Wigner distribution function formalism
for superconductors and collisionless dynamics
of the superconducting order parameter

M.H.S. Amin

D-Wave Systems Inc., 320-1985 W. Broadway, Vancouver, B.C. , V6J 4Y3 Canada

E.V. Bezuglyi, A.S. Kijko, and A.N. Omelyanchouk

*B. Verkin Institute for Low Temperature Physics and Engineering of the National Academy of Science
of Ukraine, 47 Lenin Ave., Kharkov 61103, Ukraine*
E-mail: bezugly@ilt.kharkov.ua

A technique for studying collisionless dynamics of a homogeneous superconducting system is developed which is based on Riccati parametrization of the Wigner distribution function. The quantum evolution of the superconducting order parameter, initially deviating from the equilibrium value, is calculated using this technique. The effect of a time-dependent BCS paring interaction on the dynamics of the order parameter is also studied.

1. Introduction

In this paper we study the dynamics of the superconducting order parameter within the Wigner distribution function approach. The problem of nonstationary phenomena in superconductors has been attracting attention for a long time [1,2]. The general method for description of nonstationary and nonequilibrium processes is the Keldysh technique for nonequilibrium real time Green's functions [3]. The equations for superconducting Keldysh Green's functions [4,5] are a set of quite complicated nonlinear integro-differential equations, which are nonlocal in the time and space domains. These equations can be simplified considerably in the quasiclassical approximation by integrating the Green's functions over $\xi_p = p^2/2m - \mu$ (μ is the chemical potential) [6]. The quasiclassical Larkin−Ovchinnikov equations are still nonlocal in time, but are local in space. In the stationary case, these equations transform into Eilenberger's equations [7], which are effective tools for solving stationary inhomogeneous problems.

When the time-dependent processes in superconductors are considered, three time scales are most essential. The time $t_p \sim \omega_p^{-1}$ (ω_p is the plasma fre-

quency) characterizes the scale at which the self-consistent scheme for the electromagnetic fields $\mathbf{A}(\mathbf{r},t)$, $\phi(\mathbf{r},t)$, and for the BCS pairing field $\Delta(\mathbf{r},t)$ is established. The time $t_0 \sim \Delta^{-1}$ (Δ is the modulus of the order parameter) is an intrinsic time for superconductors, during which quasiparticles with the energy spectrum $\sqrt{\Delta^2 + \xi_p^2}$ are formed in the superconductor. The stage of the relaxation of a nonequilibrium disturbance in the quasiparticle distribution is determined by the energy relaxation time τ_ε, which is caused by inelastic processes. For conventional superconductors, at temperature T not too close to the critical temperature T_c, the hierarchy of the characteristic times is $t_p \ll t_0 \ll \tau_\varepsilon$. In the time interval $t \sim \tau_\varepsilon \gg t_0$ the superconductor's dynamics is described by the quasiclassical Boltzman kinetic equation for the quasiparticle distribution function together with a self-consistent equation for $\Delta(\mathbf{r},t)$ (Aronov−Gurevich equations [8]). In the opposite case $t \ll \tau_\varepsilon$, the dynamics of the superconducting order parameter should be described by the quantum kinetic equation. Considering the collisionless evolution of the superconducting order parameter ($t \ll \tau_\varepsilon$), the equations for the Keldysh Green's functions are reduced to simpler

equations for the Green's functions at coinciding times. The latter can be transformed to the quantum kinetic equation for the Wigner distribution function (WDF). The collisionless kinetic equation for superconducting WDF can also be obtained directly from the generalized Hartree−Fock approach to the BCS pairing model [9] (see also [10,11]).

Wigner [12] has introduced a distribution function in the phase space as a quantum analog of the classical Boltzman distributions. In studying the quantum transport, the Wigner-function formalism has many advantages. It is extensively used for the description of normal metal and semiconducting electron devices whose behavior is dominated by quantum interference effects, e.g., for self-consistent treatment of transient response to a change in the applied voltage [13]. In recent years, Wigner functions are widely used in the field of quantum optics to describe the effects of quantum coherence and superposition in optical systems [14]. Such effects are of great interest in qubit (quantum bit for quantum computation) investigations [15].

The collisionless dynamics of the superconducting order parameter has gained renewed attention after the discovery of the BCS-like paired state in dilute fermionic gases [16]. The ability to control and change the strength of the pairing interaction in these systems opens up possibilities for new experimental investigations of the dynamics of the order parameter. Recently, time-dependent BCS pairing was studied theoretically in Ref. 17. The WDF technique developed in our paper provides a useful tool for studying such problems.

