

Drag of superfluid current in bilayer Bose systems

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An effect of nondissipative drag of a superfluid flow in a system of two Bose gases confined in two parallel quasi-two-dimensional traps is studied. Using an approach based on introduction of density and phase operators, we compute the drag current at zero and finite temperatures for arbitrary ratio of particle densities in the adjacent layers. We demonstrate that in a system of two ring-shaped traps the «drag force» influences the drag trap in the same way as an external magnetic flux influences a superconducting ring. This allows one to use the drag effect to control persistent current states in superfluids and opens up the possibility of implementing a Bose analog of the superconducting Josephson flux qubit.

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1. Introduction

The existence of nondissipative supercurrents is a common feature of superconducting and superfluid systems. Among various applications of superconductivity, considerable attention has been given to the use of superconducting circuits as very sensitive magnetometers (superconducting quantum interferometer devices). At present the interest in such systems is renewed in view of the possibility of using superconducting circuits with weak links as elements of quantum computers (Josephson qubits). In view of the similarity between superfluids and superconductors, one can expect that the former may also be used for implementing qubits.

Supercurrent in superconductors is coupled to the vector potential of electromagnetic fields. It allows one to control persistent current states by external fields. Obviously, there is no such a channel for control in uncharged superfluid Bose systems. In this paper we study another possibility based on a nondissipative drag effect.

The drag in normal systems has been investigated experimentally and theoretically by many authors

(see, for instance, reviews [1,2]). The main attention was given to the study of bilayer electron systems in semiconductor heterostructures. In such systems an interlayer drag effect takes place. The effect is caused by electron-electron scattering processes and it reveals itself in the appearance of a drag voltage in one layer when a normal current flows in the adjacent layer. If the former layer is in a closed circuit, the drag voltage induces a drag current flowing through the circuit. The effect is accompanied by dissipation of energy and takes place only at finite temperatures. Roughly, the drag voltage increases by a T^2 law (the deviation from this law observed experimentally [3] is connected with a phonon contribution to the interaction between the carriers [4]).

In superfluid and superconducting systems another kind of drag may take place. This drag is nondissipative and is connected with a redistribution of a supercurrent between two superfluid (superconducting) components. In contrast with the drag in a normal state, the superfluid drag has the largest value at zero temperatures and decreases with increasing temperature. The existence of nondissipative drag in superfluid systems was pointed out for the first time in

the paper by Andreev and Bashkin [5]. In that paper a three-velocity hydrodynamic model of a ^3He - ^4He mixture was developed. It has been shown that the superfluid behavior of such systems can be described by including a «drag» term in the free energy. This term is proportional to the scalar product of superfluid velocities of two components times the difference between the effective and the bare masses of ^3He atoms. The nondissipative drag effect in superconductors has been studied in the paper by Duan and Yip [6]. The authors of [6] argue that the value of drag can be obtained from the energy of zero-point fluctuations. It was shown that this energy contains a «drag» term analogous to one obtained in [5] in the hydrodynamic approach. The theory of the nondissipative drag in a bilayer system of charged bosons was developed by Tanatar and Das [7] and by Terentjev and Shevchenko [8].

The existence of nondissipative drag in a system of two one-dimensional wires in a persistent current state was predicted by Rojo and Mahan [9]. It was also shown by Duan [10] that nondissipative drag is responsible for an emergence of an interlayer Hall voltage in bilayer electron systems in the fractional quantum Hall regime.

On the basis of previous studies one can consider nondissipative drag as a fundamental property of systems with macroscopic quantum coherence. For the system of uncharged bosons this effect is especially important, since the «drag force» plays a role similar to the role of the vector potential of magnetic field in superconductors. It opens up new possibilities for observing the effects caused by phase coherence in such systems. One of the goals of this paper is to point out this analogy. In particular, we show that the nondissipative drag effect allows one to realize a superposition of flux states in a superfluid ring with a Josephson weak link.

Great attention is now being paid to the study of ultracold alkali-metal vapors confined in magnetic and optical traps, where Bose-Einstein condensation of atoms has been observed [11]. Advances in technology allow one to manipulate the parameters of such systems and make ultracold atomic gases a unique object for the study of various quantum-mechanical phenomena.

In this paper we study the effect of nondissipative drag in a system of two quasi-two-dimensional atomic Bose gases confined in two parallel traps. To describe such a situation we take into account that the densities of atoms in the drive and drag layers can be unequal. In previous studies [7,8] only the case of two layers with equal densities of the particles was considered. Another important factor is the temperature.

In atomic gases it is of the order of or higher than the energy of intralayer interactions. Previously, the dependence of nondissipative drag on the temperature was treated only qualitatively [6,8]. Here we study the temperature dependence quantitatively. We also evaluate the value of the drag for concrete mechanisms of interlayer interaction in atomic Bose gases.

2. Model and approach

The geometry of a Bose cloud can be modified significantly by varying the configuration of external fields forming the trap. When the confining potential is strongly anisotropic and the temperature and the chemical potential are smaller than the separation between the energy levels of spatial quantization in one direction, the Bose gas can be treated as a two-dimensional one. Recently, low-dimensional atomic gases have been realized experimentally [12].

Bose clouds of a ring shape can be created by using toroidal traps. A configuration of two toroidal traps situated one above the other is convenient for the study of the drag effect. It follows from the discussion below that if one excites a circulating superflow in one trap it inevitably leads to a redistribution of this superflow between the two traps, and superfluid currents appear in both rings.

