Fizika Nizkikh Temperatur, 2004, v. 30, No. 6, p. 590–595

Turbulence of second sound waves in superfluid He II

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Received January 12, 2004, revised January 23, 2004

We communicate the results of numerical studies of acoustic turbulence in a system of slightly dissipating, nonlinear second sound waves in superfluid He II. It is shown that at sufficiently high amplitude of the external driving force the power-like energy distribution over frequency is formed in the system of second sound waves. This distribution is attributed to formation of the acoustic turbulence regime in the system. The interval of frequencies in which the distribution has a power-like form is expanded to high frequencies with increasing the amplitude of the driving force. The distribution of the energy inside this interval is close to $E_\omega \sim \omega^2$. It is shown that the distribution of energy E_ω depends on the value of the nonlinearity coefficient of the second sound, but does not depend on the sign of the coefficient, i.e., the coherent structures (shock waves) do not contribute to the statistical properties of the turbulent state.

PACS: 05.20.Dd, 47.27.Eq, 67.40.Pm

1. Introduction

We report on observations in numerical experiment of Kolmogorov type spectrum of turbulence in a system of one-dimensional weakly dissipating sound waves. Studies of turbulence of sound waves (acoustic turbulence) is of importance due to many applications in physics: nonlinear waves in superfluid He II [1–3], phonon turbulence in perfect crystals [4], waves in an interstellar space [5], etc.

Traditionally the theory of weak, or wave turbulence could be used as an appropriate basis for understanding the turbulent phenomena in system of interacting waves (see the monograph [6] and references therein). For example, recent studies of turbulence of capillary waves at the surface of liquid hydrogen [7–9] and of water [10–13] have demonstrated that the experimental observations are in good accordance with the predictions of the theory [6,14] and with the results of numerical computations [15] based on this theory.

In case of the turbulence in a system of sound waves with the dispersion law

$$\omega_k = uk \tag{1}$$

the approach based on the ideas of the weak turbulence theory meets difficulties due to divergence in the perturbation theory series [18] (here ω_k is the frequency of the linear wave with the wave vector k, u is the sound velocity). These divergence appears owing to degeneration of the resonance manifold for three-wave interaction in k space in case of the linear dispersion law (1): only waves whose k vectors are collinear with each other could interact efficiently. Due to this fact the peculiarities of the nonlinear and turbulent behavior of such a system could differ from that of the system of waves in dispersive media.

Numerical calculations provide a nice opportunity to study from the first principles the peculiarities in turbulent behavior of a systems of acoustic waves, by integrating numerically the equations of motion of liquid. The results of similar studies made for dispersive systems — the gravity and capillary waves at the free surface of liquid have been published recently in [15–17].

The present paper addresses the numerical studies of acoustic turbulence in a system of sound waves with small damping. Theoretical estimations have shown that strongly anisotropic turbulent patterns could be formed in a system of sound waves [6]. In order to avoid additional difficulties related to the pattern formation we consider here the simplest model case, in which the one-dimensional sound waves in He II are considered. We suppose that the medium in which the one-dimensional sound waves propagate, is restricted from both sides by reflecting walls (waves in a resonator of finite size). This consideration corresponds in general to the conditions of the experiments [1,2] with one-dimensional nonlinear second sound waves in superfluid He II. The results of our calculations can be used, also, for qualitative treatment of the experiments [7–9] with capillary waves at the surface of liquid hydrogen in a cell of small dimensions.

Our calculations show that at high amplitudes of the external driving force, a power-law distribution of the amplitudes of the sound waves over the frequency is formed at frequencies higher the driving frequency. This distribution can be attributed to formation of the wave turbulence regime in a system of sound waves, and it is similar to the Kolmogorov spectrum of turbulence [20]. The range of frequencies in which the power-law distribution is established expands to high frequencies with increasing amplitude of the driving force. The power-law spectrum is violated at high frequencies due to transition from the regime where the nonlinear wave transformation plays the essential role in the energy transfer through the scale to the regime where the viscous damping dominates.

