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Marginal dimensions for multicritical phase transitions

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The field-theoretical model describing multicritical phenomena with two coupled order parameters with $n_{||}$ and $n_{\perp}$ components and of $O(n_{\|})\oplus O(n_{\perp})$ symmetry is considered. Conditions for realization of different types of multicritical behaviour are studied within the field-theoretical renormalization group approach. Surfaces separating stability regions for certain types of multicritical behaviour in parametric space of order parameter dimensions and space dimension d are calculated using the two-loop renormalization group functions. Series for the order parameter marginal dimensions that control the crossover between different universality classes are extracted up to the fourth order in $\varepsilon = 4 - d$ and to the fifth order in a pseudo-ε parameter using the known high-order perturbative expansions for isotropic and cubic models. Special attention is paid to a particular case of $O(1)\oplus O(2)$ symmetric model relevant for description of anisotropic antiferromagnets in an external magnetic field.

Key words: *multicritical phenomena, marginal dimensions, renormalization group*

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1. Introduction

The concept of universality plays a paradigmatic role in the modern statistical physics. Accordingly, continuous phase transitions can be grouped into universality classes (see, e. g. [1]). Systems within the same universality class are characterized by the same set of critical exponents governing the scaling behaviour of their thermodynamical functions. Therefore, one of the aims of a theoretical description of a system is to establish its universality class.

In the theory of critical phenomena it is standard now to use methods of field theoretical renormalization group (RG) [2–5]. Within these methods, a stable fixed point (FP) corresponds to the universality class. For systems with complex internal symmetries described by ϕ^4 theories with several couplings, several different nontrivial FPs may exist. Depending on global parameters of a system, these FPs can interchange their stability causing the system to trigger from one universality class to another. The lack of a stable FP can even mean that a continuous phase order transition is transformed into a discontinuous. These global parameters (that effect the FP stability) are spatial dimension d and the dimension n of the order parameter (OP). In the n–d-space, the regions of stable FPs are separated by borders and the $n(d)$ curves define the OP *marginal dimensions* that control the crossover between different universality classes.

In this paper we are interested in the stability borders and marginal dimensions for a model with two coupled OP fields, namely, the model with $O(n_{\|}) \oplus O(n_{\perp})$ symmetry [6–8]. Such a model describes, amongst other systems [9], anisotropic antiferromagnets in an external magnetic field [10–16].

Conditions for realization of different types of multicritical behaviour, that are defined by the relation between the dimensions of the OPs $n_{\|}$, $n_{\perp}$, were obtained already in the first nontrivial approximation of the field-theoretical RG for $d < 4$ [8, 17, 18]. They determine the stability regions in the parametric

$n_\parallel - n_\perp$ plane for three FPs: *isotropic Heisenberg* FP of $O(n_\parallel + n_\perp)$ symmetry, *decoupled* FP at which OPs are ordering separately, and *biconical* FP. The two-loop studies in $d = 3$ show qualitatively similar results [16, 19], although significantly changing the quantitative picture in $n_\parallel - n_\perp$ plane. Five-loop results for three-dimensional $O(n_\parallel) \oplus O(n_\perp)$ model [20] confirm the obtained picture, producing only slight corrections.

Since the previous studies of multicritical behaviour in the $O(n_\parallel) \oplus O(n_\perp)$ system concentrated on $d = 3$ case, in this paper we consider the dependence of marginal dimensions of $O(n_\parallel) \oplus O(n_\perp)$ model on space dimension d. Our motivation is caused by the fact that even a small change in d can produce crucial effects on the critical behaviour, in particular, changing the universality class of a system. The rest of the paper is organized as follows: In section 2 we present the $O(n_\parallel) \oplus O(n_\perp)$ model and its RG description. Then, our aim is to analyse the conditions for realizing different scenarios of multicritical behaviour. In section 3, we present the results obtained within the two-loop approximation based on the ε-expansion as well as on the fixed d approach. We devote the next section 4 to the results for marginal dimensions of $O(n_\parallel) \oplus O(n_\perp)$ in higher order approximations. We end the paper with section 5 where our conclusions are presented.

2. The model and RG picture of its multicritical phenomena

The model with $O(n_\parallel) \oplus O(n_\perp)$ symmetry can be obtained from the well-known $O(n)$-symmetrical model [21], splitting its n-component OP $\vec{\phi}_0$ into two: $\vec{\phi}_{\perp 0}$ and $\vec{\phi}_{\parallel 0}$ that act in orthogonal subspaces with dimensions $n_\parallel$ and $n_\perp$, respectively ($n_\parallel + n_\perp = n$):

$$\vec{\phi}_0 = \begin{pmatrix} \vec{\phi}_{\perp 0} \\ \vec{\phi}_{\parallel 0} \end{pmatrix}. \tag{2.1}$$