The main features of the drag effect can be understood from the study of a system of two uniform two-dimensional Bose gases situated in parallel layers. The Hamiltonian of the system can be presented in the form

$$H = \sum_{l=1,2} (E_l - \mu_l N_l) + \frac{1}{2} \sum_{l,l'=1,2} E_{ll'}^{\text{int}}, \quad (1)$$

where

$$E_l = \int d^2r \frac{\hbar^2}{2m} [\nabla \hat{\Psi}_l^+(\mathbf{r})] \nabla \hat{\Psi}_l(\mathbf{r}) \quad (2)$$

is the kinetic energy,

$$E_{ll'}^{\text{int}} = \int d^2r d^2r' \hat{\Psi}_l^+(\mathbf{r}) \hat{\Psi}_{l'}^+(\mathbf{r}') V_{ll'}(\mathbf{r} - \mathbf{r}') \hat{\Psi}_{l'}(\mathbf{r}') \hat{\Psi}_l(\mathbf{r}) \quad (3)$$

is the energy of the intralayer ($l = l'$) and interlayer ($l \neq l'$) interaction, and $N_l = \int d^2r \hat{\Psi}_l^+(\mathbf{r}) \hat{\Psi}_l(\mathbf{r})$. Here $\hat{\Psi}$ is the Bose field operator, l is the layer index, \mathbf{r} is the two-dimensional radius vector lying in the layer, and μ_l are the chemical potentials. To be more specific, we consider the case of point interaction between the atoms: $V_{11}(\mathbf{r}) = V_{22}(\mathbf{r}) = \gamma \delta(\mathbf{r})$, $V_{12}(\mathbf{r}) = V_{21}(\mathbf{r}) = \gamma' \delta(\mathbf{r})$ with $\gamma > 0$ and $|\gamma'| < \gamma$. Assuming the barrier between the two traps is quite high, we neglect the tunneling between the layers.

For further analysis it is convenient to use the density and phase operator approach (see, for instance, [13,14]). The approach is based on the following representation for the Bose field operators

$$\hat{\Psi}_l(\mathbf{r}) = \exp [i\varphi_l(\mathbf{r}) + i\hat{\phi}_l(\mathbf{r})] \sqrt{\rho_l + \hat{\rho}_l(\mathbf{r})}, \quad (4)$$

$$\hat{\Psi}_l^+(\mathbf{r}) = \sqrt{\rho_l + \hat{\rho}_l(\mathbf{r})} \exp [-i\varphi_l(\mathbf{r}) - i\hat{\phi}_l(\mathbf{r})], \quad (5)$$

where $\hat{\rho}_l$ and $\hat{\phi}_l$ are the density and phase fluctuation operators, $\rho_l = \langle \hat{\Psi}_l^+(\mathbf{r}) \hat{\Psi}_l(\mathbf{r}) \rangle$ is the c -number term of the density operator (one can see that it is just the density of atoms in the layer l), $\varphi_l(\mathbf{r})$ is the c -number term of the phase operator (in the approach considered, the inclusion of this term in the phase operator

allows one to describe states with nonzero average superflows).

Substituting Eqs. (4), (5) into Hamiltonian (1) and expanding it in series in powers of $\hat{\rho}_l$ and $\nabla\hat{\phi}_l$, we arrive at the expression

$$H = H_0 + H_1 + H_2 + \dots \quad (6)$$

In (6) the term

$$H_0 = \int d^2r \left\{ \sum_l \left[\frac{\hbar^2}{2m} \rho_l (\nabla\varphi_l(\mathbf{r}))^2 + \frac{\gamma}{2} \rho_l^2 - \mu_l \rho_l \right] + \gamma' \rho_1 \rho_2 \right\} \quad (7)$$

does not contain the operator part, the term

$$H_1 = \int d^2r \left\{ \sum_l \left[\frac{\hbar^2}{2m} (\nabla\varphi_l(\mathbf{r}))^2 + \gamma \rho_l + \gamma' \rho_{3-l} - \mu_l \right] \hat{\rho}_l(\mathbf{r}) + \frac{\hbar^2}{m} \rho_l [\nabla\varphi_l(\mathbf{r})] \nabla\hat{\phi}_l(\mathbf{r}) \right\} \quad (8)$$

is linear in the phase and density fluctuation operators, and the term

$$H_2 = \int d^2r \left\{ \sum_l \frac{\hbar^2}{2m} \left[\frac{(\nabla\hat{\rho}_l(\mathbf{r}))^2}{4\rho_l} + \rho_l (\nabla\hat{\phi}_l(\mathbf{r}))^2 + [\nabla\varphi_l(\mathbf{r})] (\hat{\rho}_l(\mathbf{r}) \nabla\hat{\phi}_l(\mathbf{r}) + [\nabla\hat{\phi}_l(\mathbf{r})] \hat{\rho}_l(\mathbf{r})) + \right. \right. \\ \left. \left. + \frac{i}{2} ([\nabla\hat{\rho}_l(\mathbf{r})] \nabla\hat{\phi}_l(\mathbf{r}) - [\nabla\hat{\phi}_l(\mathbf{r})] \nabla\hat{\rho}_l(\mathbf{r})) \right] + \frac{\gamma}{2} [(\hat{\rho}_1(\mathbf{r}))^2 + (\hat{\rho}_2(\mathbf{r}))^2] + \gamma' \hat{\rho}_1(\mathbf{r}) \hat{\rho}_2(\mathbf{r}) \right\} \quad (9)$$

is quadratic in the $\nabla\hat{\phi}_l$ and $\hat{\rho}_l$ operators.

If the chemical potentials are fixed, the Hamiltonian H_0 is minimized under the conditions

$$\frac{\hbar^2}{2m} (\nabla\varphi_l(\mathbf{r}))^2 + \gamma \rho_l + \gamma' \rho_{3-l} - \mu_l = 0, \quad (10)$$

$$\nabla(\rho_l \nabla\varphi_l(\mathbf{r})) = 0. \quad (11)$$

Fulfillment of Eqs. (10), (11) means that the H_1 term in the Hamiltonian (6) vanishes. One should note that, as follows from Eq. (10), the densities of the components are independent of the coordinates only when the phase gradients remain space independent as well.

The quadratic part of the Hamiltonian determines the spectrum of elementary excitations. Hereafter we will neglect the higher order terms in the Hamiltonian (6). These terms describe the scattering of the quasiparticles and they can be omitted if the temperature is much smaller than the critical temperature ($T_c \sim \hbar^2 \rho / m$).

3. Drag current

The current density operator

$$\hat{\mathbf{j}}_l = \frac{i\hbar}{2m} [(\nabla\hat{\Psi}_l^+) \hat{\Psi}_l - \hat{\Psi}_l^+ \nabla\hat{\Psi}_l], \quad (12)$$

rewritten in terms of the phase and density operators, has the form

$$\hat{\mathbf{j}}_l = \frac{\hbar}{m} \sqrt{\rho_l + \hat{\rho}_l} [\nabla(\varphi_l + \hat{\phi}_l)] \sqrt{\rho_l + \hat{\rho}_l}. \quad (13)$$

Expanding (13) in powers of the density and phase fluctuation operators and neglecting the terms of order higher than quadratic, we obtain the following expression for the mean value of the current density:

$$\mathbf{j}_l = \frac{\hbar}{m} \rho_l \nabla\varphi_l + \frac{\hbar}{2m} (\langle [\nabla\hat{\phi}_l] \hat{\rho}_l \rangle + \langle \hat{\rho}_l \nabla\hat{\phi}_l \rangle). \quad (14)$$

To derive Eq. (14) we take into account that $\langle \hat{\phi}_l \rangle = \langle \hat{\rho}_l \rangle = 0$.