2. Basic equations

In this paper we study the turbulence in a system of second sound waves in He II as an example of a nonlinear wave system. It is known that the second sound waves in He II demonstrate a highly nonlinear behavior, and they are a nice test object for studying the dynamics of nonlinear waves [1,2].

It is convenient to use in numerical calculations the Hamiltonian formulation of superfluid hydrodynamics. The Hamiltonian formalism in hydrodynamics of superfluid He II has been developed in the paper [21]. This approach has been generalized for superfluid ^{4}He–^{3}He mixtures in paper [22].

For the sake of convenience of the readers and for a statement of the notations used in the subsequent calculations we write down the known [22,23] basic equations of motion for nonlinear waves of second sound in He II in the Hamiltonian representation for planar (one-dimensional) second sound waves in a resonator. This representation is a classical limit for superfluid helium hydrodynamics formulated in terms of the first and second sound quanta. Such classical limit can be used, obviously, if the occupation number of the corresponding states is sufficiently large,

$$|b_n|^2 \gg \hbar, \tag{2}$$

where b_n is the canonical amplitude of the second sound, see below. Moreover, the term «second sound quantum» itself is correct only if the wavelength of the second sound is much larger than the mean free path of the quantum excitations l_f [23,24],

$$kl_f \ll 1, \tag{3}$$

where k is the wave vector of the second sound wave; see (6). We suppose in this paper that both conditions (2), (3) are satisfied.

An arbitrary flow of superfluid He II can be described by three pairs of conjugate variables (φ, ρ), (β, S) and (γ, f) [21]. Here φ is the superfluid velocity potential, ρ is the density of the liquid, S is the entropy of a unit mass, β is the phase variable conjugate to S, and γ and f are the Clebsch variables. The Hamiltonian function of the system is given be the total energy of the liquid

$$H = \int d^3\mathbf{r}\left(\frac{1}{2}\rho\mathbf{v}_s^2 + p\mathbf{v}_s + E_0(\rho, S, \mathbf{p})\right). \tag{4}$$

Here $E_0(\rho, S, \mathbf{p})$ is the energy of a unit of volume of the liquid in the reference frame moving with the velocity $\mathbf{v}_s$ of the superfluid component, and $\mathbf{p}$ is the momentum of the relative motion of the normal component.

In the subsequent analysis we neglect the oscillations of the density ρ in the second wave due to the fact that the thermal expansion coefficient of the liquid helium is small $(T/\rho)(\partial\rho/\partial T) \ll 1$ if the temperature of the helium bath is not very close to the temperature of the superfluid transition T_λ. In this approximation the oscillations of the two variables β and S are nonzero only in the second sound wave propagating through the unperturbed superfluid. The equations of motion in this representation are

$$\dot{S} = \frac{\delta H}{\delta \beta}, \qquad \dot{\beta} = -\frac{\delta H}{\delta S}, \tag{5}$$

with the Hamiltonian (4). The momentum of relative motion of the normal component is

$$\mathbf{p} = S\nabla\beta.$$

If the sound waves are propagating in unrestricted superfluid the Hamiltonian variables β and S could be expressed via the normal coordinates — the amplitudes $b_\mathbf{k}$ of the sound waves with the wave vector $\mathbf{k}$ [21]. In the case under study (waves in a superfluid helium in a resonator) the corresponding normal variables are given by the amplitudes b_n of standing second sound waves. The frequencies of the standing waves are equal to the resonant frequencies $\omega_n = u_{20}k_n$, where u_{20} is the second sound velocity, and the wave number k_n corresponding to the nth resonant frequency is

$$k_n = \pi n/L, \tag{6}$$

L is the length of the resonator. The number $n = 1$ will correspond to the lowest resonant frequency of the cell.

