BREAKING OF ENSEMBLES OF LINEAR AND NONLINEAR OSCILLATORS

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Some results concerning the study of the dynamics of ensembles of linear and nonlinear oscillators are stated. It is shown that, in general, a stable ensemble of linear oscillator has a limited number of oscillators. This number has been defined for some simple models. It is shown that the features of the dynamics of linear oscillators can be used for conversion of the low-frequency energy oscillations into high frequency oscillations. The dynamics of coupled nonlinear oscillators in most cases is chaotic. For such a case, it is shown that the statistical characteristics (moments) of chaotic motion can significantly reduce potential barriers that keep the particles in the capture region.

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INTRODUCTION

The ensembles of oscillators are convenient models for providing analysis of the dynamics of various physical systems. Such systems can, e.g., be ensembles of charged particles captured in various potentials, a set of eigenmodes in oscillatory systems of equivalent resonators, and many other physical systems. In general, the analysis of the dynamics of the ensemble is a complex task (e.g., [1]). The key question of this analysis is the question about the stability of the ensembles. In this paper some results are presented on the study of the dynamics of ensembles of linear and nonlinear oscillators. The most important results of the analysis can be summarized as follows: if there are a number of identical oscillators in the ensemble, and they can be grouped together, the ensemble of oscillators is transformed into an ensemble of oscillators *with a nonreciprocal connection,* and matrices of coefficients describing the kinetic and potential energy of the ensemble, ceases to be symmetrical. At that a variety of new dynamic processes can be realized, and most important of them is the possibility for such ensemble to be unstable. The criteria of instability were found. Some famous ensembles of nonlinear oscillators were investigated in detail. The most important result is that in the case when the dynamics is determined by a dynamic chaos, the oscillatory characteristics, including stability of such ensembles, are determined by the value of the statistical moments.

1. OF AN ENSEMBLE OF LINEAR OSCILLATORS

Suppose that we have a system with the Hamiltonian:

$$H=\sum_{i=0}^{N}\left(\frac{p_i^2}{2}+\omega_0^2\frac{q_i^2}{2}\right)+\mu q_0\cdot\sum_{j=1}^{N}q_j+A\sin\Omega t\cdot\sum_{j=0}^{N}q_j. \quad (1)$$

This system represents N coupled linear oscillators, which are subjected to the external periodic force. The amplitude of this force is equal to A, the frequency - Ω.

From (1) it is easy to obtain a system of equations for the oscillators:

$$\ddot{q}_i+\omega_0^2 q_i=-\mu\cdot q_0-A\cdot\sin\Omega t,$$
$$\ddot{q}_0+\omega_0^2 q_0=-\mu\cdot\sum_{i=1}^{N}q_i-A\cdot\sin\Omega t. \quad (2)$$

For simplicity, we consider a system where all oscillators are connected to each other only through the zero oscillator (Fig. 1,a). The normal frequency of such a system can be easily found. For this the solution of the system (2) when $A=0$ will be seeking in the form of:

$$q_i=a_i\exp(i\cdot\omega\cdot t),\ a_i=const. \quad (3)$$

Substituting this solution in (2), the dispersion equation can be obtained:

$$\left(-\omega^2+\omega_0^2\right)^2=\mu^2 N, \quad (4)$$

with the solution:

$$\omega=\pm\omega_0\sqrt{1\pm\mu\cdot\sqrt{N}/\omega_0^2}. \quad (5)$$

The signs + and – in the formula (5) before the root and under the root are independent. It is seen that even when the coupling coefficient is very small, but there is a large number of oscillators, one of the normal frequency can be very small (for the case of the sign "–" under the root). If inequality

$$\mu\cdot\sqrt{N}>\omega_0^2 \quad (6)$$

is fulfilled, such ensemble cannot exist and breaks down.

Suppose that the external periodic force with the frequency equals to the minimal normal frequency of the ensemble, acts on the ensemble. Then the amplitude of oscillations of the ensemble will be increasing linearly with time.