To compute the averages in (14) we rewrite the quadratic part of the Hamiltonian in terms of the operators of creation and annihilation of the elementary excitations. In the absence of the interlayer interaction ($\gamma' = 0$) this can be done by the substitution

$$\hat{\rho}_l(\mathbf{r}) = \sqrt{\frac{\rho_l}{S}} \sum_{\mathbf{k}} e^{i\mathbf{k}\mathbf{r}} \sqrt{\frac{\epsilon_{\mathbf{k}}}{E_{l\mathbf{k}}}} [b_l(\mathbf{k}) + b_l^+(-\mathbf{k})], \quad (15)$$

$$\hat{\phi}_l(\mathbf{r}) = \frac{1}{2i} \sqrt{\frac{1}{\rho_l S}} \sum_{\mathbf{k}} e^{i\mathbf{k}\mathbf{r}} \sqrt{\frac{E_{l\mathbf{k}}}{\epsilon_{\mathbf{k}}}} [b_l(\mathbf{k}) - b_l^+(-\mathbf{k})], \quad (16)$$

where the operators b_l^+ , b_l satisfy the Bose commutation relations. Here S is the area of the system, $\epsilon_k = \hbar^2 k^2 / 2m$ is the spectrum of free atoms, and $E_{lk} = \sqrt{\epsilon_k (\epsilon_k + 2\gamma\rho_l)}$ is the spectrum of elementary excitations at $\gamma' = 0$ and $\nabla\varphi_l = 0$.

In the case considered, the substitution (15), (16) reduces the Hamiltonian (9) to the form

$$H_2 = \sum_{\mathbf{k}} \left[\mathcal{E}_l(\mathbf{k}) b_l^+(\mathbf{k}) b_l(\mathbf{k}) + \frac{1}{2} (E_{lk} - \epsilon_k) \right] + \sum_{\mathbf{k}} g_k [b_1^+(\mathbf{k}) b_2(\mathbf{k}) + b_1(\mathbf{k}) b_2(-\mathbf{k}) + \text{h.c.}], \quad (17)$$

where

$$\mathcal{E}_l(\mathbf{k}) = E_{lk} + \frac{\hbar^2}{m} \mathbf{k} \nabla \varphi_l, \quad (18)$$

$$g_k = \gamma' \epsilon_k \sqrt{\frac{\rho_1 \rho_2}{E_{1k} E_{2k}}}. \quad (19)$$

The Hamiltonian (17) contains terms nondiagonal in the Bose creation and annihilation operator and it can be diagonalized using the u - v transformation

$$b_l(\mathbf{k}) = u_{l\alpha}(\mathbf{k}) \alpha(\mathbf{k}) + u_{l\beta}(\mathbf{k}) \beta(\mathbf{k}) + v_{l\alpha}(\mathbf{k}) \alpha^+(-\mathbf{k}) + v_{l\beta}(\mathbf{k}) \beta^+(-\mathbf{k}), \quad (20)$$

(see [15]) which reduces the Hamiltonian (17) to the form

$$H_2 = \sum_{\mathbf{k}} \left[\mathcal{E}_\alpha(\mathbf{k}) \left(\alpha^+(\mathbf{k}) \alpha(\mathbf{k}) + \frac{1}{2} \right) + \mathcal{E}_\beta(\mathbf{k}) \left(\beta^+(\mathbf{k}) \beta(\mathbf{k}) + \frac{1}{2} \right) - \epsilon_k \right]. \quad (21)$$

It is convenient to present the u - v coefficients and the energies of the elementary excitations as series in powers of g_k . The u - v coefficients read as

$$\begin{pmatrix} u_{1\alpha}(\mathbf{k}) & u_{1\beta}(\mathbf{k}) \\ u_{2\alpha}(\mathbf{k}) & u_{2\beta}(\mathbf{k}) \end{pmatrix} = \begin{pmatrix} A_{\mathbf{k}} & -\frac{g_k}{\mathcal{E}_1(\mathbf{k}) - \mathcal{E}_2(\mathbf{k})} \\ \frac{g_k}{\mathcal{E}_1(\mathbf{k}) - \mathcal{E}_2(\mathbf{k})} & B_{\mathbf{k}} \end{pmatrix} + \mathcal{O}(g_k^3), \quad (22)$$

$$\begin{pmatrix} v_{1\alpha}(\mathbf{k}) & v_{1\beta}(\mathbf{k}) \\ v_{2\alpha}(\mathbf{k}) & v_{2\beta}(\mathbf{k}) \end{pmatrix} =$$

$$\begin{pmatrix} -\frac{g_k^2 [\mathcal{E}_2(\mathbf{k}) + \mathcal{E}_2(-\mathbf{k})]}{[\mathcal{E}_1(\mathbf{k}) + \mathcal{E}_1(-\mathbf{k})][\mathcal{E}_1(-\mathbf{k}) - \mathcal{E}_2(-\mathbf{k})][\mathcal{E}_1(-\mathbf{k}) + \mathcal{E}_2(\mathbf{k})]} & -\frac{g_k}{\mathcal{E}_1(\mathbf{k}) + \mathcal{E}_2(-\mathbf{k})} \\ -\frac{g_k}{\mathcal{E}_1(-\mathbf{k}) + \mathcal{E}_2(\mathbf{k})} & \frac{g_k^2 [\mathcal{E}_1(\mathbf{k}) + \mathcal{E}_1(-\mathbf{k})]}{[\mathcal{E}_2(\mathbf{k}) + \mathcal{E}_2(-\mathbf{k})][\mathcal{E}_1(-\mathbf{k}) - \mathcal{E}_2(-\mathbf{k})][\mathcal{E}_1(\mathbf{k}) + \mathcal{E}_2(-\mathbf{k})]} \end{pmatrix} + \mathcal{O}(g_k^3), \quad (23)$$

where

$$A_{\mathbf{k}} = 1 - \frac{g_k^2}{2} \left(\frac{1}{[\mathcal{E}_1(\mathbf{k}) - \mathcal{E}_2(\mathbf{k})]^2} - \frac{1}{[\mathcal{E}_1(\mathbf{k}) + \mathcal{E}_2(-\mathbf{k})]^2} \right), \quad (24)$$

$$B_{\mathbf{k}} = 1 - \frac{g_k^2}{2} \left(\frac{1}{[\mathcal{E}_1(\mathbf{k}) - \mathcal{E}_2(\mathbf{k})]^2} - \frac{1}{[\mathcal{E}_1(-\mathbf{k}) + \mathcal{E}_2(\mathbf{k})]^2} \right). \quad (25)$$