Then, separating the Ginsburg-Landau-Wilson functional of $O(n)$ symmetry one can present the effective Hamiltonian of the $O(n_\parallel) \oplus O(n_\perp)$ model in the form:

$$\begin{aligned}\mathscr{H}_{\rm Bi} = \int {\rm d}^d x \Bigg[& \frac{1}{2}\mathring{r}_\perp \vec{\phi}_{\perp 0}\cdot\vec{\phi}_{\perp 0} + \frac{1}{2}\sum_{i=1}^{d} \nabla_i \vec{\phi}_{\perp 0}\cdot\nabla_i\vec{\phi}_{\perp 0} + \frac{1}{2}\mathring{r}_\parallel \vec{\phi}_{\parallel 0}\cdot\vec{\phi}_{\parallel 0} + \frac{1}{2}\sum_{i=1}^{d} \nabla_i \vec{\phi}_{\parallel 0}\cdot\nabla_i\vec{\phi}_{\parallel 0} \\ & + \frac{\mathring{u}_\perp}{4!}\left(\vec{\phi}_{\perp 0}\cdot\vec{\phi}_{\perp 0}\right)^2 + \frac{\mathring{u}_\parallel}{4!}\left(\vec{\phi}_{\parallel 0}\cdot\vec{\phi}_{\parallel 0}\right)^2 + \frac{2\mathring{u}_\times}{4!}\left(\vec{\phi}_{\perp 0}\cdot\vec{\phi}_{\perp 0}\right)\left(\vec{\phi}_{\parallel 0}\cdot\vec{\phi}_{\parallel 0}\right)\Bigg],\end{aligned} \tag{2.2}$$

where three couplings $\mathring{u}_\parallel$, $\mathring{u}_\perp$ and $\mathring{u}_\times$ should be introduced instead of the only one in the $O(n)$ symmetric model, and $\mathring{r}_\perp$ and $\mathring{r}_\parallel$ are connected with the temperature distance to the critical line for $\vec{\phi}_{\perp 0}$ and $\vec{\phi}_{\parallel 0}$, correspondingly.

The first mean-field analysis of the model with two coupled OPs was performed in order to describe the supersolids [22] (see also [17]). It shows that the character of the multicritical point in such a phase diagram depends on the sign of $\mathring{u}_\perp \mathring{u}_\parallel - \mathring{u}_\times^2$. For a positive sign, a tetracritical point is realized, while for a negative sign, it is a bicritical point. Going beyond the mean field theory, fluctuations should be taken into account. This is achieved by the field-theoretical RG approach [2], in which the large-scale behaviour of the system is connected with the stable FP of the RG transformations. The transformation of the fourth order couplings $\{\mathring{u}\}$ in (2.2) under renormalization is described by β-functions.

The β-functions for $O(n_\parallel) \oplus O(n_\perp)$ model were known in a one-loop approximation [8]. The next order approximation has been found in the massive [19] as well as in the minimal subtraction RG schemes [16]. In the minimal subtraction scheme, the β-functions were also calculated in the five-loop approximation [20], although explicit expressions were presented only for $O(3) \oplus O(2)$ symmetry [23]. Here, we work with β-functions obtained in two-loop order [16] within the minimal subtraction RG scheme [24, 25]:

$$\beta_{u_\perp} = -\varepsilon u_\perp + \frac{(n_\perp + 8)}{6}u_\perp^2 + \frac{n_\parallel}{6}u_\times^2 - \frac{(3n_\perp + 14)}{12}u_\perp^3 - \frac{5n_\parallel}{36}u_\perp u_\times^2 - \frac{n_\parallel}{9}u_\times^3, \tag{2.3}$$

$$\begin{aligned}\beta_{u_\times} = {} & -\varepsilon u_\times + \frac{(n_\perp + 2)}{6}u_\perp u_\times + \frac{(n_\parallel + 2)}{6}u_\times u_\parallel + \frac{2}{3}u_\times^2 - \frac{(n_\perp + n_\parallel + 16)}{72}u_\times^3 \\ & - \frac{(n_\perp + 2)}{6}u_\times^2 u_\perp - \frac{(n_\parallel + 2)}{6}u_\times^2 u_\parallel - \frac{5(n_\perp + 2)}{72}u_\perp^2 u_\times - \frac{5(n_\parallel + 2)}{72}u_\times u_\parallel^2,\end{aligned} \tag{2.4}$$

$$\beta_{u_\parallel} = -\varepsilon u_\parallel + \frac{(n_\parallel + 8)}{6}u_\parallel^2 + \frac{n_\perp}{6}u_\times^2 - \frac{(3n_\parallel + 14)}{12}u_\parallel^3 - \frac{5n_\perp}{36}u_\parallel u_\times^2 - \frac{n_\perp}{9}u_\times^3. \tag{2.5}$$

Here, $\{u_\perp, u_\times, u_\parallel\} = \{u\}$ are renormalized couplings and the space dimension d enters the β-functions via $\varepsilon = 4 - d$.

The FPs $\{u^*\}$ of the RG transformation are found from the zeros of the β-functions

$$\beta_{u_i}(\{u^*\}) = 0 \tag{2.6}$$

with $i = \perp, \parallel, \times$. A stable FP possesses positive eigenvalues ω_1, ω_2, ω_3 (or their real parts) of stability matrix $\partial\beta_i/\partial u_j$.