Now, if at some point in time (it is defined by the presence of decay and, accordingly, by saturation of the amplified oscillations) the connection is broken, the oscillator frequency will be a partial frequency. The amplitudes of the partial frequencies are much higher than the initial amplitude. An illustration of this fact are Figs. 2 and 3.

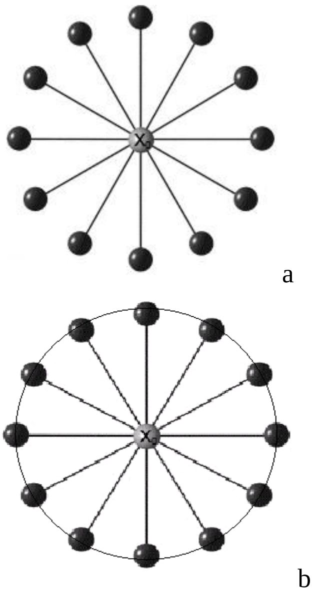


Fig. 1. Ensemble of oscillators. Connection occurs only through the central oscillator (a). Connection occurs through the central oscillator and between nearest neighbors (b)

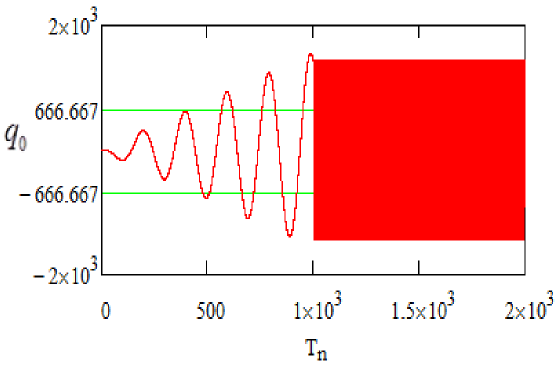


Fig. 2. The time evolution of the amplitude of the central oscillator. At time 10^3 the connection between the oscillators was torn

In these figures the time evolution of the amplitude of the central oscillator connected with nine identical oscillators are shown (see Fig. 1,a). The ensemble is under action of the external force. The dynamics of ensemble is described by equations (2). The parameters of this system were chosen as:

$\mu(t) = 0.333 \cdot \mathrm{H}(1000 - t)$, $A = -0.1$; $\Omega = 0.032$, $\omega_0 = 1$

with initial conditions equal to zero. Here $\mathrm{H}(t)$ - Heaviside function. The amplitude of the central oscillator is about 1500 (see Fig. 2). The amplitudes of the other oscillators are approximately three times lower (around 466). Note that the amplitudes of the partial oscillations depend on the moment of rupture of the connection. If to break the links at another time, for example, when the coupling coefficient is equal to $\mu(t) = 0.333 \cdot \mathrm{H}(937.67 - t)$, the amplitudes of the partial oscillations will be significantly smaller (see Fig. 3). These results show that with the help of the energy of low-frequency vibrations it is possible to excite efficiently vibrations of significantly higher frequency (in this example the frequency of the excited oscillations in more than thirty times exceeds the frequency of the external excitation).

If there is no external force ($A = 0$), but oscillators are arranged randomly in the vicinity of the bottom of the potential well, the oscillations of the oscillators are limited. However, if one slightly increases the coupling coefficient ($\mu = 0.3334$), the dynamics becomes unstable, and the breakdown criterion (6) does occur. The ensemble becomes destroyed.