The spectra of the elementary excitations are found to be

$$\mathcal{E}_\alpha(\mathbf{k}) = \mathcal{E}_1(\mathbf{k}) + g_k^2 \left[\frac{1}{\mathcal{E}_1(\mathbf{k}) - \mathcal{E}_2(\mathbf{k})} - \frac{1}{\mathcal{E}_1(\mathbf{k}) + \mathcal{E}_2(-\mathbf{k})} \right] + \mathcal{O}(g_k^4), \quad (26)$$

$$\mathcal{E}_\beta(\mathbf{k}) = \mathcal{E}_2(\mathbf{k}) - g_k^2 \left[\frac{1}{\mathcal{E}_1(\mathbf{k}) - \mathcal{E}_2(\mathbf{k})} + \frac{1}{\mathcal{E}_1(-\mathbf{k}) + \mathcal{E}_2(\mathbf{k})} \right] + \mathcal{O}(g_k^4). \quad (27)$$

One can see that the small parameter of the expansion is $g_k/|\mathcal{E}_1(\mathbf{k}) - \mathcal{E}_2(\mathbf{k})| \ll 1$. The last inequality takes place for all k , if $\gamma' \max(\rho_1, \rho_2) \ll \gamma|\rho_1 - \rho_2|$. Since in most cases of interest the interlayer interaction is much smaller than the intralayer one, for the bilayer systems with different densities in the adjacent layers one can neglect the $\mathcal{O}(g_k^3)$ and higher order terms in Eqs. (22), (23), (26), (27).

Using representation (15), (16), we obtain from (14) the following expression for the current density:

$$\mathbf{j}_l = \frac{\hbar}{m} \rho_l \nabla \varphi_l + \frac{\hbar}{mS} \sum_{\mathbf{k}} \mathbf{k} \langle b_l^+(\mathbf{k}) b_l(\mathbf{k}) \rangle. \quad (28)$$

Substituting Eq. (20) with coefficients (22), (23) into Eq. (28), computing the averages and expanding

the result in powers of the phase gradients we obtain the following expression for the currents:

$$\mathbf{j}_1 = \frac{\hbar}{m} [(\rho_{s1} - \rho_{dr}) \nabla \varphi_1 + \rho_{dr} \nabla \varphi_2], \quad (29)$$

$$\mathbf{j}_2 = \frac{\hbar}{m} [(\rho_{s2} - \rho_{dr}) \nabla \varphi_2 + \rho_{dr} \nabla \varphi_1]. \quad (30)$$

Equations (29), (30) are given in the approximation linear in $\nabla \varphi_l$. Here the terms of higher order in the phase gradients can be neglected if the phase gradients $\nabla \varphi_l$ are much smaller than the inverse healing lengths $\xi_l^{-1} \sim \sqrt{m\gamma\rho_l}/\hbar$ (this corresponds to velocities of the superflow much smaller than the critical ones $v_{cl} \sim \sqrt{\gamma\rho_l/m}$). In Eqs. (29), (30) the quantities ρ_{sl} and ρ_{dr} with an accuracy up to the g_k^2 are determined by the expressions

$$\rho_{sl} = \rho_l + \frac{1}{S} \sum_{\mathbf{k}} \varepsilon_k \frac{\partial N_{lk}}{\partial E_{lk}} - \frac{1}{S} \sum_{\mathbf{k}} g_k^2 \varepsilon_k \left[(-1)^l \left(\frac{\partial N_{1k}}{\partial E_{1k}} - \frac{\partial N_{2k}}{\partial E_{2k}} \right) \left(\frac{1}{(E_{1k} + E_{2k})^2} - \frac{1}{(E_{1k} - E_{2k})^2} \right) + \frac{\partial^2 N_{lk}}{\partial E_{lk}^2} \left(\frac{1}{E_{1k} + E_{2k}} + \frac{(-1)^l}{E_{1k} - E_{2k}} \right) \right], \quad (31)$$

$$\rho_{dr} = \frac{2}{S} \sum_{\mathbf{k}} g_k^2 \varepsilon_k \left[\frac{1 + N_{1k} + N_{2k}}{(E_{1k} + E_{2k})^3} - \frac{N_{1k} - N_{2k}}{(E_{1k} - E_{2k})^3} + \frac{1}{2} \left(\frac{\partial N_{1k}}{\partial E_{1k}} + \frac{\partial N_{2k}}{\partial E_{2k}} \right) \left(\frac{1}{(E_{1k} - E_{2k})^2} - \frac{1}{(E_{1k} + E_{2k})^2} \right) \right]. \quad (32)$$

Here $N_{lk} = [\exp(E_{lk}/T) - 1]^{-1}$ is the Bose distribution function. One can see that in the absence of the interlayer interaction ($g_k = 0$) the value of ρ_{dr} is equal to zero, and Eq. (31) for ρ_{sl} is reduced to the well-known expression for the density of the superfluid component at finite temperatures. If the interlayer interaction is switched on, the value of ρ_{dr} becomes nonzero. Then, even in the absence of the phase gradient in the drag layer the superfluid current in this layer emerges as a response on the phase gradient in the drive layer.