The stable FPs for $O(n_\parallel) \oplus O(n_\perp)$ are already known from the one-loop studies [8]. For $d < 4$ and for sufficiently low OP dimensions satisfying

$$n_\perp + n_\parallel < 4, \tag{2.7}$$

only the isotropic Heisenberg FP $\mathscr{H}$ of $O(n_\perp + n_\parallel)$ symmetry with $\{u_\perp^* = u_\times^* = u_\parallel^*\}$ is stable. When $n_\perp$ (or $n_\parallel$) increases breaking (2.7), still with

$$n_\perp n_\parallel + 2(n_\perp + n_\parallel) < 32, \tag{2.8}$$

FP $\mathscr{H}$ interchanges its stability with biconical FP $\mathscr{B}$ $\{u_\perp^* \neq u_\times^* \neq u_\parallel^*\}$. For values of $n_\perp$ and $n_\parallel$ that are above the condition (2.8), FP $\mathscr{B}$ looses its stability, while the decoupled FP $\mathscr{D}$ $\{u_\perp^* \neq 0, u_\times^* = 0, u_\parallel^* \neq 0\}$ becomes stable. According to these one-loop results, the multicritical behaviour of the $O(1) \oplus O(2)$ model is governed by FP $\mathscr{H}$ (connected with bicriticality) for all space dimensions $d < 4$. However, within the higher order calculations, the stability of FPs depends not only on $n_\parallel$, $n_\perp$ but also on d. Using resummation procedures for the two-loop RG functions at $d = 3$, one can show that the conditions of the FPs stability (2.7) and (2.8) are drastically shifted to smaller values of OP components [16, 19]. In particular, in the case $n_\parallel = 1$, $n_\perp = 2$ FP $\mathscr{B}$ (connected with tetracriticality) appears to be stable in a two loop order [16]. Resummation of higher orders ε-expansion [20] does not change this result.

3. Stability border-surfaces within a two-loop order approximation

As noted above, the stability of FPs $\mathscr{D}$, $\mathscr{B}$, $\mathscr{H}$ is dependent on three parameters $n_\parallel$, $n_\perp$ and d. Therefore, the borders between regions for which one or another FP is stable, form surfaces in the parametric space $n_\parallel - n_\perp - d$: $f(n_\parallel, n_\perp, d) = 0$. We call them *border-surfaces* (BSs).

Two alternative ways are used in practice to analyze RG functions and to get universal quantities, in particular, marginal dimensions. In one approach, i.e., the ε-expansion, the solutions are obtained as a series in ε and then they are evaluated at the value of interest (for instance, at $\varepsilon = 1$ for $d = 3$ theories). Alternatively, one may fix the space dimension d to a certain value and directly solve a system of non-linear equations obtaining the FP coordinates numerically [26]. In the next two subsections we use these approaches to obtain marginal dimensions of the $O(n_\parallel) \oplus O(n_\perp)$ model within a two-loop RG approximation.

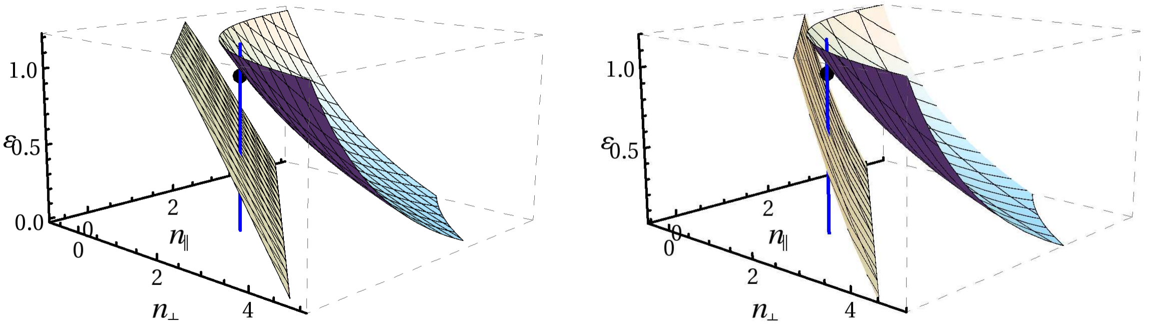
3.1. BSs from ε-expansion

We start our analysis with establishing the border between the stability regions of the decoupled FP $\mathscr{D}$ and the biconical FP $\mathscr{B}$. As it was noted in [16], two of the FP $\mathscr{D}$ stability exponents correspond to the stability exponent of the $O(n)$ model $\omega^{\mathscr{H}(n)}$: $\omega_1^{\mathscr{D}} = \omega^{\mathscr{H}(n_\parallel)}$, $\omega_3^{\mathscr{D}} = \omega^{\mathscr{H}(n_\perp)}$, while the remaining one is defined by

$$\omega_2^{\mathscr{D}} = \partial\beta_{u_\times}/u_\times\big|_{\mathscr{D}}\,. \tag{3.1}$$

Since $\omega^{\mathscr{H}(n)}$ is always positive, only $\omega_2^{\mathscr{D}}$ governs the stability of the FP $\mathscr{D}$, changing its sign depending on $n_\parallel$, $n_\perp$, d. Therefore, the surface between stability regions of FPs $\mathscr{D}$ and $\mathscr{B}$ can be extracted from the condition of (3.1) vanishing. Substituting the ε-expansion for the FP $\mathscr{D}$ coordinates into (3.1) one collects terms up to ε^2 and sets the result equal to zero:

$$\varepsilon\left[\frac{(13n_\parallel + 44)(n_\parallel + 2)}{2(n_\parallel + 8)^3} + \frac{(n_\perp + 2)(13n_\perp + 44)}{2(n_\perp + 8)^3}\right] + \left[\frac{n_\parallel - 4}{2(n_\parallel + 8)} + \frac{n_\perp - 4}{2(n_\perp + 8)}\right] = 0. \tag{3.2}$$