In Sec. 2, following Kulik's approach [9], we derive a quantum kinetic equation for the superconducting WDF in $(\mathbf{r}, t)$-space. This equation is simplified for the case of a homogeneous state (Sec. 3) and then used to study the collisionless dynamics of the order parameter in small superconducting systems (Sec. 4). We consider the problem of the time evolution of the order parameter, initially deviated from the equilibrium value, and found that on times much smaller than τ_ε, the time dependence of Δ has an oscillatory character. Earlier, such a problem was studied by other authors using a linear response approach [18], assuming small deviation from equilibrium. In the present paper, the time dependences were obtained under arbitrary initial perturbations (not only small). The time dependent response of the order parameter to a time-varying pairing potential is also studied. A numerical method for solving the equation for the WDF, which is based on the Maki−Schopohl transformation [19], is developed.

2. Wigner distribution function formalism for the superconducting state

We write the Hamiltonian of the superconductor as $H = H_0 + H_1$, where H_0 includes electron interactions with external fields, the vector potential $\mathbf{A}(\mathbf{r})$ and the scalar potential $\phi(\mathbf{r})$, as well as with the pairing field $\Delta(\mathbf{r})$,

$$H_0 = \sum_{\sigma=\uparrow,\downarrow} \int d\mathbf{r}\, \psi_\sigma^\dagger(\mathbf{r})[\varepsilon - \mu + e\phi(\mathbf{r})]\psi_\sigma(\mathbf{r}) -$$

$$- \int d\mathbf{r}\, [\Delta(\mathbf{r})\psi_\uparrow^\dagger(\mathbf{r})\psi_\downarrow^\dagger(\mathbf{r}) + \Delta^*(\mathbf{r})\psi_\downarrow(\mathbf{r})\psi_\uparrow(\mathbf{r})], \quad (1)$$

$$\varepsilon = \frac{1}{2m}\left[\frac{\nabla}{i} - \frac{e}{c}\mathbf{A}(\mathbf{r})\right]^2 \quad (2)$$

(we use the system of units where $\hbar = k_B = 1$). Here

$$\psi_\sigma(\mathbf{r}) = \frac{1}{\sqrt{V}}\sum_{\mathbf{p}} a_{\mathbf{p}\sigma}(t)\, e^{i\mathbf{p}\mathbf{r}}$$

is the annihilation operator of an electron with the spin σ. The Hamiltonian H_1 describes impurity, electron−phonon, electron−electron, etc. scattering providing relaxation processes.

The pairing field $\Delta(\mathbf{r})$ is to be determined from the self-consistency equation

$$\Delta^*(\mathbf{r}) = V_0 \langle \psi_\uparrow^\dagger(\mathbf{r})\psi_\downarrow^\dagger(\mathbf{r})\rangle, \quad (3)$$

where V_0 is the pairing potential. The electromagnetic potentials obey Maxwell's equations,

$$\nabla \times \mathbf{A}(\mathbf{r}) = \frac{4\pi}{c}\mathbf{j}(\mathbf{r}), \quad (4)$$

$$\nabla^2\phi + \frac{1}{c}\frac{\partial}{\partial t}\nabla \cdot \mathbf{A} = -4\pi\rho(\mathbf{r}), \quad (5)$$

where $\rho(\mathbf{r})$ and $\mathbf{j}(\mathbf{r})$ are the charge and current densities, respectively:

$$\rho(\mathbf{r}) = e\sum_\sigma \langle \psi_\sigma^\dagger(\mathbf{r})\psi_\sigma(\mathbf{r})\rangle, \quad (6)$$

$$\mathbf{j}(\mathbf{r}) = -\frac{ie}{m}\sum_\sigma \langle \psi_\sigma^\dagger(\mathbf{r})\nabla\psi_\sigma(\mathbf{r}) - (\nabla\psi_\sigma^\dagger(\mathbf{r}))\psi_\sigma(\mathbf{r})\rangle -$$

$$- \frac{2e^2}{mc}\mathbf{A}(\mathbf{r})\sum_\sigma \langle \psi_\sigma^\dagger(\mathbf{r})\psi_\sigma(\mathbf{r})\rangle, \quad (7)$$

and the angle brackets denote statistical averaging.