Equations (31), (32) were derived under assumption of $\rho_1 \neq \rho_2$ (and, consequently, $E_{1k} \neq E_{2k}$). The case $\rho_1 \approx \rho_2$ required more rigorous consideration since in this case the mixing of the modes is strong even for a weak interlayer interaction. One can find that the expressions (31), (32) remain finite at $\rho_1 \rightarrow \rho_2$:

$$\lim_{\rho_1 \rightarrow \rho_2} \rho_{s1} = \lim_{\rho_1 \rightarrow \rho_2} \rho_{s2} = \frac{1}{S} \sum_{\mathbf{k}} \varepsilon_k \frac{\partial N_k}{\partial E_k} - \frac{1}{2S} \sum_{\mathbf{k}} \frac{g_k^2 \varepsilon_k}{E_k} \left(\frac{\partial^2 N_k}{\partial E_k^2} - E_k \frac{\partial^3 N_k}{\partial E_k^3} \right), \quad (33)$$

$$\lim_{\rho_1 \rightarrow \rho_2} \rho_{dr} = \frac{1}{4S} \sum_{\mathbf{k}} \frac{g_k^2 \varepsilon_k}{E_k^3} \left(1 + 2N_k - 2E_k \frac{\partial N_k}{\partial E_k} + \frac{2}{3} E_k^3 \frac{\partial^3 N_k}{\partial E_k^3} \right), \quad (34)$$

where E_k is the energy of the elementary excitations at $\rho_1 = \rho_2$ and $\gamma' = 0$. Using the exact expressions for the spectra and the u - v coefficients we obtain that for the case of two layers with equal densities and in the weak interlayer interaction limit $\gamma' \ll \gamma$ the quantities ρ_{sl} and ρ_{dr} are determined just Eqs. (33), (34). It al-

lows us to conclude that Eqs. (31), (32) are valid for an arbitrary ratio between the densities.

Let us first consider the case of zero temperature. We define the drag current as the current in the drag layer (e.g., layer 1) in the absence of the phase gradient in this layer. At $T = 0$ the drag current is equal to

$$j_{\text{dr}} = C_{\text{dr}} \left(\frac{\gamma'}{\gamma} \right)^2 \left(\frac{m\gamma}{2\pi\hbar^2} \right) \frac{\hbar}{m} \rho_1 \nabla \varphi_2, \quad (35)$$

where

$$C_{\text{dr}} = \int_0^{\infty} dx \frac{x^{1/2}}{\sqrt{x+1} \sqrt{x+\rho_1/\rho_2} (\sqrt{x+1} + \sqrt{x+\rho_1/\rho_2})^3}. \quad (36)$$

The factor C_{dr} is an increasing function of the ratio ρ_2/ρ_1 (at $\rho_2/\rho_1 \rightarrow 0$ the factor C_{dr} approaches zero, at $\rho_2 = \rho_1$ it is equal to $1/12$, and it approaches $1/4$ at $\rho_2/\rho_1 \rightarrow \infty$). Thus, the drag current increases under increasing the density of the particles in the drive layer.

At finite temperatures the drag current decreases. At small T one can use the long-wave approximation for the spectra $E_{1(2)k}$ in the temperature dependent part of Eq. (32) and evaluate this part analytically. This yields the following relation:

$$j_{\text{dr}}(T) = j_{\text{dr}}(0) \left[1 - \frac{16\zeta(3)}{C_{\text{dr}}} \frac{\rho_1}{\rho_2} \left(\frac{T}{2\gamma\rho_1} \right)^3 \right]. \quad (37)$$

But, actually, this approximation is valid only at very low temperatures. The results of numerical evaluation of Eq. (32) are shown in Fig. 1. This figure demonstrates that at $T \sim 2\gamma\rho_1$ the temperature decrease of the drag current is much slower. Based on the results presented in Fig. 1, we also conclude that the temperature reduction of the drag current becomes smaller with increasing density of the particles in the drive layer.

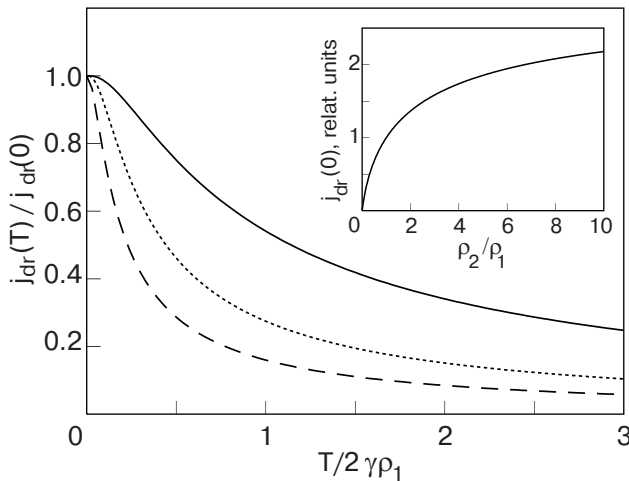


Fig. 1. The dependence of the drag current on the temperature at $\rho_2/\rho_1 = 5.0$; 1.0 ; 0.2 (solid, dashed, and dotted curves, respectively) normalized to its value at $T = 0$. The dependence of $j_{\text{dr}}(0)$ on the ratio ρ_2/ρ_1 normalized to its value at $\rho_2 = \rho_1$ is shown in the inset.

In a Bose cloud confined in a trap the density is nonuniform. This results in a modification of the spectrum of elementary excitations. We may argue that this modification reveals itself in only minor changes of the value of the drag. One can find that the main contribution to the sum in Eq. (32) comes from the excitations with wave vectors of the order of or larger than the inverse healing lengths ξ_l^{-1} . In systems with the healing lengths much smaller than the linear size of the Bose clouds the spectrum at $q \gtrsim \xi_1^{-1}, \xi_2^{-1}$ is well described by the quasi-uniform approximation. Therefore, in such systems the local drag current is given by the same equations (29)–(32) as in the uniform case, with the only modification that the densities ρ_1, ρ_2 in these formulas should be understood as functions of \mathbf{r} . In particular, we predict that for two Bose gases confined in harmonic traps having the same Thomas–Fermi radius (and this radius is much larger than the average healing length), the spatial distribution of the superflow in the drag trap at $T = 0$ will replicate (with a drag factor) the spatial distribution of the superflow in the drive trap. At finite temperatures one can expect a reduction of the drag factor near the edge of the Bose cloud, where the density is low.

