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On the extension of Helmert transform

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The article deals with the method of comparison of the coordinate systems used by two quite different scientific disciplines: stellar astronomy and the geodesy. Geodesic Helmert transform is analysed along with a series of stellar astronomy kinematic models: Kovalsky-Airy, Lindbladt-Oort and Ogorodnikov-Milne. An analogy was built allowing us to propose an extension to the Helmert transform. In the second part of the article, three different approaches to the solution of the correlation problem are compared, and the results of the numerical experiment are presented.

Key words: stellar astronomy, geodesy, mathematical methods

INTRODUCTION

Let us state several definitions first. The construction of a coordinate systems for the sky (galaxies, quasars) or for the Earth (cities, geodetic markers, GPS navigators) leads to a catalogue of coordinates of objects. Thus, the word "catalogue" in this article always refers to a set of object coordinates, while catalogue objects are referred as points.

An essential method for building catalogues is through observations and subsequent processing. There are no possibilities to build absolute catalogues, as objects tend to change their positions. There are individual object velocities, due to galactic rotation in the case of celestial coordinate systems, and due to tectonic movements and secular tidal effects in the case of Earth coordinate systems. That is why one should always specify the catalogue epoch and precision, meaning its systematic (averaged over all objects) and random (every object has own error) precisions. The only way to determine such errors is through the comparison of different catalogues.

Furthermore, a catalogue should not be interpreted as just a list of point coordinates. Every catalogue defines its own coordinate system. This should be kept in mind when using the catalogue. Throughout the years, geodesy has constructed of numerous catalogues of terrestrial objects, and has defined numerous Terrestrial Reference Frames (TRF). Astronomy has done the same with respect to catalogues of celestial bodies, and has similarly defined numerous Celestial Reference Frames (CRF). Space geodynamics makes use of both types of catalogues. Determination of the transformation between CRF and TRF is its main task. In this article we will not analyse geodynamic transformations, but the TRFs and CRFs themselves only.

Let $\vec{r}_i$ be the coordinate of some object given in i-th xRF (x might be C or T, but C and T may never be present simultaneously in the same formula). Spherical coordinates in TRF are λ — longitude, and φ — lattitude. If r_i if the distance from the centre, then:

$$\vec{r}_i = r_i \left(\cos\lambda\cos\varphi, \sin\lambda\cos\varphi, \sin\varphi\right)^T.$$

For TRF $r_i \approx R_\oplus$, the Earth' radius.

Spherical coordinates in CRF are α — right ascension and δ — declination. Following the spherical astronomy definition where all sources are placed on the same celestial sphere of any useful radius $R_\odot$ we will have the same coordinate definitions:

$$\vec{r}_i = r_i \left(\cos\alpha\cos\delta, \sin\alpha\cos\delta, \sin\delta\right)^T.$$

For CRF $r_i = R_\odot$, usually 1.

Helmert transform was proposed for the comparison of different geodesic catalogues (read: TRFs) by Friedrich Robert Helmert (1843–1917), director of Potsdam geodetic institute and Professor at the University of Berlin [11]. He introduced shift plus rotation transform. It is widely used until now, see for example [2, 3, 10], and not only for TRF/CRF comparison [4].

But there are other possibilities to be analysed.

SHIFTED CENTRES, PARALLEL AXES

In this case for any point:

$$\vec{r}_2 = \vec{r}_1 + \vec{b}, \qquad (1)$$

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where $\vec{b}$ is the shift of the centres. In stellar astronomy we may suppose that $\vec{r}_1$ and $\vec{r}_2$ are coordinates in catalogues built for two distinct epochs with time interval Δt between them. That is:

$$\vec{b} = \vec{V}\Delta t = \vec{r}_2 - \vec{r}_1.$$

Point coordinates may change:

$$\vec{r}_2 = \vec{r}_1 + \left.\frac{d\vec{r}}{dt}\right|_1 \Delta t = \vec{r}_1 + \vec{\mu}\Delta t,$$

where $\vec{\mu}$ is a proper motion vector.