Oscillations of the variables β and S in the one-dimensional second sound wave are expressed via the normal coordinates as follows

$$\beta(x,t) = \sum_n \bar{\beta}_n \cos(k_n x)(b_n - b_n^*), \tag{7}$$

$$\delta S(x,t) = \sum_n \bar{S}_n \cos(k_n x)(b_n + b_n^*), \tag{8}$$

where the normalization factors are

$$\bar{\beta}_n = -i\left[\omega_n L\left(\frac{\partial S}{\partial T}\right)\right]^{-1/2}, \quad \bar{S}_n = \left[\frac{\omega_n}{L}\left(\frac{\partial S}{\partial T}\right)\right]^{1/2}. \tag{9}$$

The equations of motion to be integrated are presented by the Hamiltonian equations for the normal coordinates b_n,

$$i\dot{b}_n = \frac{\partial H}{\partial b_n^*} - i\gamma_n b_n + f_n. \tag{10}$$

The dissipation and the interaction with the external driving force are included phenomenologically in these equations, H is the Hamiltonian function of the system, γ_n is the damping coefficient of the nth standing wave, f_n is the driving force acting on the nth resonance, a dot denotes the derivative with respect to time t, and a star denotes the complex conjugate.

The standard way to describe the dynamics of the system of interacting waves is to use an expansion of the Hamiltonian H in a series in the normal coordinates b_n

$$H = H_2 + H_3. \tag{11}$$

The term H_2 is quadratic function of the normal coordinates b_n

$$H_2 = \sum_n \omega_n |b_n|^2; \tag{12}$$

it describes the propagation of linear second sound waves. The term H_3 is cubic in the amplitude b_n and corresponds to mutual interaction of three second sound waves — splitting of one wave into two waves and confluence of two waves into one wave. The higher order terms are omitted in the expansion (11), so the nonlinear processes of fourth and higher order are disregarded. These processes involve the interaction of four or more of waves. They are important for isotropization of the spectrum in the three-dimensional case, and they should be taken into account in a more general theory. The general form of the interaction Hamiltonian is

$$H_3 = \sum_{n_1,n_2,n_3} V_{n_1,n_2,n_3} b_{n_1}^* b_{n_2} b_{n_3} \delta_{n_1-n_2-n_3} + \text{c.c}. \tag{13}$$

Here V_{n_1,n_2,n_3} is the amplitude of nonlinear interaction of three second sound waves. It is supposed that nonresonant terms in (13), for which the resonance condition

$$\omega_1 - \omega_2 - \omega_3 = 0 \tag{14}$$

is not satisfied, are excluded from the Hamiltonian function by canonical transformation (see [6] for details). For the second sound waves with the dispersion relation (1) it follows from Eq. (6) that the resonance condition (14) is equivalent to the condition

$$n_1 - n_2 - n_3 = 0.$$

This corresponds to the presence of the Kroneker delta in the right-hand side of Eq. (13).

The amplitude of interaction of second sound waves in He II with wave vectors k_1, k_2 and k_3 is equal to [22]

$$V(k_1,k_2,k_3) = \frac{\alpha}{\sqrt{2}}\frac{\sigma}{(\partial\sigma/\partial \mathrm{T})}\sqrt{\frac{\rho_s u_{20}}{\rho\rho_n}}\sqrt{k_1 k_2 k_3} = $$

$$= \bar{V}\sqrt{k_1 k_2 k_3}. \tag{15}$$

Here α is the nonlinearity coefficient of the second sound waves, σ is the entropy per unit mass, ρ_s and ρ_n are the superfluid and normal density, and u_{20} is the second sound velocity. In case of interaction of three standing second sound waves in a resonator the amplitude of interaction acquires the form