One can ask to what extent the two-dimensionality of the system studied may influence the results obtained. It is known that in 2D systems fluctuations of the phase of the order parameter are large and at non-zero temperature the off-diagonal one-particle density matrix $\langle \hat{\Psi}^\dagger(\mathbf{r})\hat{\Psi}(0) \rangle$ goes to zero in the limit $|\mathbf{r}| \rightarrow \infty$. This implies the absence of long-range order in the systems at $T \neq 0$. But since the asymptotic behavior of the density matrix is described by a power-law dependence on r (not an exponential one), at temperatures lower than the critical one (the Kosterlitz–Thouless transition temperature) the superfluid density becomes nonzero. The drag of the superflow between two 2D Bose gases, considered in this paper, is connected with a finite value of the superfluid density, and that is why it decreases with increasing temperature. The density and phase operator approach, used in this paper, is not based on the existence of the Bose–Einstein condensate. Moreover, the power-law asymptotic behavior of the density matrix can be easily derived in this approach with inclusion of the thermal excitations described by the Hamiltonian H_2 . But at the same level of approximation we do not find any crucial influence of the two-dimensionality on the drag phenomena.

4. The value of the drag in atomic Bose gases

Let us present some estimates for the value of the drag in atomic Bose gases. For simplicity we specify

the case of $\rho_1 = \rho_2 = \rho$ and $T = 0$. It is convenient to introduce the drag factor $f_{dr} = \rho_{dr}/(\rho - \rho_{dr})$, which gives the ratio between the currents in the drag and in the drive traps in the absence of the phase gradient in the drag trap. Taking into account that $\rho_{dr} \ll \rho$, we use the approximate expression $f_{dr} = \rho_{dr}/\rho$ for further analysis.

The value of f_{dr} depends on the interaction parameters γ and γ' . The parameter γ can be expressed through the dimensionless effective «scattering length» \tilde{a}

$$\gamma = \frac{2\sqrt{2\pi}\hbar^2}{m}\tilde{a}. \quad (38)$$

In a quasi-two-dimensional trap the effective scattering length is connected with the 3D scattering length a and the oscillator length in the z direction $l_z = \sqrt{\hbar/m\omega_z}$ by the relation $\tilde{a} = a/l_z$, which is valid for $a \ll l_z$ [16]. Let us introduce the interlayer dimensionless effective «scattering length» \tilde{a}' that is connected with the interlayer interaction parameter γ' by the relation

$$\gamma' = \frac{2\sqrt{2\pi}\hbar^2}{m}\tilde{a}'. \quad (39)$$

Substituting Eqs. (38), (39) into Eq. (34), we obtain the drag factor in the form

$$f_{dr} = \frac{1}{12}\sqrt{\frac{2}{\pi}}\frac{(\tilde{a}')^2}{\tilde{a}}. \quad (40)$$

Equation (40) is valid for $|\tilde{a}'| \ll \tilde{a}$, but one can expect that it is approximately correct at $|\tilde{a}'| \approx \tilde{a}$ (we emphasize that in any case the stability condition requires that the inequality $|\tilde{a}'| < \tilde{a}$ be satisfied).

To evaluate the value of $|\tilde{a}'|$ we should specify a mechanism of the interlayer interaction. Let us first consider the interaction that corresponds to the «tail» of the van der Waals potential:

$$V_{12}^{vdW}(r) = -\frac{C_6}{(r^2 + d^2)^3}. \quad (41)$$

Here C_6 is the van der Waals constant and d is the interlayer distance. The Fourier component of the potential (41) is

$$V_{12}^{vdW}(k) = \int d^2r V_{12}^{vdW}(r)e^{i\mathbf{k}\mathbf{r}} = -\frac{\pi C_6}{4d^2}k^2 K_2(kd), \quad (42)$$

where $K_2(x)$ is the modified Bessel function of the second kind. Taking into account that the van der Waals interaction is a short-ranged one, we can evaluate γ' as $\gamma' = V_{12}^{vdW}(k \rightarrow 0) = -\pi C_6/(2d^4)$. It yields $|\tilde{a}'_{vdW}| \approx \sqrt{\pi/2}C_6 m/(4\hbar^2 d^4)$.

This result can be obtained in a more rigorous way. Our approach is easily generalized for the case of an

arbitrary central force interlayer interaction potential. To do this we should redefine the quantity g_k as

$$g_k = \mathcal{V}_{12}(k)\epsilon_k \sqrt{\frac{\rho_1 \rho_2}{E_{1k} E_{2k}}} \quad (43)$$

and substitute this definition (instead of Eq. (19)) into the formulas for ρ_{sl} and ρ_{dr} obtained in the previous Section. Using Eq. (34), one can present the drag factor (for $T = 0$ and $\rho_1 = \rho_2$) in the form

$$f_{dr} = \frac{1}{16\pi^2} \frac{m^2}{\hbar^4 \tilde{a}} \sqrt{\frac{\pi}{2}} \int_0^\infty dx \frac{x^2 [\mathcal{V}_{12}(q_0 x)]^2}{(x^2 + 1)^{5/2}} \quad (44)$$

with $q_0 = \sqrt{8\sqrt{2\pi}\rho\tilde{a}}$. Substituting Eq. (42) into Eq. (44), we find

$$f_{dr} = \frac{1}{12} \left(\frac{C_6 m}{4\hbar^2 d^4} \right)^2 \frac{1}{\tilde{a}} \sqrt{\frac{\pi}{2}} F_{vdW}(dq_0). \quad (45)$$

Here the function

$$F_{vdW}(x) = \frac{3x^4}{4} \int_0^\infty dy \frac{y^6}{(1+y^2)^{5/2}} K_2^2(xy) \quad (46)$$

describes the dependence of the drag factor on the density ρ . Comparing Eqs. (46) and (40) we obtain the following expression for the modules of the effective interlayer scattering length

$$|\tilde{a}'_{vdW}| = \sqrt{\frac{\pi}{2}} \frac{C_6 m}{4\hbar^2 d^4} \sqrt{F_{vdW}(dq_0)}. \quad (47)$$

The dependence of the factor $\sqrt{F_{vdW}}$ on the parameter dq_0 is shown in Fig. 2. One can see from this figure that at $dq_0 \ll 1$ (which corresponds to the low