In spherical coordinates:

$$\vec{V}\Delta t = \left.\frac{d\vec{r}}{dt}\right|_1 \Delta t,$$

or

$$\vec{V} = \begin{pmatrix} \dot{r}\cos\alpha\cos\delta - r\sin\alpha\dot{\alpha}\cos\delta - r\cos\alpha\sin\delta\dot{\delta} \\ \dot{r}\sin\alpha\cos\delta + r\cos\alpha\dot{\alpha}\cos\delta - r\sin\alpha\sin\delta\dot{\delta} \\ \dot{r}\sin\delta + r\cos\delta\dot{\delta} \end{pmatrix}.$$

In matrix notations:

$$\vec{V} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} r\cos\alpha\cos\delta & -r\sin\alpha & -r\cos\alpha\sin\delta \\ r\sin\alpha\cos\delta & r\cos\alpha & -r\sin\alpha\sin\delta \\ r\sin\delta & 0 & r\cos\delta \end{pmatrix} \times \begin{pmatrix} \dot{r}/r \\ \dot{\alpha}\cos\delta \\ \dot{\delta} \end{pmatrix}. \quad (2)$$

With generally used stellar astronomy designations: $\dot{r} = V_r$, $\dot{\alpha}\cos\delta = \mu_\alpha$, $\dot{\delta} = \mu_\delta$ having in mind $\Delta t = 1$ year and after inverting the matrix in (2):

$$\begin{pmatrix} V_r/r \\ \mu_\alpha \\ \mu_\delta \end{pmatrix} = \begin{pmatrix} \cos\alpha\cos\delta & \sin\alpha\cos\delta & \sin\delta \\ \sin\alpha & -\cos\alpha & 0 \\ -\cos\alpha\sin\delta & \sin\alpha\sin\delta & \cos\delta \end{pmatrix} \times \begin{pmatrix} X/r \\ Y/r \\ Z/r \end{pmatrix}. \quad (3)$$

Or without matrices:

$$\begin{cases} V_r/r = X/r\cos\alpha\cos\delta + Y/r\sin\alpha\cos\delta + Z/r\sin\delta, \\ \mu_\alpha = X/r\sin\alpha - Y/r\cos\alpha, \\ \mu_\delta = -X/r\cos\alpha\sin\delta + Y/r\sin\alpha\sin\delta + Z/r\cos\delta, \end{cases} \quad (4)$$

which is more essential for classical classroom presentation.

The two last equations in (4) are identical to the Kovalsky-Airy model. The first one is used to determine the Sun's velocity from radial stellar velocities only [9], here $(X, Y, Z)^T = \vec{V}_\odot$. In geodesy we may use (1), or the Kovalsky-Airy model (3), for example for comparison of geocentre positions determined from VLBI and SLR. Both relations are identical and differ only in mathematical writing style.

SHIFTED CENTRES, NON-PARALLEL AXES

Following Helmert we imply that two RFs have different centres and their axes might coincide through three rotations around three coordinate axes:

$$\vec{r}_2 = \mathbb{B}\vec{r}_1 + \vec{b} = \mathbb{R}(n)\mathbb{Q}(m)\mathbb{P}(l)\vec{r}_1 + \vec{b}, \quad (5)$$

where $\mathbb{P}, \mathbb{Q}, \mathbb{R}$ are three elementary rotation matrices around x, y, z axes respectively. In the general case, angles l, m, n are small and their cosines might be replaced with unity, and sinuses with their arguments. It gives us:

$$\mathbb{B} = \begin{pmatrix} 1 & l & 0 \\ -l & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -m \\ 0 & 1 & 0 \\ m & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & n \\ 0 & -n & 1 \end{pmatrix},$$

and:

$$\vec{r}_2 = \begin{pmatrix} 1 & l & -m \\ -l & 1 & n \\ m & -n & 1 \end{pmatrix} \vec{r}_1 + \vec{b}. \quad (6)$$

While the angles remain small, the result does not depend upon the order of multiplications.