By introducing the «particle−hole» (Gor'kov−Nambu) representation of the electron creation and annihilation operators in terms of 2-vectors

$$A_{\mathbf{p}} = \begin{pmatrix} a_{\mathbf{p}\uparrow} \\ a_{-\mathbf{p}\downarrow}^\dagger \end{pmatrix}, \quad A_{\mathbf{p}}^\dagger = \begin{pmatrix} a_{\mathbf{p}\uparrow}^\dagger & a_{-\mathbf{p}\downarrow} \end{pmatrix}, \quad (8)$$

$$\Psi(\mathbf{r}) = \begin{pmatrix} \psi_\uparrow(\mathbf{r}) \\ \psi_\downarrow^\dagger(\mathbf{r}) \end{pmatrix}, \ \Psi^\dagger(\mathbf{r}) = (\psi_\uparrow^\dagger(\mathbf{r}) \ \psi_\downarrow(\mathbf{r})) , \quad (9)$$

we define the matrix $\hat{f}_{\mathbf{pq}}(t)$ in the «particle−hole» space,

$$f_{\mathbf{pq}}^{\alpha\beta}(t) = \left\langle A_{\mathbf{p}-\frac{\mathbf{q}}{2},\beta}^\dagger(t) A_{\mathbf{p}+\frac{\mathbf{q}}{2},\alpha}(t) \right\rangle,$$

where $\alpha, \beta = 1, 2$ are the indices of the vectors $A_{\mathbf{p}}$. The function $f_{pq}^{\alpha\beta}$ is the Fourier transform of the Wigner distribution function $f_{\alpha\beta}(\mathbf{p}, \mathbf{r}, t)$ generalized to the superconducting case,

$$f_{\alpha\beta}(\mathbf{p}, \mathbf{r}, t) = \sum_{\mathbf{q}} e^{i\mathbf{qr}} \left\langle A_{\mathbf{p}-\frac{\mathbf{q}}{2},\beta}^\dagger(t) A_{\mathbf{p}+\frac{\mathbf{q}}{2},\alpha}(t) \right\rangle. \quad (10)$$

Correspondingly, the components of the matrix $\hat{f}(\mathbf{p}, \mathbf{r}, t)$ are expressed in terms of the Nambu operators $\Psi_\alpha(\mathbf{r}, t)$ in the Heisenberg representation as

$$f_{\alpha\beta} = \int d\mathbf{r}' \, e^{-i\mathbf{pr}'} \langle \Psi_\alpha^\dagger(\mathbf{r} + \mathbf{r}'/2, t) \Psi_\beta(\mathbf{r} - \mathbf{r}'/2, t) \rangle . \quad (11)$$

It follows from Eq. (11) that f_{11} and f_{22} are real functions, and $f_{12} = f_{21}^*$. The self-consistency relations, Eqs. (3), (6), and (7), can be written in terms of $\hat{f}$ as

$$\Delta = V_0 \int \frac{d\mathbf{p}}{(2\pi)^3} \operatorname{Tr} \tau_- \hat{f}(\mathbf{p}), \quad (12)$$

$$\rho = e \int \frac{d\mathbf{p}}{(2\pi)^3} \operatorname{Tr} \tau_3 \hat{f}(\mathbf{p}), \quad (13)$$

$$\mathbf{j} = \int \frac{d\mathbf{p}}{(2\pi)^3} \operatorname{Tr} \tilde{\mathbf{p}} \, \hat{f}(\mathbf{p}), \quad (14)$$

where $\tilde{\mathbf{p}} = \mathbf{p} - e\tau_3 \mathbf{A}/c$, $\tau_- = (1/2)(\tau_1 - i\tau_2)$, and τ_i are the Pauli matrices.

The evolution equation for the WDF can be derived from the equation of motion for the electron field operators $\psi = \psi_\sigma(\mathbf{r}, t)$:

$$i\frac{\partial\psi}{\partial t} = [\psi, H]. \quad (15)$$

Restricting our consideration to the collisionless stage of the evolution, we neglect the interaction part $\hat{H}_1$ of the Hamiltonian and obtain, from Eq. (15), the equations of motion for the Nambu operators $\Psi(\mathbf{r}, t)$,

$$\left[i\frac{\partial}{\partial t} - \tau_3(\hat{\xi} + e\phi) + \hat{\Delta} \right]\Psi = 0, \quad \hat{\Delta} = \begin{pmatrix} 0 & \Delta \\ \Delta^* & 0 \end{pmatrix}, \quad (16)$$

where $\hat{\xi} = -(\nabla + ie\tau_3 \mathbf{A}/c)^2/2m - \mu$. By making use of the definition of the WDF in Eq. (11), we arrive, after some algebra, at the following dynamic equation for $\hat{f}(\mathbf{p}, \mathbf{r}, t)$:

$$\frac{\partial\hat{f}}{\partial t} + i\left[\frac{(\tilde{\mathbf{p}} - i\tilde{\nabla}/2)^2}{2m} \tau_3, \hat{f} \right] + i\,[e\phi\tau_3 - \hat{\Delta}, \hat{f}]_\otimes = 0, \quad (17)$$