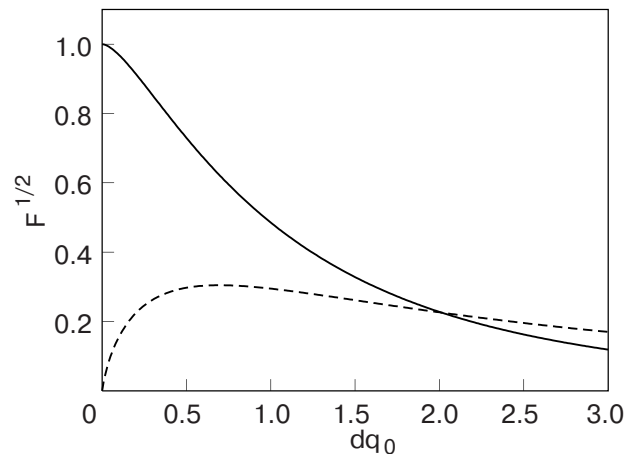


Fig. 2. The density dependent factors in the effective interlayer scattering length versus the parameter $dq_0 = \sqrt{8\sqrt{2\pi}\rho d^2 \tilde{a}}$. Solid curve – $\sqrt{F_{vdW}}$ (van der Waals interaction), dashed curve – $\sqrt{F_{d-d}}$ (dipole-dipole interaction).

density limit) the factor $\sqrt{F_{\text{vdW}}}$ in Eq. (47) is close to unity, and we arrive at the expression for $|\tilde{a}'_{\text{vdW}}|$ given above.

Due to the short-range nature of the van der Waals interaction the interlayer effective scattering length decreases quickly under increasing of d . Therefore, the interlayer distance d should be rather small to achieve an observable value of the drag. Using a typical value of C_6 ($C_6 \approx 3 \cdot 10^{-57}$ erg·cm⁶) and taking $d \approx 10$ nm, $m = 87$ a.u. (Rb), we evaluate $|\tilde{a}'_{\text{vdW}}| \approx 10^{-1}$.

The quantities l_z and a can be controlled in experiments. The first of these is controlled by changing the profile of the confining potential in the z direction, and the latter, by tuning the magnetic field to the value close to the Feshbach resonance field [17–19]. Near this resonance the scattering length changes sign, and a situation with rather small 3D scattering length a (much smaller than the value of l_z , which, in turn, has to be smaller than $d/2$) can be realized. Using this possibility, one can tune the quantity \tilde{a} close to the value of $|\tilde{a}'_{\text{vdW}}|$ and obtain the drag factor $f_{\text{dr}} \approx 7 \cdot 10^{-3}$.

Off resonance the typical values of the 3D scattering length lie in the interval 3–5 nm and for $l_z < d/2$ and $d \approx 10$ nm the estimate $\tilde{a} = a/l_z$ is not applicable. In the ultra-2D limit ($l_z/a \ll 1$) the interaction parameter can be evaluated by using the formula [20]

$$\gamma = \frac{4\pi\hbar^2}{m} \frac{1}{|\ln(\rho a^2)|}.$$

For typical densities $\rho = 10^8$ – 10^{10} cm⁻² it yields $\tilde{a} = 0.2$ – 0.4 , and the drag factor $f_{\text{dr}} \approx (2-3) \cdot 10^{-3}$.

At $d \gtrsim 100$ nm the drag caused by the van der Waals interaction becomes negligibly small. But in the last case the dipole–dipole interaction may give an essential contribution to the drag. Let us consider the situation where the dipole moments of the atoms are aligned in a direction perpendicular to the layers. Then the interaction potential has the form

$$\mathcal{V}_{12}^{d-d}(r) = D^2 \frac{r^2 - 2d^2}{(r^2 + d^2)^{5/2}}, \quad (48)$$

where D is the dipole moment. The Fourier component of the potential (48) reads as

$$\mathcal{V}_{12}^{d-d}(k) = -2\pi D^2 k e^{-kd}. \quad (49)$$

Substituting Eq. (49) into Eq. (44), we obtain

$$f_{\text{dr}} = \frac{1}{12} \left(\frac{D^2 m}{\hbar^2 d} \right)^2 \frac{1}{\tilde{a}} \sqrt{\frac{\pi}{2}} F_{d-d}(dq_0), \quad (50)$$

where

$$F_{d-d}(x) = 3x^2 \int_0^\infty dy \frac{y^4}{(1+y^2)^{5/2}} e^{-2xy}. \quad (51)$$

One can see that Eq. (50) is reduced to Eq. (40) under the definition

$$|\tilde{a}'_{d-d}| = \frac{D^2 m}{\hbar^2 d} \sqrt{\frac{\pi}{2}} \sqrt{F_{d-d}(dq_0)}. \quad (52)$$

The dependence $\sqrt{F_{d-d}(dq_0)}$ is also shown in Fig. 2. In contrast to with the previous case the value of \tilde{a}'_{d-d} approaches zero in the low density limit. But at $dq_0 > 0.1$, which corresponds to $\rho > 10^{-2} d^{-2} / (8\sqrt{2}\pi\tilde{a})$, one can neglect the dependence of f_{dr} on the density and put the factor $\sqrt{F_{d-d}} \approx 0.2$. For the estimates given below we assume that the condition $dq_0 > 0.1$ is fulfilled.

For the magnetic dipole–dipole interaction D is the magnetic dipole moment of the atoms. The magnetic dipoles can be aligned in the same direction if a constant magnetic field is applied to the system. Taking $d = 100$ nm, $D = \mu_B$ (the Bohr magneton), and $m = 87$ a.u., we obtain $|\tilde{a}'_{d-d}| \approx 3 \cdot 10^{-4}$. In the case when \tilde{a} is tuned to the value $|\tilde{a}'_{d-d}|$, one can achieve the drag factor $f_{\text{dr}} \approx 2 \cdot 10^{-5}$.

For Bose atoms with large magnetic dipole moments this value can be much larger. A good candidate atom is Cr ($D = 6\mu_B$). The possibility of realizing a Cr Bose–Einstein condensate is discussed in [21]. For $m = 52$ a.u., $D = 6\mu_B$, and $d = 100$ nm we evaluate $|\tilde{a}'_{d-d}| \approx 6 \cdot 10^{-3}$, and, consequently, the maximum drag factor $f_{\text{dr}} \approx 4 \cdot 10^{-4}$.