$$V_{n_1,n_2,n_3} = V_0\sqrt{n_1 n_2 n_3}, \tag{16}$$

where $V_0 = \bar{V}/u_{20}^{3/2}$. The sign of the amplitude of interaction V_{n_1,n_2,n_3} coincides with the sign of the nonlinearity coefficient α of second sound and it could be negative or positive, depending on the temperature and pressure in the superfluid liquid. It is known [23] that in superfluid ^{4}He at saturated vapor pressure the nonlinearity coefficient α of the roton second sound is negative at temperatures $T_\alpha < T < T_\lambda$ (where T_λ is the temperature of the superfluid transition) and is positive at $T < T_\alpha$. At $T = T_\alpha = 1.88$ K the nonlinearity coefficient α is equal to zero. Increase of the pressure in a superfluid, as well as introduction of ^{3}He atoms into He II, should lead to lowering the temperature T_α [2,22]. For example, in superfluid He II with 10% of ^{3}He atoms impurity the temperature T_α is lowered to 1.7 K at saturated vapor pressure. So both two cases $V_0 > 0$ and $V_0 < 0$ in Eq. (16) have physical meaning.

The damping coefficient of the second sound wave is chosen as $\gamma_n = Cn^2$. In a real experiment the value of the constant C is determined by viscous damping of the sound waves in the bulk of liquid and, also, by the energy losses at the walls of the resonator. The constant C is estimated roughly as $C \sim 1/Q$, where Q is the quality factor of the resonator. The typical value of the Q factor in experiments [1] with second sound waves was $Q \sim 10^2$–10^3. In the calculation reported we put $C = 10^{-2}$ (which corresponds to $Q \sim 10^2$ at $n = 1$), so $\gamma_n = 10^{-2} n^2$.

The basic system of equations (10) to be integrated numerically is obtained by collecting the formulas (10)–(13), (16):

$$i\dot{b}_n = V_0 \sum_{n_1, n_2} (n n_1 n_2)^{1/2} \, (b_{n_1} b_{n_2} \delta_{n - n_1 - n_2} + $$
$$+ 2 b_{n_1} b^*_{n_2} \delta_{n_1 - n_2 - n}) - i\gamma_n b_n + f_n . \tag{17}$$

The linear term $\omega_n b_n$, which arises in right-hand side of Eq. (17) from the quadratic part H_2 of the Hamiltonian (11), is eliminated by the change of variables $b_n \to b_n \exp(-i\omega_n t)$. The simplest initial conditions for the second sound wave amplitude is used in this paper:

$$b_n(t = 0) = 0.$$

We integrate numerically the equations of motion (17) in two cases: a) the second sound in a resonator is excited by the harmonic external driving force with a frequency equal to one of the resonant frequencies, and b) the second sound is driven by a quasiperiodic force, the frequency spectrum of which contains several harmonics.

The distribution of the amplitudes of the second sound waves

$$P(n) = \langle |b_n(t)|^2 \rangle \tag{18}$$

over the wave number n averaged over time t is calculated from the results of integration. The energy distribution over frequency can be found from Eq. (11). In the present case of one-dimensional waves as a first approximation it is equal to

$$E_\omega = \omega_n P(n). \tag{19}$$

We evaluate the frequency dependence of the energy distribution E_ω of the second sound wave system from the calculated $P(n)$ dependence.

3. Results and discussion

Driving at a single resonance frequency

First, we consider the case where a periodic driving force excites a standing second sound wave at the frequency equal to the lowest resonant frequency ω_1 of the resonator,

$$f_1(t) = f_0 \exp(-i\omega_1 t),$$

f_0 is the amplitude of the driving force. As it was pointed above, the dependence on time t of the amplitudes of second sound waves b_n is determined by numerical integration of the equations (10). At large time (when the relaxation processes are finished) the amplitudes b_n tend to some constant values. This means that we may omit the averaging over time in the definition (18) in this case.

We calculate the spectrum $P(n)$ for given amplitude f_0 and study the evolution of the distribution $P(n)$ with increasing driving amplitude f_0. Figure 1 shows the distributions $P(n)$ calculated for three values of the driving force $f_0 = 0.01$ (circles), 0.1 (triangles) and 1.0 (boxes). Lines connecting the symbols are drawn as a guide to the eye. The straight line corresponds to the dependence $P(n) \sim n^{-3}$.