(a) (b)

Figure 1. (Color online) BSs between different universality classes of the $O(n_{||}) \oplus O(n_\perp)$ model obtained (a) by applying an ε-expansion and (b) by using a resummation procedure at a fixed d to two-loop RG functions. The left hand (lower) surface separates the stability region of FP $\mathscr{H}$ (on the left from the surface) and FP $\mathscr{B}$ (on the right from the surface). The right hand (upper) surface separates the stability regions of FP $\mathscr{B}$ (on the left from surface) and $\mathscr{D}$ (on the right from surface). The vertical line shows the position of a system with $n_\parallel = 1$, $n_\perp = 2$. The disc on the line indicates the position at $d = 3$.

This is analytically solved for $\varepsilon = \varepsilon(n_{||}, n_\perp)$. The result is shown as the right hand surface in figure 1 (a).

The BS between the regions of stability of the FPs $\mathscr{B}$ and $\mathscr{H}$ can be derived from the condition that FP $\mathscr{H}$ changes its stability. Only one of the three eigenvalues of the stability matrix changes its sign in the region considered. Calculating this eigenvalue up to the ε^2 order we get the equation for the surface:

$$\frac{\left[-5(n_\perp + n_\parallel + 8)^2 + 66(n_\perp + n_\parallel + 8) - 360\right]\varepsilon}{(n_\perp + n_\parallel + 8)^3} + \left(\frac{12}{n_\perp + n_\parallel + 8} - 1\right) = 0. \tag{3.3}$$

The surface is also shown in figure 1 (a) (the lower left hand surface).

The limiting borderlines in the plane $\varepsilon = 0$ ($d = 4$), are equivalent to the case when the one loop order inequalities (2.7), (2.8) are transformed into equalities, from which one obtains

$$n_\parallel^{\mathscr{D}}(n_\perp) = \frac{2(16 - n_\perp)}{n_\perp + 2}, \qquad n_\parallel^{\mathscr{H}}(n_\perp) = -n_\perp + 4. \tag{3.4}$$

The vertical line in figure 1 (a) presents a system with $n_\parallel = 2$, $n_\perp = 1$, indicating which FP governs the multicritical behavior of this system with the change of ε. Note that the FP $\mathscr{B}$ is stable in the region $0.51 \lesssim \varepsilon \lesssim 1.04$. We are interested in this case, since it describes anisotropic ferro- and antiferromagnets in space dimension $d = 3$.

3.2. BSs from resummed β-functions

Another way to obtain the BSs, is to calculate them from the β-functions (2.3)–(2.5) fixing d at certain values. Since the RG expansions have divergent [2] nature, the special resummation techniques are needed to get convergent results [29]. The two-loop β-functions (2.3)–(2.5) $\beta = \beta_{u_i}$ have a form of polynomials in renormalized couplings:

$$\beta(\{u\}) = \sum_{1\leqslant i,j,k\leqslant 3} c_{ijk} u_\perp^i u_\parallel^j u_\times^k . \tag{3.5}$$

We first represent (3.5) in the form of a resolvent series [30] in one auxiliary variable t:

$$F(\{u\}, t) = \sum_{1\leqslant i,j,k\leqslant 3} c_{ijk} u_\perp^i u_\parallel^j u_\times^k t^{i+j+k-1} = \sum_{0\leqslant\alpha\leqslant 2} a_\alpha(\{u\},\{c\}) t^\alpha, \tag{3.6}$$

where the expansion coefficients a_α in (3.6) explicitly depend on the couplings and on the coefficients c_{ijk} (3.5). Obviously, $F(\{u\}, 1) = \beta(\{u\})$. We resume the function (3.6) as a single variable function using the Padé-Borel technique [31] and writing its Borel image as:

$$F^{\mathrm{B}}(t) = \sum_{0\leqslant\alpha\leqslant 2} \frac{a_\alpha t^\alpha}{\alpha!}. \tag{3.7}$$

Analytical continuation of the function (3.7) is achieved by representing it in the form of a Padé approximant [32]. In our case, we use the diagonal Padé approximant [1/1]:

$$F^{\mathrm{B}}(t) \simeq [1/1](t). \tag{3.8}$$

Finally, the resummed function is obtained via an inverse Borel transform:

$$F^{\mathrm{res}} = \int_0^\infty [1/1](t)\mathrm{e}^{-t}. \tag{3.9}$$

Applying the above procedure to the two-loop β-functions (2.3)–(2.5) at a fixed d and searching for their FP solutions with $u_\times^* = 0$ and together with expression (3.1), where $\omega_2^{\mathscr{D}} = 0$, we find a BS, separating the stability region of the FP $\mathscr{D}$ from the FP $\mathscr{B}$ stability region. It is depicted as the upper surface in figure 1 (b). Searching for the FP solutions with $u_\perp^* = u_\times^* = u_\parallel^*$ at which the determinant of the stability matrix vanishes we derive the BS between the stability regions of the FPs $\mathscr{H}$ and $\mathscr{B}$. This is the upper left hand surface in figure 1 (b).