For the next step, let us extract a unity matrix from the right side, move it to the left side:

$$\vec{r}_2 - \vec{r}_1 = \begin{pmatrix} 0 & l & -m \\ -l & 0 & n \\ m & -n & 0 \end{pmatrix} \vec{r}_1 + \vec{b} = A\vec{r}_1 + \vec{b},$$

which may be rewritten in yet another form (here x_1, y_1, z_1 are components of the vector $\vec{r}_1$):

$$\vec{r}_2 - \vec{r}_1 = \begin{pmatrix} ly_1 - mz_1 \\ nz_1 - lx_1 \\ mx_1 - ny_1 \end{pmatrix} + \vec{b} = [\vec{\omega} \times \vec{r}_1] + \vec{b}, \quad (7)$$

where $\vec{\omega} = (n, m, l)^T = (\omega_1, \omega_2, \omega_3)^T$ is an angular velocity vector which is generally used to explain rotation from RF_1 to RF_2. It is a well-known result, as any rotations around the main coordinate axes can be replaced with only one around some specially selected axis.

Inserting sphericals into (7) gives:

$$\begin{pmatrix} r\cos\alpha\cos\delta & -r\sin\alpha & -r\cos\alpha\sin\delta \\ r\sin\alpha\cos\delta & r\cos\alpha & -r\sin\alpha\sin\delta \\ r\sin\delta & 0 & r\cos\delta \end{pmatrix} \times \begin{pmatrix} V_r/r \\ \mu_\alpha \\ \mu_\delta \end{pmatrix} = \begin{pmatrix} \omega_2 z - \omega_3 y \\ \omega_3 x - \omega_1 z \\ \omega_1 y - \omega_2 x \end{pmatrix} + \begin{pmatrix} X \\ Y \\ Z \end{pmatrix},$$

or after matrix inversion, rewriting without matrices:

$$\begin{aligned} V_r/r &= X/r\cos\alpha\cos\delta + Y/r\sin\alpha\cos\delta + Z/r\sin\delta, \\ \mu_\alpha &= -X/r\sin\alpha + Y/r\cos\alpha - \omega_1\cos\alpha\sin\delta - \\ &\quad -\omega_2\sin\alpha\sin\delta + \omega_3\cos\delta, \\ \mu_\delta &= -X/r\cos\alpha\sin\delta - Y/r\sin\alpha\sin\delta + Z/r\cos\delta + \\ &\quad +\omega_1\sin\alpha - \omega_2\cos\alpha. \end{aligned} \tag{8}$$

Thus, starting from (5) and supposing that rotations are small, we came to the Lindblat-Oort model of stellar astronomy [8].

Let us have a look at (6) again, where $[\vec{\omega}\times\vec{r}_1] = A\vec{r}_1$, A is an antisymmetric matrix, and $A\vec{r}_1$ or alternatively $[\vec{\omega}\times\vec{r}_1]$ defines the velocity field essential for the first catalogue. That velocity field moves the points of the first catalogue to the positions of the second one in full analogy with liquid flow, with the points immersed into that liquid. The points are the partices flowing from the first catalogue positions to the second catalogue positions. This analogy lead us to the third model.

DEFORMABLE VELOCITY FIELD WITH SHIFTED CENTRES AND NON-PARALLEL AXES

If the velocity filed is defined by matrix (or tensor) A, can we add another part to that tensor? Since the transformation matrix (or velocity field tensor) A is antisymmetric, let us add the symmetric part S as well. A new model arises:

$$\vec{r}_2 - \vec{r}_1 = [\vec{\omega}\times\vec{r}_1] + \vec{b} + S\vec{r}_1. \tag{9}$$

A symmetric matrix explains additional deformations of the velocity field. Interpretation of these values are as follows: diagonal elements explain scaling along their axes, while non-diagonal elements explain the pressure effects within their planes.