In the inset in Fig. 1 the dependences of the effective exponent

$$s(n) = \frac{d \log P(n)}{d \log n} \tag{20}$$

on the resonance number n are shown for $f = 0.01$ and $f_0 = 1$. It is seen that in the case $f_0 = 1$ there is an interval of resonance numbers n in which the effective exponent s is close to $s = -3$.

From Fig. 1 it is seen that at sufficiently high amplitudes of the driving force there exists a range of frequencies higher the driving frequency, in which the distribution $P(n)$ of the amplitudes of weakly dissipating acoustic waves can be described by a

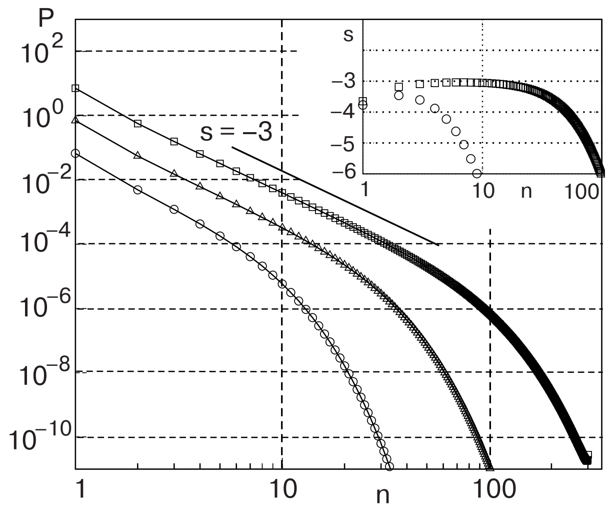

Fig. 1. The spectrum $P(n)$ of second sound oscillations calculated for the amplitudes of the monochromatic driving force $f_0 = 0.01$ (○), 0.1 (△), and 1 (□). The driving is applied at the lowest resonance frequency. The straight line corresponds to the power distribution $P(n) \sim n^{-3}$. The inset shows the dependence of the effective scaling exponent s on the resonance number n for $f_0 = 0.01$ and $f_0 = 1$.

power-like function $P(n) \sim n^{-3}$. This dependence corresponds to an energy distribution in the frequency scale (see Eq. (19))

$$E_\omega = \mathrm{const} \cdot \omega^{-2}. \tag{21}$$

The interval of frequencies, in which the power spectrum (21) is established, is enlarged to high frequencies with increasing amplitude of the driving force.

Driving by a quasiperiodic force

In this Section we present the results of calculations in which the driving force is not monochromatic, but the frequency spectrum of the force has several harmonics (quasiperiodic force).

In the calculations it is assumed that the external driving force excites the three lowest resonant modes directly, and $f_n = 0$ for $n > 3$. For each resonant mode n ($n = 1, 2, 3$) the corresponding driving force f_n is the sum of three periodic harmonics, the frequencies of which are incommensurate with each other and with the resonant frequency of the given resonance. The integral power that is pumped by the driving force into the system can be characterized in this case by the effective dispersion

$$D = \sum_n \langle f_n f_n^* \rangle,$$

where the angle brackets denote averaging over time.

In the case of pumping of the system by a quasiperiodic force the amplitudes b_n do not tend to some limiting values at large time but are fluctuating at all t. In calculations the distribution $P(n)$ averaged over time t is determined from the results of numerical integration of the equations (17). Figure 2 demonstrates the evolution of the distribution $P(n)$ with increasing