where $[\ldots]$ denotes usual commutator, in which we consider $\tilde{\nabla}$ as an integral operator with the kernel $\nabla_{\mathbf{r}}\delta(\mathbf{r} - \mathbf{r}')$, thus $(\tilde{\nabla}\hat{f}) = -(\hat{f}\tilde{\nabla}) = \nabla\hat{f}$. The quantity $[\ldots]_\otimes$ is defined as $[A, B]_\otimes \equiv A \otimes B - B \otimes A$, where $(A \otimes B)(\mathbf{p}, \mathbf{r}, t)$ is the Fourier transformation of the spatial convolution $(AB)(\mathbf{r}_1, \mathbf{r}_2) = \int d\mathbf{r} \, A(\mathbf{r}_1, \mathbf{r}) B(\mathbf{r}, \mathbf{r}_2)$:

$$(A \otimes B)(\mathbf{p}, \mathbf{r}) = \int d\mathbf{r}' e^{-i\mathbf{pr}'}(AB)(\mathbf{r} + \mathbf{r}'/2, \mathbf{r} - \mathbf{r}'/2) =$$

$$= \exp\left\{ \frac{i}{2}[\partial_{\mathbf{r}}^A \partial_{\mathbf{p}}^B - \partial_{\mathbf{p}}^A \partial_{\mathbf{r}}^B] \right\} A(\mathbf{p}, \mathbf{r}) B(\mathbf{p}, \mathbf{r}) . \quad (18)$$

By making use of the transformation $\hat{f} \rightarrow \exp(i\tau_3\chi/2)\hat{f}\exp(-i\tau_3\chi/2)$, we can exclude the phase χ of the superconducting order parameter and proceed to gauge invariant quantities, i.e., the momentum of the superfluid condensate $\mathbf{p}_s$ and the potential Φ defined by

$$\mathbf{p}_s = m\mathbf{v}_s = \frac{1}{2}\left(\nabla\chi - \frac{2e}{c}\mathbf{A} \right), \Phi = \frac{1}{2}\left(\frac{\partial\chi}{\partial t} + 2e\phi \right). \quad (19)$$

The electromagnetic fields are related to $\mathbf{p}_s$ and Φ through

$$e\mathbf{E} = \frac{\partial\mathbf{p}_s}{\partial t} - \nabla\Phi, \ e\mathbf{H} = -\nabla \times \mathbf{p}_s. \quad (20)$$

This results in the substitutions $\tilde{\mathbf{p}} \rightarrow \mathbf{p} + \tau_3\mathbf{p}_s$ and $e\phi \rightarrow \Phi$ in the dynamical equation (17), as well as in the definition of the current in Eq. (14). Note that the anisotropic term $\mathbf{p} \cdot \mathbf{v}_s$ arising from $\tilde{\mathbf{p}}$ in Eq. (17) commutes with $\hat{f}$ and thus drops out from this equation.

While the physical sense of $\mathbf{p}_s$ is obvious, the interpretation of the gauge invariant potential Φ is less evident. Within the framework of the two-fluid model, it can be interpreted as the difference $\Phi = \mu_s - \mu_n$ between the electrochemical potentials of the condensate of Cooper pairs, $\mu_s = \mu + (1/2)\partial\chi/\partial t$, and quasiparticles, $\mu_n = \mu - e\phi$; thus, a nonzero value of Φ means the nonequilibrium of the electrons in the superconductor. In bulk superconductors, Φ and $\mathbf{p}_s$ decay within their corresponding lengths: London (skin) depth δ for $\mathbf{p}_s$ and the electric field penetration depth λ_E for Φ.

3. Wigner distribution function for homogeneous superconducting systems

In what follows, we focus on homogeneous superconducting systems in the clean limit, assuming the scattering rate is much smaller than Δ. To be more specific, we assume the magnitude of the order parameter Δ and the gradient of its phase, $\nabla\chi$, to be uniform inside the superconductor. Using an appropriate gauge transformation, we include the spatially varying part of the phase of Δ into the homogeneous $\mathbf{p}_s$. A «residual» spatially uniform phase is kept to describe the dynamics of the phase of the order parameter. It can be related to, e.g., possible (time-dependent) phase on either side of a Josephson junction. In this case, the equation for the WDF takes the form

$$\frac{\partial \hat{f}}{\partial t} + i[\tilde{\xi}_p \tau_3 - \hat{\Delta}, \hat{f}] + \nu(\hat{f} - \hat{f}_0) = 0, \quad (21)$$

where $\tilde{\xi}_p = \xi_p + \Phi + m\upsilon_s^2/2$. The phenomenological collision term $\nu(\hat{f} - \hat{f}_0)$ qualitatively describes slow relaxation of the WDF to its equilibrium value $\hat{f}_0$ which is associated with the interaction Hamiltonian H_1. In the collisionless limit considered below, we will assume $\nu \to +0$, in order to provide correct analytical behavior of the WDF at $t \to +\infty$.