Substitution of stellar astronomy formulas into (9) leads to the classical Ogoridnikov-Milne model [6, 7] in matrix form:

$$\begin{pmatrix} r\cos\alpha\cos\delta & -r\sin\alpha & -r\cos\alpha\sin\delta \\ r\sin\alpha\cos\delta & r\cos\alpha & -r\sin\alpha\sin\delta \\ r\sin\delta & 0 & r\cos\delta \end{pmatrix} \times \begin{pmatrix} V_r/r \\ \mu_\alpha \\ \mu_\delta \end{pmatrix} = \begin{pmatrix} \omega_2 z - \omega_3 y \\ \omega_3 x - \omega_1 z \\ \omega_1 y - \omega_2 x \end{pmatrix} + \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + \begin{pmatrix} S_{11}x + S_{12}y + S_{13}z \\ S_{12}x + S_{22}y + S_{23}z \\ S_{13}x + S_{23}y + S_{33}z \end{pmatrix},$$

or without matrices:

$$\begin{aligned} V_r/r &= X/r\cos\alpha\cos\delta + Y/r\sin\alpha\cos\delta + \\ &\quad +Z/r\sin\delta + S_{12}\sin 2\alpha\cos^2\delta + \\ &\quad +S_{13}\cos\alpha\sin 2\delta + S_{23}\sin\alpha\sin 2\delta + \\ &\quad +S_{11}\cos^2\alpha\cos^2\delta + S_{22}\sin^2\alpha\cos^2\delta + \\ &\quad +S_{33}\sin^2\delta, \\ \mu_\alpha &= -X/r\sin\alpha + Y/r\cos\alpha - \\ &\quad -\omega_1\cos\alpha\sin\delta - \omega_2\sin\alpha\sin\delta + \\ &\quad +\omega_3\cos\delta + S_{12}\cos 2\alpha\cos\delta - \\ &\quad -S_{13}\sin\alpha + S_{23}\cos\alpha\sin\delta - \\ &\quad -(S_{11} - S_{22})/2\sin 2\alpha\cos\delta, \\ \mu_\delta &= -X/r\cos\alpha\sin\delta - Y/r\sin\alpha\sin\delta + \\ &\quad +Z/r\cos\delta + \omega_1\sin\alpha - \omega_2\cos\alpha - \\ &\quad -S_{12}\sin 2\alpha\sin 2\delta/2 + \\ &\quad +S_{13}\cos\alpha\cos 2\delta + S_{23}\sin\alpha\cos 2\delta - \\ &\quad -(S_{11} - S_{22})/2\cos^2\alpha\sin 2\delta + \\ &\quad +(S_{33} - S_{22})/2\sin 2\delta. \end{aligned}$$

A geodesic transformation of type (9) is an extended Helmert transform.

We should understand "deformation" in a very weak sense. There is no real deformation of the space with embedded sources. The only deformed field is the virtual field of velocities, which moves the objects of the catalogue to their positions in another catalogue, in the best way.

CASE OF CORRELATED RANDOM ERRORS

Typically, the Helmert transform is used as the first step of a multiple-step procedure of constructing the combined catalogue. Parameters of (9) together form the model of systematic errors between RF_i and RF_j. Random errors might be estimated as a total residual error after removing of the systematic one. There is some probability that the combined catalogue will have a lower level of random errors, as compared with the raw ones. Let us define RF_0 as a combined catalogue. The classical method of building them from M raw ones postulates that [12]:

$$\vec{r}_0 = \frac{\sum\limits_{i=1}^{M} p_i\vec{r}_i}{\sum\limits_{i=1}^{M} p_i}, \qquad \sigma_0^2 = \frac{1}{\sum\limits_{i=1}^{M} p_i},$$

where $\vec{r}_0$ is the best position, estimated with error σ_0^2, p_i are weights.

To estimate the random errors let us suppose that all numbers in (9) are already determined with Least Squares procedure applied to N shared points of RF_i and RF_0. It means that now we are able to transform all the shared points from RF_i to RF_0 applying (9) to points coordinates in RF_i. It leads us to:

$$\vec{r}_i^{(*)} = [\omega\times\vec{r}_i] + \vec{r}_i + \vec{b} + S\vec{r}_i. \tag{10}$$

The total residual dispersion is then:

$$\sigma_{i0}^2 = \sum \left(\vec{r}_i^{(*)} - \vec{r}_0\right)^2 / N, \qquad (11)$$

and it is the mean random error of the RF_i and RF_0. According to statistics for any two RF_i and RF_j:

$$\sigma_{ij}^2 = \sigma_i^2 + \sigma_j^2 - 2\rho_{ij}\sigma_i\sigma_j, \qquad (12)$$

where σ_i^2 is the dispersion of random errors of the RF_i, and ρ_{ij} is the correlation of random errors of two $RF's$.