5. The «drag force» as an analog of the vector potential

In Sec. 3 we compute the drag current directly. The same results can be obtained from an analysis of the dependence of the free energy of the system on the phase gradients. The free energy of the system can be found from the common thermodynamic relation

$$F = H_0 + E_{\text{zero}} + T \sum_{\lambda=\alpha,\beta} \sum_{\mathbf{k}} \ln \left[1 - \exp \left(-\frac{\mathcal{E}_\lambda(\mathbf{k})}{T} \right) \right]. \quad (53)$$

Here the quantity H_0 given by Eq. (7) is the classical energy of the system, and

$$E_{\text{zero}} = \frac{1}{2} \sum_{\lambda=\alpha,\beta} \sum_{\mathbf{k}} [\mathcal{E}_\lambda(\mathbf{k}) - \epsilon_k] \quad (54)$$

is the energy of the zero-point fluctuations.

Substituting the spectra (26), (27) into Eq. (53) and expanding the final expression in powers of the phase gradients, we find the following expression for the free energy:

$$F = F_0 + \int d^2r \frac{\hbar^2}{2m} [\rho_{s1}(\nabla\varphi_1)^2 + \rho_{s2}(\nabla\varphi_2)^2 - \rho_{dr}(\nabla\varphi_1 - \nabla\varphi_2)^2] + \text{higher order terms}, \quad (55)$$

where F_0 does not depend on the phase gradients, and the quantities ρ_{sl} , ρ_{dr} are determined by the expressions (31), (32). One can see that the answer (29)–(32) obtained in Sec. 3 by another method can also be found from Eq. (55) using the relation

$$\mathbf{j}_l = \frac{1}{\hbar S} \frac{\partial F}{\partial (\nabla\varphi_l)}. \quad (56)$$

The relation (55) is more instructive in the sense that it demonstrates an analogy between the drag effect in superfluids and the exciting of a supercurrent by an external magnetic field in superconductors. To illustrate this analogy let us consider two ring-shaped traps and fix the phase gradient in the drive trap (trap 2 in the notation used below). Then the free energy as a function of the phase gradient in the drag trap (trap 1) can be presented in the form

$$F = \text{const} + \frac{\pi\hbar^2\omega}{mR} \tilde{\rho}_{s1}(\Phi + \Phi_{dr})^2, \quad (57)$$

where R is the radius of the ring, ω is its width, $\tilde{\rho}_{s1} = \rho_{s1} - \rho_{dr}$,

$$\Phi = \frac{1}{2\pi} \oint_C d\mathbf{V}\varphi_1 \quad (58)$$

(here C is a contour around the ring) is the winding number for the phase φ_1 , and

$$\Phi_{dr} = \frac{\rho_{dr}}{\tilde{\rho}_{s1}} \frac{1}{2\pi} \oint_C d\mathbf{V}\varphi_2 \quad (59)$$

is the winding number for the phase φ_2 times the drag factor. In deriving (57) we, for simplicity, neglect the dependence of the densities on the coordinate inside the traps.

Since the value of Φ must be an integer, the minimum of the free energy at $|\Phi_{dr}| < 1/2$ is reached for $\Phi = 0$. In this case the phase gradient in the drag trap is equal to zero, and the superfluid current in the drag trap flows in the same direction as in the drive trap. If $|\Phi_{dr}| > 1/2$ the free energy reaches its minimum at nonzero Φ , and a phase gradient is induced in the drag trap. Then, together with the drag current, a counterflow current appears in the drag trap (depending on the value of Φ_{dr} the total current in this trap can be

parallel or antiparallel to the current in the drag trap). Just the same situation takes place in a superconducting ring with nonzero magnetic flux inside the ring. Thus, in two-ring Bose systems the quantity Φ_{dr} plays the same role as an external magnetic flux (measured in units of flux quanta) in superconducting circuits.

To realize this situation experimentally one should create a circulating superflow in the drive trap. It can be done by elliptic rotating deformation of this trap. The rotation can be switched off when a superflow has been created. The value of the drag current can be found from measurement of the angular momentum of the drag trap. At present a number methods for measuring this quantity have been realized experimentally [22–25]. The methods are based on the study of the dynamics of collective excitations, on the investigation of interference phenomena under hyperfine state transitions, and on the observation of the dynamics of expansion of the Bose cloud.

To extend the analogy with superconductors, let us consider the case where a drag trap of ring geometry contains a Josephson weak link. Then the free energy as a function of the phase shift $\Delta\varphi$ across the link reads as

$$F = \text{const} - E_J \cos(\Delta\varphi) + \frac{\pi\hbar^2\omega}{mR} \tilde{\rho}_{s1} \left(\frac{\Delta\varphi}{2\pi} + \Phi_{dr} \right)^2, \quad (60)$$

where E_J is the Josephson energy. At $E_J > (E_J)_c = \hbar^2\omega\tilde{\rho}_{s1}/(2\pi Rm)$ and $|\Phi_{dr}| = 1/2, 3/2, \dots$ the dependence $F(\Delta\varphi)$ has two degenerate minima. If $E_J/(E_J)_c - 1 \ll 1$ these minima are very shallow, and one can expect an observation of a superposition of two quantum states with different phase shifts (and with the superfluid currents flowing in opposite directions) will be entangled. This is the same regime that is required for implementing the superconducting Josephson flux (persistent current) qubit [26]. While in alkali-metal Bose gases the drag factor is rather small, and even under the most favorable conditions the maximum value that can be reached is of order $10^{-2} - 10^{-3}$ (see Sec. 4), the case $|\Phi_{dr}| \approx 1/2$ can be realized in ring-shape traps of large radius ($10^2 - 10^3 \mu\text{m}$).

In closing, we would like to mention another system in which the effects described in this paper may take place, namely, excitonic or electron–hole Bose liquids in electron bilayers. In these systems electron–hole pairs with components belonging to adjacent layers may form a superfluid state. For the first time the effect was predicted in [27,28], and recently it was confirmed experimentally [29]. The superfluid drag effect may emerge in two parallel bilayers (the four-

layer system). In the four-layer system the intralayer (in the same bilayer) and interlayer (between bilayers) interactions are of the same order: both of them are determined by the dipole–dipole mechanism. In such a case the dipole moment of the pair is large. Therefore, one can expect that nondissipative drag in these systems will be rather strong.

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