Equation (21) has several important properties which can be derived from the equations for the matrix elements of $\hat{f}$,

$$i\frac{\partial f_{11}}{\partial t} = -i\frac{\partial f_{22}}{\partial t} = -(\Delta f_{21} - \Delta^* f_{12}), \quad (22)$$

$$i\frac{\partial f_{12}}{\partial t} = 2\tilde{\xi}_p f_{12} + \Delta(f_{11} - f_{22}), \quad (23)$$

$$-i\frac{\partial f_{21}}{\partial t} = 2\tilde{\xi}_p f_{21} + \Delta^*(f_{11} - f_{22}). \quad (24)$$

First, we note that only the difference $f_{11} - f_{22}$ of the diagonal components of the matrix $\hat{f}$ enters the equations for the off-diagonal components f_{12} and f_{21}. Furthermore, from Eq. (22), one finds that the sum of the diagonal components $f_{11} + f_{22} = \text{const}$. This allows us to present the function $\hat{f}$ in the following form:

$$\hat{f} = \frac{1}{2}\left[\hat{1}(1 - \mathcal{F}_-) - \hat{\tilde{f}}\mathcal{F}_+\right], \quad \hat{\tilde{f}} = \begin{pmatrix} -g & f \\ f^* & g \end{pmatrix}, \quad (25)$$

where f and g are isotropic functions, and the time-independent quantities $\mathcal{F}_\pm$ have the meaning of quasiparticle distribution functions which are conserved during the stage of the collisionless evolution. Assuming the system to be initially in equilibrium and

comparing Eq. (25) with the equilibrium form of the WDF, which can be directly obtained from the definition in Eq. (10)

$$\hat{f}_0 = \frac{1}{2}\left\{\hat{1}(1 - \mathcal{F}_-) - \frac{1}{\tilde{\varepsilon}_p}(\tilde{\xi}_p \hat{\tau}_3 - \hat{\Delta})\mathcal{F}_+\right\}, \quad (26)$$

we find the distribution functions

$$\mathcal{F}_\pm = \frac{1}{2}\left[\tanh\frac{\tilde{\varepsilon}_p + \mathbf{p}\cdot\mathbf{v}_s(0)}{2T} \pm \tanh\frac{\tilde{\varepsilon}_p - \mathbf{p}\cdot\mathbf{v}_s(0)}{2T}\right],$$

$$\tilde{\varepsilon}_p = \sqrt{\tilde{\xi}_p^2 + |\Delta|^2}, \quad (27)$$

and the equilibrium values of the functions f and g

$$f_0 = \frac{\Delta}{\tilde{\varepsilon}_p}, \quad g_0 = \frac{\tilde{\xi}_p}{\tilde{\varepsilon}_p}. \quad (28)$$

In this representation, the dynamic equation (21) for the WDF reduces to the following system of scalar equations for the functions g and f:

$$\frac{\partial g}{\partial t} = i(\Delta^* f - \Delta f^*), \quad \frac{\partial f}{\partial t} = 2i(\Delta g - \tilde{\xi}_p f), \quad (29)$$

which, together with Eq. (28), lead to the normalization condition $\hat{\tilde{f}}^2 = 1$ or

$$g^2 + ff^* = 1. \quad (30)$$

The self-consistency equation has the form

$$\Delta(t) = \frac{\lambda}{2}\int\frac{d\Omega_p}{4\pi}\int_{-\omega_D}^{\omega_D}d\xi_p f(\xi_p, t)\mathcal{F}_+, \quad (31)$$

where ω_D is the Debye frequency, $\lambda = N(0)V_0$ is the dimensionless pairing constant, $N(0)$ is the electron density of states per spin at the Fermi level, and Ω_p denotes angle variables associated with the momentum vector. The charge and current densities are given by

$$\rho(t) = -eN(0)\int\frac{d\Omega_p}{4\pi}\int d\xi_p\, g(\xi_p, t)\mathcal{F}_+, \quad (32)$$

$$\mathbf{j}(t) = en\mathbf{v}_s(t) - eN(0)\int\frac{d\Omega_p}{4\pi}\mathbf{p}\int d\xi_p\,\mathcal{F}_-, \quad (33)$$

where n is the net electron density. Equation (33) shows that the electric current is governed directly by the superfluid velocity and has nothing to do with the evolution of the WDF,