The case considered above is a simplest model. The more realistic is the ensemble of oscillators presented in Fig .1,b. In this ensemble, as in the previous, all oscillators are connected to the central oscillator, but they are also bound with one of the nearest neighbors. In addition, the frequency of the central oscillator differs from the frequency of other oscillators. Then the system of equations describing the dynamics of such ensemble is:

$$\ddot{q}_i + \omega_0^2 q_i = -\mu \cdot q_0 - \mu_1 (q_{i+1} + q_{i-1}) - A \cdot \sin \Omega t,$$

$$\ddot{q}_0 + \omega_1^2 q_0 = -\mu \cdot \sum_{i=1}^{N} q_i - A \cdot \sin \Omega t \quad . \tag{7}$$

To find the conditions for existence of such ensemble it is convenient to rewrite the system (7) as:

$$\ddot{Q} + \omega^2 Q = -\mu \cdot N q_0,$$

$$\ddot{q}_0 + \omega_1^2 q_0 = -\mu \cdot Q \quad , \tag{8}$$

where $\omega^2 = \omega_0^2 + 2\mu_1$; $Q = \sum_{i=1}^{N} q_i$; $A = 0$.

Similarly to the way as the condition (6) was obtained, it is easy to find the condition for breaking of such ensemble, namely:

$$\mu^2 N > \omega^2 \omega_1^2 \quad . \tag{9}$$

All main results, obtained in the analysis of the previous model, are also valid for this model. Note that the ensemble of identical oscillators, in which each oscillator is connected to all other oscillators with the same bond, is *stable*.

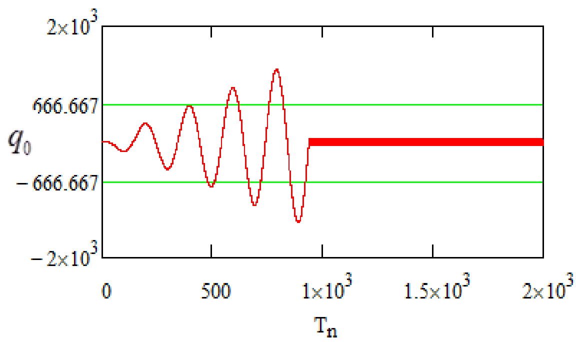


Fig. 3. The time evolution of the amplitude of the central oscillator. The moment of breaking the bonds is equal 937.67

2. DYNAMICS OF SYSTEMS OF NONLINEAR OSCILLATORS

The oscillations of the linear oscillators correspond to oscillations in an infinite parabolic potential well. The amplitudes of these oscillations are unlimited, and the limitations of amplitudes are associated with either attenuation or nonlinearities. Below we consider the effect of nonlinearities.

Let us consider an ensemble of coupled N nonlinear oscillators, which is under influence of external regular periodic force. The Hamiltonian of such system can be represented as:

$$H=\sum_j\sum_{i=1}^{N}\left[\frac{\dot{x}_i^2}{2}+\Phi(x_i)+G(x_i,x_j)-\varepsilon(\tau)\cdot x_i\right], \qquad (10)$$

to which the following system of second order equations corresponds:

$$\ddot{x}_i=F_0(x_i)+F_1(x_i,x_j)+\varepsilon(\tau)\ . \qquad (11)$$

The second term $F_1=-\sum_i^N \partial G/\partial x_i$ on the right side of the system describes the interaction between nonlinear oscillators. If it is missing, the system describes the dynamics of an ensemble of nonlinear oscillators independent from each other. The nature of the nonlinearity is determined by the function $F_0(x_i)=-\partial\Phi/\partial x_i$. The third term describes the external periodic force.

Let's assume, for definiteness, that each of the nonlinear oscillators is a charged particle that moves in a nonlinear potential. Suppose additionally that due to interaction between oscillators, or due to impacts of external regular force, their dynamics is chaotic. In this case, the displacement of each of these oscillators can be represented as:

$$x_i=\bar{x}+\delta_i\ , \qquad (12)$$

where $\bar{x}=\sum_{i=1}^{N}x_i/N$ is average coordinate nonlinear oscillators displacement; δ_i - is random deviation from the normal distribution, such, that $\langle\delta_i\rangle=0$. For an ensemble of oscillators the value $\bar{x}$ corresponds to the "the center of inertia" of the ensemble. With this assumption, the average value over the ensemble of functions F_0 and F_1 is convenient to decompose in a series of moments:

$$\langle F_0(\bar{x}+\delta_i)\rangle=F_0(\bar{x})+\sum_{n=1}^{\infty}\frac{M_n}{n!}\left(\frac{d}{dx_i}\right)^n\cdot F_0\,|_{\delta_i=0}\,, \qquad (13)$$

where $\mathrm{M}_n=\langle(\delta)^n\rangle$ are moments.