It is technically difficult to extract the data from the resummed function in the limit $\varepsilon \to 0$. Therefore, we present BSs for $0.002 \leqslant \varepsilon \leqslant 1.2$, and $n_\parallel, n_\perp$ in the range from –0.56 to 5. Limiting borderlines in the plane $\varepsilon \to 0$ described by (3.4) give us the one-loop (thin) borderlines of figure 1 of [16], while the intersections of the surfaces with the plane $\varepsilon = 1$ give the two-loop (thick) borderlines of figure 1 of [16].

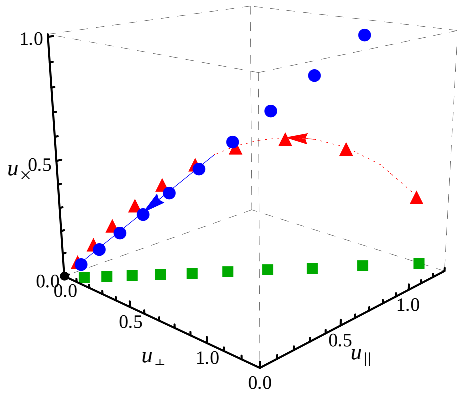


Figure 2. (Color online) Locations of the FPs $\mathscr{B}$ (triangles), $\mathscr{H}$ (discs) and $\mathscr{D}$ (squares) for $n_\perp = 2$, $n_\parallel = 1$ and ε changing from 1 (end right hand marks) to 0 with a step size of 0.1. Arrows show the direction in which ε decreases. Results are obtained using resummation (3.6)–(3.9). The line shows the track of a stable FP. Dotted part of the line indicates that FP $\mathscr{B}$ is stable, while the solid part indicates that FP $\mathscr{H}$ is stable. Black disc at the origin indicates the Gaussian FP.

Similarly to what we did it in figure 1 (a), we present in figure 1 (b) the line indicating the stability regions for FPs of the $O(1)\oplus O(2)$ model. In this approximation for the β-functions, FP $\mathscr{B}$ is stable in the region $0.66 \lesssim \varepsilon \lesssim 1.06$, in particular at $\varepsilon = 1$ ($d = 3$). Let us check how the FP picture changes along the line with an increasing d. Varying the space dimension d from 3 to 4 with stepsize 0.1 we can observe the drift of FPs $\mathscr{B}$, $\mathscr{H}$, $\mathscr{D}$ towards to the Gaussian FP. The tracks are shown in figure 2 by triangles, discs and squares indicating the change of locations of FPs $\mathscr{B}$, $\mathscr{H}$ and $\mathscr{D}$ in $u_\perp - u_{||} - u_\times$ space with the change of d. Numerical values of the coordinates of these FPs are listed in table 1. FP $\mathscr{B}$ is stable up to the intersection of traces of FP $\mathscr{B}$ and FP $\mathscr{H}$, which happens at $d \approx 3.34$, where it interchanges its stability with FP $\mathscr{H}$. Thus, only the FP $\mathscr{H}$ is stable starting from the intersection point and up to the Gaussian FP. This would mean that in higher space dimensions and at $n_{||} = 1$, $n_\perp = 2$, the phase diagram contains a bicritical point instead of a tetracritical point.

Table 1. Coordinates of the FPs $\mathscr{B}$, $\mathscr{H}$, and $\mathscr{D}$ at $n_\| = 1$, $n_\perp = 2$ depending on ε as found from the resumed two loop β-functions.

ε	$u_\perp^{\mathscr{B},*}$	$u_\|^{\mathscr{B},*}$	$u_\times^{\mathscr{B},*}$	$u^{\mathscr{H},*}$	$u_\perp^{\mathscr{D},*}$	$u_\|^{\mathscr{D},*}$
1.	1.1277	1.2874	0.3013	1.0016	1.1415	1.3146
0.9	0.9112	1.0039	0.5273	0.8434	0.9569	1.0971
0.8	0.7313	0.7739	0.5799	0.7026	0.7939	0.9063
0.7	0.5834	0.5934	0.5518	0.5771	0.6496	0.7386
0.6	0.4596	0.4502	0.4863	0.4650	0.5214	0.5906
0.5	0.3543	0.3348	0.4048	0.3647	0.4075	0.4600
0.4	0.2634	0.2405	0.3183	0.2749	0.3061	0.3444
0.3	0.1843	0.1627	0.2323	0.1944	0.2158	0.2420
0.2	0.1149	0.0982	0.1497	0.1223	0.1354	0.1513
0.1	0.0539	0.0446	0.0720	0.0578	0.0637	0.0710

4. High loop order results for marginal dimensions

The marginal dimensions of the $O(n_\|) \oplus O(n_\perp)$ models can be defined based on the high order RG results for simpler isotropic and cubic models. In particular, exact scaling arguments [18] connect the FP $\mathscr{D}$ stability with the critical exponents of the $O(n_\perp)$ and $O(n_\|)$ models:

$$\omega_2^{\mathscr{D}} = -\frac{1}{2}\left[\frac{\alpha(n_\perp)}{\nu(n_\perp)} + \frac{\alpha(n_\|)}{\nu(n_\|)}\right] = d - \frac{1}{\nu(n_\perp)} - \frac{1}{\nu(n_\|)}, \tag{4.1}$$

where $\alpha(n)$ and $\nu(n)$ are the heat capacity and correlation length critical exponents of the $O(n)$ model.