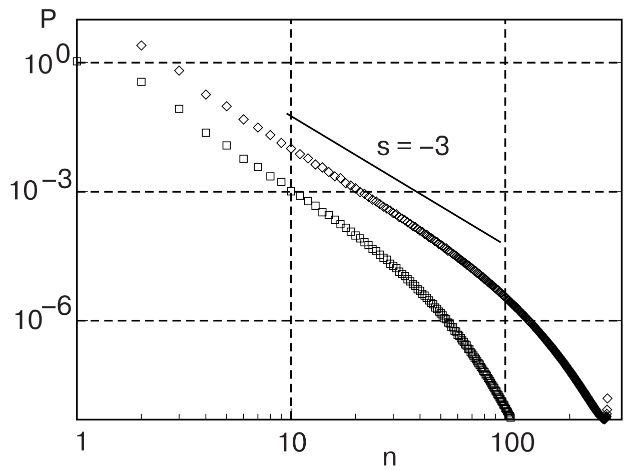

Fig. 2. Evolution of the spectrum with increasing of the effective dispersion of the quasiperiodic driving force from $D = 0.63$ to 4.2. The straight line corresponds to the power distribution $P(n) \sim n^{-3}$.

D from 0.63 to 4.2. It is seen from Fig. 2 that at high D the power-like distribution

$$P(n) \sim n^{-3} \tag{22}$$

is formed in some region of wave numbers, similarly to the case of pumping by a monochromatic driving force.

The observed power-like spectrum of oscillations could be attributed to the formation of the acoustic turbulent state in the system of second sound waves. The range of frequencies (or the region of the wave numbers), in which the scale-invariant distribution (22) is established, can be called «the inertial range» in analogy with the interval of frequencies where the scaling law is valid in Kolmogorov's picture of turbulence. At low frequencies the inertial range is limited by the characteristic frequency of driving, and at high frequencies the inertial range is limited by the transition from the regime of nonlinear transfer of the energy of waves over scales to the regime where the viscous damping dominates.

The calculations are performed for both positive and negative signs of the nonlinearity coefficients α (i.e., for positive and negative V_0 in Eq. (16)). It is observed that the phases of the high-frequency waves generated due to nonlinearity are different in cases $V_0 > 0$ and $V_0 < 0$ (with the same absolute value $|V_0|$), but the averaged distributions $P(n)$ are the same in these two cases. This indicates that formation of the coherent structures (like shock waves) did not affect the turbulent distribution of acoustic waves in a resonator. The physical sense of this fact is that the mutual nonlinear interaction of the second sound waves reflected from the resonator's walls prevents the shock front formation. This fact could be important for future analyses of the acoustic turbulence in superfluid helium, because earlier the sound turbulence has been considered mainly as a statistic of shock waves propagating in an unrestricted medium.

Note that in case of one-dimensional waves with the nondecay dispersion law, for which the four-wave interaction plays the main role, the coherent effects (the wave collapse or the soliton formation) are quite important in a turbulent regime [19].

We should note that the scaling index of the turbulent distribution $s = -3$ observed in these calculations is distinguished from the index $s_0 = -2.5$ which could be calculated by using the kinetic equation of the weak acoustic turbulence theory [1] in case of one-dimensional acoustic waves. This could indicate that the high-order corrections to the standard kinetic equations are important in the case under study. So, applicability of the kinetic equations (or some of its improved variants) for description of turbulence in a system of one-dimensional sound waves could be a question for further studies.

4. Conclusions

Numerical studies of the dynamics of the nonlinearly interacting one-dimensional sound waves shows that a power-like distribution of energy is formed at some range of frequencies if the amplitude of the low-frequency driving force is sufficiently high. In this inertial range of frequencies the distribution of energy over frequency is close to $E_\omega \sim \omega^2$. This spectrum can be attributed to formation of a turbulent state in the system of acoustic waves. The inertial range is expanded toward high frequencies with increasing amplitude of the driving force. At high frequencies the inertial range is limited by a change of the mechanism of energy transfer from nonlinear wave transformation to viscous damping. The shape of the energy spectrum depends on the absolute value of the nonlinearity coefficient of the sound waves (does not depend on its sign), which manifests the fact that the formation of the coherent structures does not influence the energy distribution in this turbulent system.