$$\mathbf{j}(t) = en\mathbf{v}_s(t) + e(n_s - n)\mathbf{v}_s(0) =$$
$$= \mathbf{j}(0) + en[\mathbf{v}_s(t) - \mathbf{v}_s(0)], \quad (34)$$

where n_s is the condensate density calculated for the initial superfluid velocity $\mathbf{v}_s(0)$. This property reflects the specifics of the collisionless regime, in which the normal component of the current flow is not affected by scattering, and therefore the velocities of both the superfluid and normal components of the electron fluid undergo equal changes $\mathbf{v}_s(t) - \mathbf{v}_s(0)$: $\mathbf{v}_s(0) \rightarrow \mathbf{v}_s(t)$, $\mathbf{v}_n(0) = 0 \rightarrow \mathbf{v}_s(t) - \mathbf{v}_s(0)$. From this we conclude that at nonzero temperature, when the density of the normal component, $n_n \equiv n - n_s$, is nonzero, the current reverses its direction with respect to $\mathbf{v}_s(t)$ if the latter becomes smaller than $\mathbf{v}_s(0) n_n / n$.

4. Collisionless evolution of the order parameter in superconductors

In the paper by Volkov and Kogan [18], the problem of evolution of the order parameter $\Delta(t)$ at given initial value of the WDF (and corresponding initial self-consistent value of $\Delta = \Delta(0)$) was analyzed within a linear approximation, assuming small deviations of $\Delta(t)$ and $\hat{f}(\xi, t)$ from their equilibrium values. It was shown that the time variations of Δ have the form of harmonic oscillations with a period of the order of Δ^{-1} and an amplitude decreasing slowly as $t^{-1/2}$. At large $t \gg t_0 = \Delta^{-1}(0)$, the order parameter approaches a constant value $\Delta_\infty \equiv \Delta(t \rightarrow \infty)$, which is determined by the initial conditions and coincides neither with $\Delta(0)$, nor with the equilibrium value Δ_0.

In this paper, we address a more general nonlinear problem, with arbitrary initial conditions, which may essentially differ from the equilibrium state. In particular, this allows us to consider formation of the superconducting state from the initial normal state at low enough temperatures, or destruction of the initial superconducting state at high temperatures. To this end, we apply a numerical procedure, by making use of the so-called Riccati parametrization of the functions f and g. Due to the normalization condition (30), these functions can be expressed through a single function $a(\xi_p, t)$,

$$g = \frac{1 - aa^*}{1 + aa^*}, \quad f = \frac{2a}{1 + aa^*}, \tag{35}$$

which satisfies a nonlinear Riccati-type equation,

$$\frac{\partial a}{\partial t} = i(-2\tilde{\xi}_p a - \Delta^* a^2 + \Delta). \tag{36}$$

In the stationary limit ($\Delta = \text{const}$), the solution of Eq. (36) is

$$a_0 = \frac{\Delta}{\tilde{\xi}_p + \tilde{\varepsilon}_p}. \tag{37}$$

In the general nonstationary case, one needs to integrate Eq. (36), together with the self-consistency equation (31). Thus, proceeding to the discrete time variable, $t = n\delta t$, $n = 0, 1, \ldots$, one has to calculate the new value of Δ from Eq. (31) after each time step δt, and then use it for the next step. For sufficiently small δt, Δ can be approximately considered as constant between t and $t + \delta t$, which allows us to apply an analytical solution of Eq. (36) within this time interval,

$$a(t + \delta t) = a(t) + \frac{\Delta(t) - 2\tilde{\xi}_p a(t) - \Delta^*(t)a^2(t)}{\Delta^*(t)a(t) + \tilde{\xi}_p - i\tilde{\varepsilon}_p \cot(\tilde{\varepsilon}_p \delta t)}, \tag{38}$$

and thus to calculate $a(t + \delta t)$ explicitly. As a result, the numerical procedure reduces to the numerical solution of the self-consistency equation at each step of the calculations.