There are numerous unknowns in (12) which make the equation system underdetermined. In the case of comparison of M different $RF's$, there are $M(M-1)$ equations with $M(M-1)$ unknown correlation coefficients ρ_{ij} and M unknown σ_i. The solution of these equations is possible only with additional assumptions.

To solve (12) one needs to determine the correlations ρ_{ij} in some way. It might be done directly:

$$\rho_{ij} = \frac{\sum \left(\vec{r}_i - \hat{\vec{r}}_i\right)\left(\vec{r}_j - \hat{\vec{r}}_j\right)}{\sigma_i\sigma_j}, \qquad (13)$$

or by the method proposed in [1], where ρ_{ij} are the solutions of a linear equation system, built on $(RF_i - RF_j)$ and $(RF_i + RF_j)$ dispersions. Here $\vec{r}_i^{(*)}$ are points of RF_i after applying systematic correction between RF_i and combined RF_0. If d_{ij}^2 is the estimation of the difference between the dispersions of the catalogues $(RF_i - RF_j)$, and s_{ij}^2 is an estimation of the sum of the dispersions of the catalogues $(RF_i + RF_j)$. For three catalogues [1] gives the linear equations:

$$\begin{cases} \sigma_1^2 = (s_{12}^2 + s_{13}^2 - s_{23}^2 - \sigma_{12}^2 - \sigma_{13}^2 + \sigma_{23}^2)/4, \\ \sigma_2^2 = (s_{23}^2 + s_{12}^2 - s_{13}^2 - \sigma_{23}^2 - \sigma_{12}^2 + \sigma_{13}^2)/4, \\ \sigma_3^2 = (s_{23}^2 + s_{13}^2 - s_{12}^2 - \sigma_{23}^2 - \sigma_{13}^2 + \sigma_{12}^2)/4, \end{cases} \qquad (14)$$

and

$$\rho_{ij} = \frac{s_{ij}^2 - d_{ij}^2}{2\sigma_i\sigma_j}.$$

Yet another solution method of (13) uses the optimization procedure in the space of $\rho's$. For the case of $M = 3$ let us build a three-dimensional space ($\vec{\rho}$-space) where $\rho_{12}, \rho_{23}, \rho_{31}$ are coordinates. Then, let us rewrite (12) as a 3-dimensional vector equation:

$$\vec{f} = \begin{pmatrix} \sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2 - \sigma_{12}^2 \\ \sigma_2^2 + \sigma_3^2 - 2\rho_{23}\sigma_2\sigma_3 - \sigma_{23}^2 \\ \sigma_3^2 + \sigma_1^2 - 2\rho_{31}\sigma_3\sigma_1 - \sigma_{31}^2 \end{pmatrix} = \vec{0}, \qquad (15)$$

and proceed with the solution of (15) for σ's on the lattice of $\rho's$ in $\vec{\rho}$-space. Newton method of tangents:

$$\vec{\sigma}_{new} = \vec{\sigma}_{old} - \mathbb{J}^{-1}\vec{f}(\vec{\sigma}_{old}),$$

is fully sufficient with starting value $\vec{\sigma}_0$ and Jakobi matrix $\mathbb{J}$:

$$\vec{\sigma}_0 = \begin{pmatrix} \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{pmatrix}, \qquad \mathbb{J} = \left[\frac{\partial\vec{f}}{\partial\vec{\sigma}}\right].$$

Unfortunately, the proposed method generates a large number of solutions. For example, if we built a lattice in $\vec{\rho}$-space with 0.1 step, we will have $21^3 = 9261$ solutions.