3. THE ENSEMBLES OF MATHEMATICAL PENDULUMS AND DUFFING OSCILLATORS

If every oscillator is a mathematical pendulum $F_0(x_i)=-\sin(x_i)$, the average value of this function will look like:

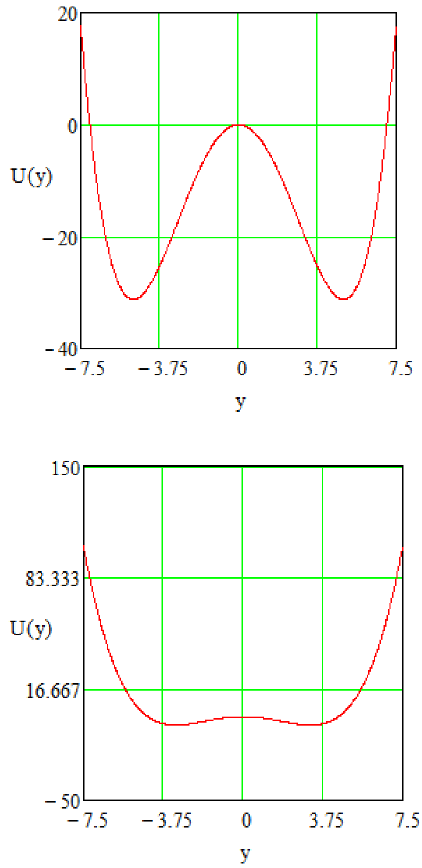
$$\langle F_0(x_i)\rangle=-\langle\sin(x_i)\rangle=-\left[1-\sum_{m=1}^{\infty}\frac{\mathrm{M}_{2m}}{(2m!)}\right]\sin\bar{x}\ .$$

For this particular case, the most important results can be obtained by the use of the formula (13): the characteristics of a mathematical pendulum, engaged in random motion, will be significantly different from the known characteristics. For illustration of this fact we start from the system (11) for finding an equation that describes the dynamics of the average deviation. For simplicity, the external force and the connection between oscillators are neglected:

$$\ddot{\bar{x}}+\left[1-\sum_{m=1}^{\infty}\frac{\mathrm{M}_{2m}}{(2m!)}\right]\sin\bar{x}=0\ . \qquad (14)$$

Equation (14) describes the dynamics of a mathematical pendulum. The potential of this mathematical pendulum, and accordingly, its oscillatory characteristics, do essentially depend on the statistical characteristics of the oscillators. Importantly, that the square of the frequency of small oscillations of a pendulum depends essentially on the moments of chaotic dynamics. It can be seen that this value is reducing with increasing moments. As a result, the depth of the potential well, in which the particles moves, does also reduce, and, thus, even small magnitude external forces can easily throw particles out from the capture area. The trapped particles are becoming untrapped. Especially visually this influence of the random dynamics is seen when one is considering the system of Duffing oscillators. For such oscillators $F_0(x_i)=\alpha\cdot x_i-\beta\cdot x_i^3$, $\alpha>0$ и $\beta>0$. The equation for the center of inertia will look as:

$$\ddot{\bar{x}}-(\alpha-3\cdot\beta\cdot M_2)\bar{x}+\beta\cdot\bar{x}^3=0\,. \qquad (15)$$



a

b

Fig. 4. Potential of Duffing oscillator at: M_2=0; β=0.2 (a) and M_2=5; β=0.2 (b)

Equation (15) is the Duffing equation. The potential of this equation is presented in Fig. 4. In Fig. 4,a the potential corresponds to a regular dynamics of the oscillator. In Fig. 4,b – the oscillator, the dynamics of which is chaotic. As can be seen, the chaotic dynamics can significantly change the potential. The presence of the second moment (variance) reduces the potential barrier between the minima. The particles move almost like in a single-well potential. Numerical calculations confirm all these results.