As it was indicated in [20], the stability of the FP $\mathscr{H}$ is defined by the marginal dimension n_c of the cubic model. Since in the FP $\mathscr{H}$, the RG functions depend only on the combination $n = n_\perp + n_\|$, the resulting marginal dimension can be presented in the form $n_\perp^{\mathscr{H}}(n_\|,\varepsilon) = n_\mathrm{c}(\varepsilon) - n_\|$.

In the following two subsections we present an analysis of the marginal dimensions $n_\perp^{\mathscr{D}}(n_\|,\varepsilon)$, $n_\perp^{\mathscr{H}}(n_\|,\varepsilon)$ based on the five-loop minimal subtraction series for the RG functions of isotropic [34] and cubic models [35], as well as for the case $d = 3$ based on the six-loop series for these models [36, 37] obtained within the massive scheme [38, 39].

4.1. Five-loop ε-expansions for marginal dimensions

Let us start with the calculation of $n_\perp^{\mathscr{D}}(n_\|,\varepsilon)$. Substituting the five-loop ε-expansions of the $O(n)$ theory [34] into (4.1) and putting $\omega_2^{\mathscr{D}}$ equal to zero, we get the equation for the BS. Keeping $n_\|$ as a parameter and expanding in ε, we get $n_\perp^{\mathscr{D}}(n_\|,\varepsilon)$ in the following form:

$$\begin{aligned}
n_\perp(n_\|,\varepsilon)\cdot(n_\|+2) \quad = \quad & 2(16-n_\|)-48\varepsilon+8\Big[3\zeta(3)\left(n_\|^2+34n_\|+100\right)+\left(n_\|^2+58n_\|+148\right)\Big]R_{n_\|}^2\varepsilon^2 \\
& +\Big\{\left(11n_\|^4-920n_\|^3-528n_\|^2+21376n_\|+51584\right)/3 \\
& -4\left(106592+64480n_\|+13548n_\|^2+1258n_\|^3+17n_\|^4\right)\zeta(3)/3 \\
& +\Big[18\left(100+34n_\|+n_\|^2\right)\zeta(4)-40\left(550+163n_\|+7n_\|^2\right)\zeta(5)/3\Big]R_{n_\|}^{-2}\Big\}R_{n_\|}^4\varepsilon^3 \\
& +\Big[\left(3n_\|^6+170n_\|^5-43120n_\|^4-442864n_\|^3-2069072n_\|^2-4512896n_\|-3457280\right)/6 \\
& -50\left(550+163n_\|+7n_\|^2\right)\zeta(6)R_{n_\|}^{-4}/3
\end{aligned}$$

$$+\zeta(3)\left(1816192+5011904n_\parallel+3131936n_\parallel^2+623376n_\parallel^3+41740n_\parallel^4+1486n_\parallel^5-n_\parallel^6\right)/3$$
$$-4\left(151424+131552n_\parallel+15728n_\parallel^2-1250n_\parallel^3-17n_\parallel^4-5n_\parallel^5\right)\zeta^2(3)R_{n_\parallel}^{-1}/3$$
$$-\left(106592+64480n_\parallel+13548n_\parallel^2+1258n_\parallel^3+17n_\parallel^4\right)\zeta(4)R_{n_\parallel}^{-2}$$
$$+\left(4822640+2331088n_\parallel+416334n_\parallel^2+51745n_\parallel^3+1103n_\parallel^4\right)\zeta(5)R_{n_\parallel}^{-2}/9$$
$$+49\left(66320+31792n_\parallel+5826n_\parallel^2+535n_\parallel^3+17n_\parallel^4\right)\zeta(7)R_{n_\parallel}^{-2}/2\Bigg]R_{n_\parallel}^6\varepsilon^4, \tag{4.2}$$

where $R_n=(n+8)^{-1}$. Expressions for certain physical values of $n_\parallel$ are less cumbersome:

$$n_\perp(1,\varepsilon) = 10.-16.\varepsilon+22.84224\varepsilon^2-44.06758\varepsilon^3+113.6428\varepsilon^4, \tag{4.3}$$
$$n_\perp(2,\varepsilon) = 7.-12.\varepsilon+17.76523\varepsilon^2-34.18402\varepsilon^3+84.07657\varepsilon^4, \tag{4.4}$$
$$n_\perp(3,\varepsilon) = 5.2-9.6\varepsilon+14.43837\varepsilon^2-28.00490\varepsilon^3+67.23923\varepsilon^4. \tag{4.5}$$

The obtained ε-expansion diverges, as it can be seen from the growth of the expansion coefficients in (4.3)–(4.5) as well as it follows from the Padé table [32] for $n_\perp(1,1)$:

$$n_\perp(1,1)=\begin{pmatrix} 10.0000 & 3.8462 & 3.4773 & 2.4576 & 9.6637\\ -6.0000 & 3.4092 & 3.9728 & 3.1128 & o\\ 16.8422 & 1.7981 & 2.8846 & o & o\\ -27.2253 & 4.5288 & o & o & o\\ 86.4175 & o & o & o & o \end{pmatrix}. \tag{4.6}$$

The element MN of the table (4.6) is the value of $n_\perp(1,1)$ given by the $[M/N]$ Padé approximant at $\varepsilon=1$. Here and below, symbol o denotes the approximants which can not be constructed within the order of perturbation theory considered here. Usually, the best convergence of the results is observed along the main diagonal and the closest sub-diagonals of the Padé table [32]. However, it appears that the value of $n_\perp(1,1)$ given by the Padé-aproximant [2/2] differs from those given by [1/2] and [2/1] by an order of one, leading to an uncertainty of the numerical estimate.