The investigations are supported in part by INTAS (grant 2001-0618), by RFBR (grants 03-02-16865 and 03-02-16121), and by the Ministry of Industry, Science and Technology of the RF in frames of contract No. 40.012.1.1.11.64, theme «Quantum phenomena at low and ultralow temperatures», and by the grant NSh-2169.2003.2. G.K also acknowledges support from the Science Support Foundation (Russia).

1. I.Yu. Borisenko, V.B. Efimov, and L.P. Mezhov-Deglin, *Fiz. Nizk. Temp.* **14**, 1123 (1988) [*Sov. J. Low Temp. Phys.* **14**, 619 (1988)].
2. V.B. Efimov, G.V. Kolmakov, E.V. Lebedeva, L.P. Mezhov-Deglin, and A.B. Trusov, *J. Low Temp. Phys.* **119**, 309 (2000).
3. G.V. Kolmakov, *Physica* **D86**, 470 (1995).
4. V.S. Tsoi, *Centr. Europ. J. Phys.* **1**, 72 (2003).
5. S. Grzedzielski and R. Lallement, *Space Sci. Rev.* **78**, Nos. 1–2 (1996).
6. V. Zakharov, V. L'vov, and G. Fal'kovich, *Kolmogorov Spectra of Turbulence*, Springer-Verlag, Berlin (1992), Vol. 1.
7. M.Yu. Brazhnikov, G.V. Kolmakov, A.A. Levchenko, and L.P. Mezhov-Deglin, *JETP Lett.* **73**, 398 (2001).
8. M.Yu. Brazhnikov, G.V. Kolmakov, A.A. Levchenko, and L.P. Mezhov-Deglin, *JETP Lett.* **74**, 583 (2001).
9. M.Yu. Brazhnikov, G.V. Kolmakov, and A.A. Levchenko, *JETP* **95**, 447 (2002).
10. W.B. Wright, R. Budakian, and S.J. Putterman, *Phys. Rev. Lett.* **76**, 4528 (1996).
11. E. Henry, P. Alstrom, and M.T. Levinsen, *Europhys. Lett.* **52**, 27 (2000).
12. M.Yu. Brazhnikov, G.V. Kolmakov, A.A. Levchenko, and L.P. Mezhov-Deglin, *Europhys. Lett.* **58**, 510 (2002).
13. M. Lommer and M.T. Levinsen, *J. Fluoresc.* **12**, 45 (2002).
14. V.E. Zakharov and N.N. Filonenko, *J. Appl. Mech. Techn. Phys.* **5**, 62 (1967).
15. A.N. Pushkarev and V.E. Zakharov, *Physica* **D135**, 98 (2000).
16. A.I. Dyachenko, A.O. Korotkevich, and V.E. Zakharov, *JETP Lett.* **77**, 477 (2003).
17. A.I. Dyachenko, A.O. Korotkevich, and V.E. Zakharov, *JETP Lett.* **77**, 546 (2003).
18. V. L'vov, Yu. L'vov, A.C. Newell, and V.E. Zakharov, *Phys. Rev.* **E56**, 390 (1997).
19. V.E. Zakharov, P. Guyenne, A.N. Pushkarev, and F. Dias, *Physica* **D152–153**, 573 (2001).
20. A.N. Kolmogorov, *Dokl. Akad. Nauk SSSR* **30**, 299 (1941).
21. V.L. Pokrovskii and I.M. Khalatnikov, *JETP* **44**, 1036 (1976).
22. G.V. Kolmakov, *Fiz. Nizk. Temp.* **29**, 667 (2003) [*Low Temp. Phys.* **29**, 495 (2003)].
23. I.M. Khalatnikov, *An Introduction to the Theory of Superfluidity*, Perseus Publishing (1989).
24. E. Lifshits and L. Pitaevskii, *Statistical Physics*, part II, Nauka, Moscow (1978), p. 121.