CONCLUSIONS

Now let's formulate the most important results of this work:

1. In the general case, the ensemble of linear oscillators may have a limited number of coupled oscillators. With the number of oscillators exceeding the critical number the ensembles are destroyed. From the above analysis it is clear that this is happening when the ensemble has some number of identical oscillators that can be combined into a group. The amplitudes of the normal vibrations of the stable ensemble are limited only by attenuation and nonlinearity.

2. The distinctive features of the dynamics of linear ensembles can be used to convert the energy of low-frequency oscillations into the energy of high-frequency oscillations (see also [2, 3]).

3. In most cases, the dynamics of coupled nonlinear oscillators is chaotic. It is also chaotic under influence of external perturbation on the oscillators.

In such cases the dynamics of ensemble is in many respects determined by statistical properties of chaotic motion of the oscillators.

The most significant feature of this dynamics is the reduction of particle capture area. Thus, it is easy to see from (14) that the width of the nonlinear resonance is significantly reduced to:

$$4\cdot\sqrt{\left[1-\sum_{m=1}^{\infty}\mathrm{M}_{2m}/(2m!)\right]}.$$

The motion of particles in the Duffing potential occurs, practically, without influence of a potential barrier which exists in the regular dynamics mode (compare Fig. 4,a and Fig. 4,b). It is easy to show that in this case, even small external disturbances may knock out particles from the capture area, i.e. turn the trapped particles into the untrapped particles.

REFERENCES

1. L.D. Landau, E.M. Lifshits. *Mechanics*. M.: "Science". 1965, p. 204.
2. V.A. Buts, D.M. Vavriv, O.G. Nechayev, D.V. Tarasov. A Simple Method for Generating Electromagnetic Oscillations *// IEEE Transactions on circuits and systems – II. Express Briefs*. 2015, v. 62, № 1, p. 36-40.
3. V.A. Buts. Mechanisms for increasing the frequency and degree of coherence of radiation // *Theoretical Problems Physics. Series "Problems of theoretical and mathematical physics. Proceedings. Scientific works". KhSU*. 2014, v. 1, p. 82-247.

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РАЗРУШЕНИЕ АНСАМБЛЕЙ ЛИНЕЙНЫХ И НЕЛИНЕЙНЫХ ОСЦИЛЛЯТОРОВ

В.А. Буц

Изложены некоторые результаты исследования динамики ансамблей линейных и нелинейных осцилляторов. Показано, что в общем случае устойчивый ансамбль линейных осцилляторов имеет ограниченное число осцилляторов. Для простых моделей определено это количество. Показано, что особенность динамики линейных осцилляторов может быть использована для преобразования энергии низкочастотных колебаний в высокочастотные. Динамика связанных нелинейных осцилляторов в большинстве случаев хаотична. В этом случае показано, что статистические характеристики (моменты) хаотического движения могут существенно уменьшать потенциальные барьеры, которые удерживают частицы в области захвата.

РУЙНУВАННЯ АНСАМБЛІВ ЛІНІЙНИХ ТА НЕЛІНІЙНИХ ОСЦИЛЯТОРІВ

В.О. Буц

Викладено деякі результати дослідження динаміки ансамблів лінійних та нелінійних осциляторів. Показано, що в загальному випадку стійкий ансамбль лінійних осциляторів має обмежене число осциляторів. Для простих моделей визначена ця кількість. Показано, що особливість динаміки лінійних осциляторів може бути використана для перетворення енергії низькочастотних коливань у високочастотні. Динаміка пов'язаних нелінійних осциляторів у більшості випадків хаотична. У цьому випадку показано, що статистичні характеристики (моменти) хаотичного руху можуть істотно зменшувати потенційні бар'єри, які утримують частинки в області захвату.