To obtain a reliable estimate of $n_\perp(1,\varepsilon)$ we rely on the Padé-Borel resummation described in subsection 3.2. We obtain a resolvent series by a substitution $\varepsilon\to\varepsilon t$. For the obtained expression we build the Borel-image, then approximate it by the [3/1] Padé approximant. Performing an integration of the inverse Borel transform, we arrive at the result shown in figure 3 with a solid line. In a similar way, we

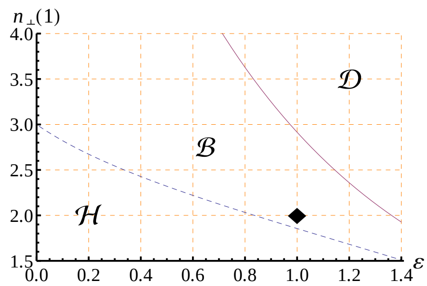


Figure 3. (Color online) The dependence of the marginal dimensions $n_\perp^{\mathcal{D}}(1)$ (solid line) and $n_\perp^{\mathcal{H}}(1)$ (dashed line) on ε. The results are obtained based on the five-loop expansion for isotropic and cubic models using Padé-Borel resummation with [3/1] Padé approximant (see the text). The diamond denotes the location of the three-dimensional $O(1)\oplus O(2)$ system.

get $n_\perp^{\mathcal{H}}(n_\parallel,\varepsilon)$ using the available five-loop ε-expansion for the marginal dimension of the cubic model [35]:

$$n_c = 4-2\varepsilon+\left(-\frac{5}{12}+\frac{5\zeta(3)}{2}\right)\varepsilon^2+\left(\frac{15\zeta(4)}{8}-\frac{25\zeta(5)}{3}+\frac{5\zeta(3)}{8}-\frac{1}{72}\right)\varepsilon^3$$
$$+\left(\frac{15\zeta(4)}{32}-\frac{125\zeta(6)}{12}+\frac{11515\zeta(7)}{384}-\frac{3155\zeta(5)}{1728}-\frac{229\zeta(3)^2}{144}+\frac{93\zeta(3)}{128}-\frac{1}{384}\right)\varepsilon^4. \tag{4.7}$$

The dependence of $n_{\perp}^{\mathscr{H}}(1,\varepsilon) = n_{\mathrm{c}}(\varepsilon) - 1$ on ε is obtained as above by the Padé-Borel resummation with [3/1] Padé approximant. The result is shown in figure 3 with a dashed line.

As it can be seen from figure 3, that FP $\mathscr{B}$ for the $O(1) \oplus O(2)$ model is stable in the region $0.84 \lesssim \varepsilon \lesssim 1.36$. The value of $n_{\perp} = 2$, $d = 3$ (denoted by a diamond) is located very close to the boundary $n_{\perp}^{\mathscr{H}}(1,\varepsilon)$. Thus, one concludes a very slow approach to the FP. Measurements in $O(1) \oplus O(2)$ systems may show an effective critical behaviour with the exponents close to the $O(3)$ case [16]. Anyway, in recent Monte-Carlo simulations of the Heisenberg ferromagnet with uniaxial exchange anisotropy, only a bicritical point with Heisenberg symmetry was obtained [40].

4.2. Marginal dimension for of $d = 3$ in a six-loop order

Fixing the spatial dimension to $d = 3$, we can analyze $n_{\perp}^{\mathscr{D}}(n_{\|})$ using pseudo-ε expansions (for details see [33]). Introducing the pseudo-ε parameter τ into 6-loop RG functions of the massive scheme at a fixed $d = 3$ [37] for the $O(n)$ model, one can derive critical exponents in the form of pseudo-ε expansions and substitute them into equation (4.1). Similar to the former subsection, we extract the pseudo-ε expansion for $n_{\perp}^{\mathscr{D}}(n_{\|})$ and present as an example

$$n_{\perp}(1) = 10 - 10.66667\tau + 5.13069\tau^2 - 2.30752\tau^3 + 1.69527\tau^4 - 1.98282\tau^5, \tag{4.8}$$

$$n_{\perp}(2) = 7 - 8\tau + 3.99473\tau^2 - 1.76429\tau^3 + 1.140396\tau^4 - 1.20818\tau^5, \tag{4.9}$$

$$n_{\perp}(3) = 5.2 - 6.4\tau + 3.24817\tau^2 - 1.46248\tau^3 + 0.84832\tau^4 - 0.84314\tau^5. \tag{4.10}$$