In our calculations, we use time steps $\delta t = 0.02 t_0$. After each step, the values of the modulus and the phase of $\Delta(t)$ were recalculated by means of the self-consistency equation (31). In Fig. 1, we present time variations of the order parameter modulus with the initial values $\Delta(0)$ essentially different from the
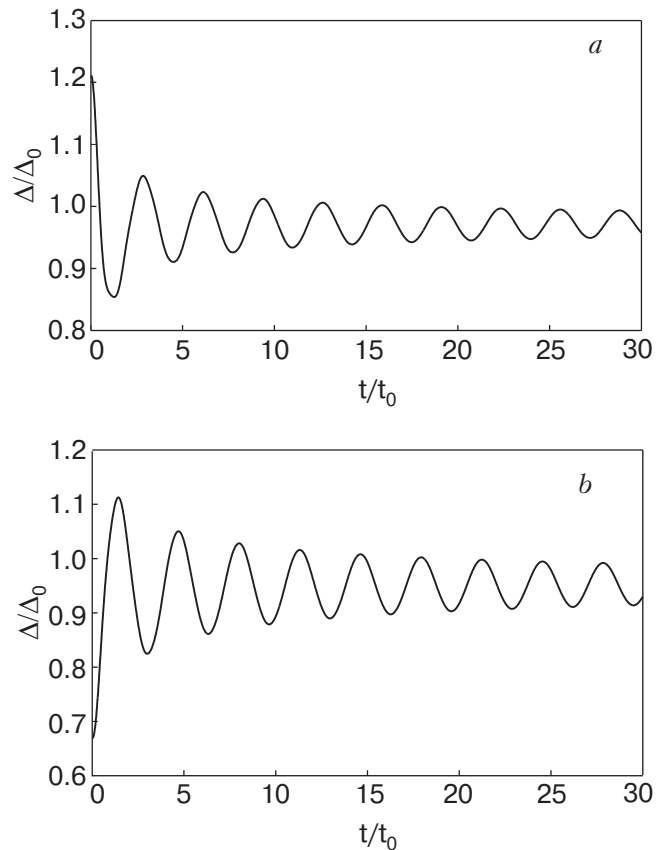

Fig. 1. Collisionless time evolution of the order parameter for an initial value $\Delta(0)$ larger (*a*) or smaller (*b*) than the equilibrium value Δ_0 at $T = 0$. In all figures the time is normalized on $t_0 = 1/\Delta_0$.

equilibrium value Δ_0 at $T = 0$. It is obvious that equal values of $\Delta(0)$ may be obtained for different forms of the initial Wigner distribution function $\hat{f}(0)$. In our evaluation, we use the equilibrium form of $\hat{f}(0)$ given by Eq. (26) at $T = 0$, with a formal parameter Δ_{in}, which, however, appears to be slightly different from the initial self-consistent value $\Delta(0)$. This difference depends weakly on the value of the pairing constant λ, for which in the following we put $\lambda = 0.5$. The initial value of $\Delta_{in} = 1.5\Delta_0$ leads to $\Delta(0) \approx 1.3\Delta_0$ (Fig. 1,*a*), whereas $\Delta_{in} = 0.5\Delta_0$ yields self-consistent $\Delta(0) \approx$ $\approx 0.67\Delta_0$ (Fig. 1,*b*).

Another type of perturbation in the system is the switching of λ from one value to another, or, more generally, the case of time-dependent BCS pairing. We have used the equations (35), (36), (38), and (31) with time-dependent $\lambda = \lambda(t)$ to study this problem numerically. In Fig. 2 the collisionless evolution of the order parameter under the changing of λ in time is shown.

It is interesting to note that the initial BCS form of the WDF automatically leads to conservation of arbitrary initial value of the order parameter phase χ. Actually, such property is associated with the definite symmetry of the initial WDF with respect to ξ_p, which holds during the time evolution, $f(\xi_p, t) =$ $= f(-\xi_p, t)$, $g(\xi_p, t) = -g(-\xi_p, t)$, and manifests equality of the populations of the electron- and hole-like excitations with equal energies $\tilde{\varepsilon}_p$. The introduction of an imbalance between the electron and hole branches of the excitation spectrum (i.e., violation of the above-mentioned symmetry) produces an excess charge in the quasiparticle subsystem which, due to electroneutrality of the metal, should be com-
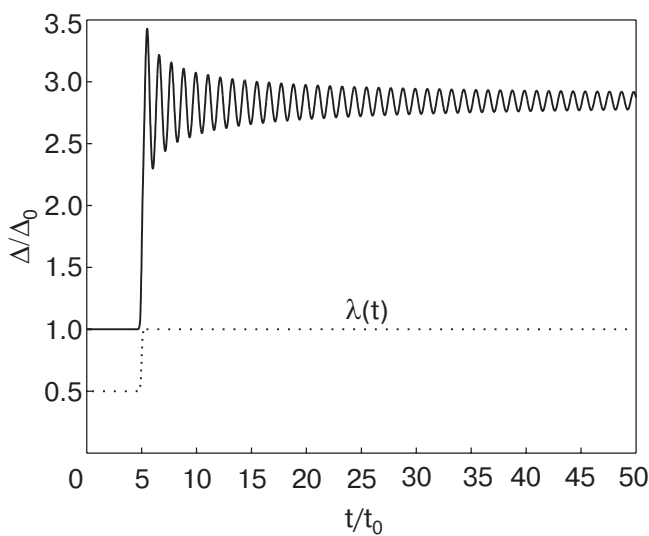
pensated by the opposite charge of the superfluid condensate. This means the appearance of the difference $\delta\mu$ between the electrochemical potentials μ_n and μ_s of excitations and the condensate, respectively, which produces time variations of the order parameter phase according to the relationship $d\chi/dt = 2\delta\mu$. For a given constant $\delta\mu$, we find continuous variation of the phase with a constant rate.

