right) +$$
$$+ \frac{u_i u_j}{u^2 \sqrt{\xi^2 + r_m^2}} ,$$

$\mathbf{u} = \mathbf{v} - \mathbf{v}_0$, $\xi = u(t - t')$, r_{min} is the minimum distance between two particles used to eliminate the divergence on integration over $d\mathbf{r}_0$ in Eq.(8) [7, 8].

After the integration on t′, the expression for the frictional force and change in time of the mean-square momentum deviation from equilibrium value, becomes

$$A_i = 2\pi q^4 \frac{\partial}{\partial p_j} \int d\mathbf{p}_0 f_1(\mathbf{p}_0) G_{ij}(u, \zeta) , \quad (13)$$

$$\frac{d}{dt}\left\langle \Delta p_i \Delta p_j \right\rangle = 4\pi q^4 \int d\mathbf{p}_0 f_1(\mathbf{p}_0) G_{ij}(u, \zeta) , \quad (14)$$

where

$$G_{ij} = \left(\delta_{ij} - 3\frac{u_i u_j}{u^2}\right)\frac{B(\zeta)}{u} + 2\frac{u_i u_j}{u^3}\ln\left(\zeta + \sqrt{\zeta^2+1}\right),$$

$$B(x) = \left(1 + \frac{1}{2x^2}\right)\ln\left(x + \sqrt{x^2+1}\right) - \frac{\sqrt{1+x^2}}{2x},$$

$\zeta = u\tau/r_{\min}$, $\tau = t - t_0$, δ_{ij} is the Kronecker delta.

3. DISCUSSION

The Eqs. (13) and (14) describe the change in time of the mean value of the momentum and mean square spread in the momentum of particles on all considered time interval, from the initial instant t_0.

Let's consider the initial evolution stage of the system, when time τ is less than characteristic time τ_0 randomization of particles motion in the considered system, where $\tau_0 = r_m/\bar{u}$, $\bar{u}$ is the mean speed of the particles. For small values of the time $\tau \ll \tau_0$ expanding the functions G_{ij} in powers of ς as far as the third-order terms, we find

$$G_{ij} = \frac{2\tau}{3r_{\min}}\delta_{ij} - \frac{\tau^3}{15r_{\min}^3}\left(u^2\delta_{ij} + 2u_i u_j\right). \qquad (15)$$

Substituting (15) in (13) and (14) we get:

$$\mathbf{A} = -\frac{4\pi q^4\tau^3}{3mr_{\min}^3}\int \mathbf{u} f_1(\mathbf{p}_0)d\mathbf{p}_0, \qquad (16)$$

$$\frac{d}{dt}\left\langle \Delta p_i \Delta p_j \right\rangle = \frac{8\pi q^4\tau}{3r_{\min}}\int d\mathbf{p}_0 f_1(\mathbf{p}_0)\left[\delta_{ij} - \right.$$
$$\left. - \frac{\tau^2}{10r_{\min}^2}\left(u^2\delta_{ij} + 2u_i u_j\right)\right] \cong \frac{8\pi q^4}{3r_{\min}} n\tau\,\delta_{ij}. \qquad (17)$$

It is easy to see that the dynamical friction experienced by the particles in the case of their pre-Brownian motion is defined by the second term in Eq. (15).

The mean square spread in the momentum of particles at this stage of the evolution of the system is governed mainly by the first term in the Eq. (15) and increases proportionally to the square of the time [7]

$$\left\langle (\Delta p)^2 \right\rangle = \frac{4\pi q^4}{r_{\min}} n\tau^2.$$

It should be noted that quadratic dependence of the mean-square spread in velocity of particles on time at the initial stage of the charged particles system evolution was observed in many numerical experiments [14].

From Eqs. (13) and (14) it follows that friction force and the change in the mean-square spread in momenta per unit time are connected by the relation

$$A_i = \frac{\partial}{2\partial p_j}\frac{d}{dt}\left\langle \Delta p_i \Delta p_j \right\rangle, \qquad (18)$$

which is valid for all times both at the pre-Brownian particles motion and the kinetic stage of the system evolution. For the kinetic stage such relation between frictional force and diffusion coefficients also follows from the collision integral.

For isotropic initial distribution of particles in momenta from the Eq. (16) follows that the more the velocity of the test particle, the more dynamical friction is experienced by this particle

$$\mathbf{A} = -\frac{4\pi q^4\tau^3 n}{3mr_{\min}^3}\mathbf{v},$$

where $n = \int f_1(p_0)d\mathbf{p}_0$.

At the kinetic stage of the system evolution of the charged particles with Coulomb interaction the frictional force decreases as the particle velocity increases [15].

From the formula (16) also follows that if there is a stream of particles in the system, the particles with the velocity higher than mean velocity of particles reduce the velocity, and the particles with velocity low than mean velocity increase the velocity.

At the pre-Brownian stage of the system evolution of the charged particles with Coulomb interaction the change in time of mean value of the momentum of the test particle due to the fluctuations of the field is proportional to the third power of time and can be smaller than the changes in time of a mean-square momenta spread at this stage, which is proportional to the time.

For $\tau \gg \tau_0$ at the kinetic stage of the system evolution, when motion of particles is completely random, asymptotic expression of the friction force (13) derived for $\varsigma \gg 1$, becomes

$$A_i = 2\pi q^4 \frac{\partial}{\partial p_j}\int d\mathbf{p}_0 f_1(\mathbf{p}_0) \times$$
$$\times\left[\left(\delta_{ij} - 3\frac{u_i u_j}{u^2}\right)\frac{1}{u}\left(\Lambda - \frac{1}{2}\right) + 2\frac{u_i u_j}{u^3}\Lambda\right], \qquad (19)$$

where $\Lambda = \ln\frac{2u\tau}{r_{\min}}$.

When $\Lambda \gg 1$ the expression (19) agrees with the corresponding formulas of [1].

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СИЛЫ ПАРНОГО ВЗАИМОДЕЙСТВИЯ И КОЭФФИЦИЕНТЫ ТРЕНИЯ И ДИФФУЗИИ ЧАСТИЦ В ПРОСТРАНСТВЕ ИМПУЛЬСОВ

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Рассмотрены процессы диффузии частиц в пространстве импульсов для системы, состоящей из большого числа частиц. Коэффициент трения и тензор диффузии получены исходя непосредственно из динамики движения отдельных частиц под действием сил парного взаимодействия со стороны каждой из них. Получено выражение для силы трения в случае предброуновского движения кулоновски взаимодействующих заряженных частиц.

СИЛИ ПАРНОЇ ВЗАЄМОДІЇ І КОЕФІЦІЄНТИ ТЕРТЯ І ДИФУЗІЇ ЧАСТИНОК У ПРОСТОРІ ІМПУЛЬСІВ

В.В. Огнівенко

Розглянуто процеси дифузії частинок у просторі імпульсів для системи, що складається з великої кількості частинок. Коефіцієнт тертя й тензор дифузії отримані виходячи безпосередньо з динаміки руху окремих частинок під дією сил парної взаємодії з боку кожної з них. Отримано вираз для сили тертя у випадку передброунівського руху кулонівськи взаємодіючих заряджених частинок.