The pseudo-ε expansions have better convergent properties, as it is known from other studies [36, 41, 42]. This is also seen from the coefficients in the series (4.8)–(4.10), as well as from a comparison of the Padé table (4.6) with the one that follows from the pseudo-ε expansion (4.8):

$$n_{\perp}(1) = \begin{pmatrix} 10. & 4.8387 & 3.7156 & 3.2882 & 3.1541 & 3.0391 \\ -0.6667 & 2.7977 & 2.8683 & 3.0805 & 2.0645 & o \\ 4.4640 & 2.8724 & 2.7743 & 2.9854 & o & o \\ 2.1565 & 3.1338 & 3.0073 & o & o & o \\ 3.8518 & 2.9379 & o & o & o & o \\ 1.8690 & o & o & o & o & o \end{pmatrix}. \tag{4.11}$$

However, the convergence of the results might be spoiled if a pole in the denominator of a Padé approximant appears. We demonstrate this below by the Padé table for $n_{\perp}(3)$:

$$n_{\perp}(3) = \begin{pmatrix} 5.2 & 2.3310 & 1.6662 & 1.3945 & 1.2670 & 1.1823 \\ -1.2 & 0.9546 & 1.0319 & 1.1042 & 0.9528 & o \\ 2.0482 & 1.0397 & {\scriptstyle -0.4474} & 1.0623 & o & o \\ 0.5857 & 1.1226 & 1.0705 & o & o & o \\ 1.4340 & 1.0112 & o & o & o & o \\ 0.5909 & o & o & o & o & o \end{pmatrix}, \tag{4.12}$$

where by small digits we indicate a result for the approximant [2/2] with a pole for $\tau = 0.944$.

From the Padé-Borel procedure with [4/1] approximant, we get: $n_{\perp}^{\mathscr{D}}(1) = 2.981$. An estimate for $n_{\perp}^{\mathscr{H}}(1)$ readily follows from the known result obtained based on the six-loop pseudo-ε-expansion $n_{\mathrm{c}} = 2.862$ [36]. Subtraction of 1 leads to the following result $n_{\perp}^{\mathscr{H}}(1) = 1.862$.

5. Conclusion

In the present paper we have studied the conditions under which different types of multicritical behaviour are realized for the $O(n_{\|}) \oplus O(n_{\perp})$ model. These types are related to the three FPs ($\mathscr{H}$, $\mathscr{D}$, $\mathscr{B}$), and their stability defines the regions in the space of the dimensions of the OPs as well as in the spatial dimension where the corresponding multicritical behavior manifests itself. Using the ε-expansion for the two-loop β-functions obtained in the minimal subtraction scheme we derived the BSs separating these regions. We obtained similar BSs applying the resummation procedure. In the particular case of $O(1)\oplus O(2)$ symmetry, we confirm the previous studies finding that the biconical FP associated with a tetracritical behaviour is stable for the case $d=3$. In higher space dimensions, the $O(n_\parallel+n_\perp)$ symmetrical FP associated with the bicritical behaviour is stable.

Our analysis also made use of the results of higher order approximations within the field-theoretical RG approach. At this stage, there were used the scaling arguments connecting the stability of the FPs of $O(n_\parallel)\oplus O(n_\perp)$ model with the universal quantities of the $O(n)$ and the cubic models. Exploiting five-loop expressions for the $O(n)$ model, we derived an ε-expansion for the marginal dimension $n^{\mathscr{D}}(n_\parallel,\varepsilon)$ separating the regions of stability for the FPs $\mathscr{D}$ and $\mathscr{B}$. Applying the resummation procedure to this result, we have analyzed the dependence of $n^{\mathscr{D}}(1,\varepsilon)$ on ε. Exploiting the five-loop expressions for the cubic model we obtained the value of $n^{\mathscr{H}}(1,\varepsilon)$ separating the regions of stability for the FPs $\mathscr{H}$ and $\mathscr{B}$. Finally, we complete our results by three-dimensional estimates of $n^{\mathscr{D}}(1)$ and $n^{\mathscr{H}}(1)$ based on the pseudo-ε expansions derived within a six-loop RG approximation.

These results are also important for the critical dynamics [43–47]. The type of a dynamical FP in such systems depend, of course, on the static FP values. In order to extend our results to the dynamics of antiferromagnets in an external field, further work is necessary. One has to extend this analysis to the statics of the corresponding model C [48–50].

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Граничні вимірності для мультикритичних фазових переходів

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Розглядається теоретико-польова модель, що описує мультикритичні явища і має два зв’язані параметри порядку з $n_{||}$ і $n_{\perp}$ компонентами та $O(n_{\|})\oplus O(n_{\perp})$ симетрію. У рамках теоретико-польової ренормгрупи вивчаються умови реалізації різних типів мультикритичної поведінки. Використовуючи двопетлеві ренормгрупові функції, розраховуються поверхні, що розділяють області стійкості для певних типів критичної поведінки, в параметричному просторі вимірностей параметрів порядку та просторової вимірності d. Використовуючи розклади для ізотропної та кубічної моделей, відомі у високих порядках теорії збурень, отримуються ряди для граничної вимірності параметра порядку, яка контролює кросовер між різними класами універсальності, до четвертого порядку за $\varepsilon = 4-d$ та до п’ятого порядку за псевдо-ε параметром. Особлива увага приділяється випадку $O(1)\oplus O(2)$ симетричної моделі, яка властива для опису анізотропних антиферомагнетиків у зовнішньому магнітному полі.

Ключові слова: *мультикритичні явища, граничні вимірності, ренормгрупа*