The processes of formation and destruction of the superconducting state can be also analyzed within the nonlinear collisionless approach. By starting evaluations from a very small value of Δ_{in} ($\sim 10^{-3}\Delta_0$) in Eq. (28) at $T = 0$, which approximately represents an initial normal state, we observe a rapid increase in $\Delta(t)$ at the time $t \sim t_0$ up to $\Delta \sim \Delta_0$, followed by an oscillatory approach to a stable superconducting state (see Fig. 3). We note that the asymptotic value Δ_∞ appears to be noticeably lower than Δ_0, which means that the real equilibrium value of Δ at the superconducting transition is formed via the relaxation processes.

Strictly speaking, at any temperature, including the region $T < T_c$, the self-consistency equation (3) always has a trivial solution $\Delta = 0$, which corresponds to the normal state. However, at $T < T_c$, the normal state is associated with a maximum of the free energy, and therefore Fig. 3 actually illustrates the thermodynamic instability of the normal state with respect to an infinitesimal Δ, which develops through the quantum kinetic process described by Eqs. (3) and (29). It is interesting to note that, despite the strong nonlinearity of the process, the oscillations of $\Delta(t)$ have an almost purely harmonic shape.

The instability of the superconducting state at temperatures $T > T_c$ is illustrated by Fig. 4, which was obtained by starting evaluations from the initial superconducting state in Eq. (28) at the rather high temperature $T = 2.5\Delta_0$. The order parameter decreases
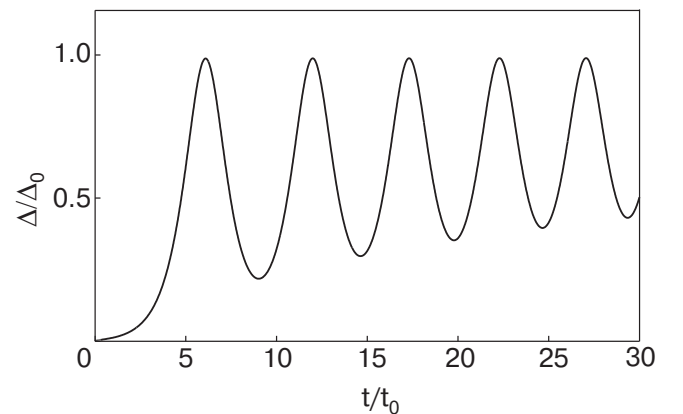

Fig. 2. Collisionless time evolution of the order parameter under the changing of the coupling constant λ from the value 0.5 to 1.0.

Fig. 3. Instability of the equilibrium normal state at $T = 0$. We start from $\Delta(0) = 0.001\Delta_0$.
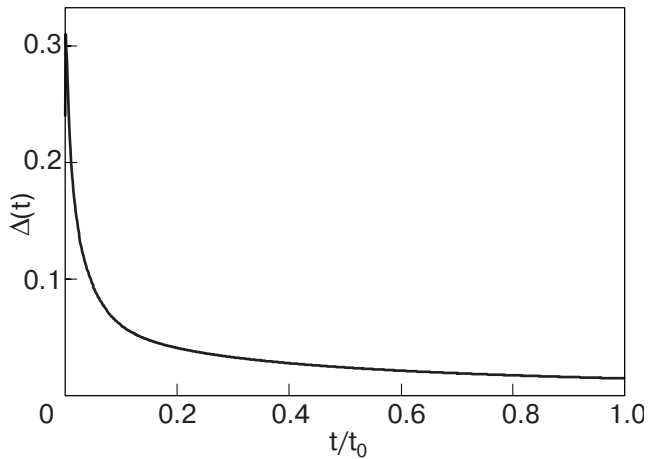
Fig. 4. Instability of the superconducting state at $T = 2.5\Delta_0 > T_c$, with the initial condition $\Delta(0) = 0.31\Delta_0$.

approximately exponentially with the characteristic decay time $0.42t_0$ without any oscillations. In the final stage of the evolution, the order parameter enters the fluctuation regime, which is outside of the framework of our self-consistent approach.

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