Since the Newton method is unrestricted, some solutions have negative σ_i and therefore should be dropped. Other solutions demonstrate known behaviour: the greater negative $\rho's$ are, the lower $\sigma's$ are as a result. This is why we apply an additional restriction: the solution of (12) in $\vec{\rho}-$space should be a minimum-length positive component vector $\vec{\sigma}$ corresponding to the minimum value of $\rho's$ modules.

SHIFTED CORRELATED RANDOM ERRORS

Let us return to (12) and (10), which are the definition of σ_{ij}. If one estimates Helmert parameters from (11), then calculates σ_{ij} for transformation RF_i to RF_j, and then calculates σ_{ji}^2 for transformation from RF_j to RF_i, then one will have $\sigma_{ij}^2 \neq \sigma_{ji}^2$ and even $\rho_{ij} \neq \rho_{ji}$ in general.

Our explanation of this fact is that there is an uncompensated systematic error still present in the data (but not accounted with (9)) which distorts and shifts the dispersion estimation (12). If we suppose that the systematic part is not correlated with random one, we can rewrite (12) in an another form:

$$\begin{cases} \sigma_{ij}^2 = \sigma_i^2 + \sigma_j^2 + k_{ij} - 2\rho_{ij}\sigma_i\sigma_j, \\ \sigma_{ji}^2 = \sigma_i^2 + \sigma_j^2 - k_{ij} - 2\rho_{ij}\sigma_i\sigma_j, \end{cases} \qquad (16)$$

where k_{ij} is an additional member, while the dispersion of a portion of the systematic errors is still present in random residuals.

It is thus evident that:

$$k_{ij} = \left(\sigma_{ij}^2 - \sigma_{ji}^2\right)/2.$$

This term is absolutely artificial and its inclusion in (16) still needs mathematical approvement, however its value might serve as goodness-of-fit criteria of the applicability of Helmert transform.

NUMERICAL EXPERIMENT

The specially assigned software was created for the purpose of numerical experiments with the Helmert transform. The software builds three artificial catalogues consisting of points on the Earth's surface, equally distributed along longitude and latitude with 5° steps; each catalogue containing 2664 points. Then Gaussian correlated random noise was added to each points in the catalogues. To generate Gaussian random noise values, the following well-known transformation was used:

$$(u_1, u_2) \to$$
$$\to (n_1 = u_1\sqrt{-2\log s/s}, n_2 = u_2\sqrt{-2\log s/s}),$$

where $s = u_1^2 + u_2^2$, and $0 < s < 1$ with u_1 and u_2 are random values uniformly distributed on $[-1, 1]$. Before they were applied, the values of (n_1, n_2) were correlated with one another according to the rule:

$$(n_1, n_2) \to (n_1, n_1\rho + n_2\sqrt{1-\rho^2})$$

with predefined ρ. Generation of uniformly distributed values was conducted with $x128()$ algorithm [5].

After comparison of the catalogues, building the combined one, values of errors and correlations were calculated using three different approaches. These are a) standard formulae (11) and (13); b) linear solution from [1]; c) optimization in $\vec{\rho}$-space.

The most interesting for us are values of correlations, calculated through different methods. In most cases the correlations according to (14) are 5%–10% less than correlations according to (13). In contrary to this, values of σ do not show any similarities.

The following example might show the correlation differences: (0.261, 0.677, 0.383) from (14) and (0.331, 0.700, 0.442) from (13), whilst sigmas are (7.68, 1.16, 4.79) with (14) correlations and (3.72, 4.72, 0.89) with (13) ones. The most surprising third method, as it can converge to both of the results depending of starting conditions.

CONCLUSION

Extended Helmert transform was used for the comparison of model TRF catalogues, and it was shown that systematic errors are accounted for in a more precise way than with the classical Helmert transform.

Can the extended transform be extended once again? We may add spherical functions to the right side of (9), like in many astrometric texts on catalogue comparison, e. g. [12].

However, as we used throughout the article the analogy between geodesy and stellar astronomy, we can suppose that there is a time to start searching for "tectonic plates" in the sky, like it was done in geodesy years ago.

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