Condensed Matter Physics 2008, Vol. 11, No 4(56), pp. 755–760

CONDENSED MATTER PHYSICS

Local solutions to Darboux problem with a discontinuous right-hand side

P.Pikuta

Marie Curie-Skłodowska University, Institute of Mathematics
pl. Marii Curie-Skłodowskiej 1, 20–031 Lublin, Poland

Received January 31, 2008

The existence of a local solution to the Darboux problem $u_{xy}(x,y) = g(u(x,y))$, $u(x,0) = u(0,y) = 0$, where g is Lebesgue measurable and has at most polynomial growth, is proved.

Key words: *Darboux problem, discontinuous differential equations*

PACS: *02.30.Jr*

1. Introduction

In this paper we deal with the Darboux problem

$$u_{xy}(x,y) = g(u(x,y)) \text{ a.e. in } [0,T]\times[0,T], \tag{1}$$

$$u(x,0) = u(0,y) = 0, \tag{2}$$

where g is not assumed to be continuous. The problem arises as a natural extension of the Cauchy problem for an autonomous equation $x'(t) = f(x(t))$ with a discontinuous right-hand side, see [1].

Definition 1 *We say that a continuous function $u : [0,T]\times[0,T] \to \mathbb{R}$ is a solution to the Darboux problem*

$$u_{xy}(x,y) = g(x,y,u(x,y)), \tag{3}$$

$$u(x,0) = u(0,y) = 0, \tag{4}$$

if u is a C^1 function on $(0,T)\times(0,T)$, satisfying the equation (3) a.e. in $[0,T]\times[0,T]$ and the initial condition (4) for all $x,y \in [0,T]$.

In most papers devoted to the discontinuous Darboux problem, see e.g. [2,3], the right-hand side of equations

$$u_{xy}(x,y) = g(x,y,u) \quad \text{or} \quad u_{xy}(x,y) = g(x,y,u,u_x,u_y)$$

is usually assumed to satisfy Carathéodory-type conditions:

- f is measurable with respect to the first two variables,
- f is continuous or Lipschitz continuous with respect to other variables,
- $|f|$ is bounded by a constant or by an integrable function $M(x,y)$.

Recently, another approach has been presented (see [4,5]) for the equations

$$u_{xy}(x,y) = g(xy, u(x,y)) \quad \text{and} \quad u_{xy}(x,y) = g(u(x,y)),$$

showing that strong assumptions of continuity or Lipschitz continuity can be replaced by measurability conditions.

In this paper, the existence of a local solution to the Darboux problem (1)–(2), where g is Lebesgue measurable and has at most polynomial growth, is proved.

Although the main purpose of this paper is to establish an existence theorem for (1)–(2), the method used here (see also [5]) involves functional differential equations, namely, the problem

$$\begin{aligned} q'(t) &= g\left(\int_0^t \frac{q(\sigma)}{\sigma}\mathrm{d}\sigma\right) \text{ a.e. in } [0,T], \\ q(0) &= 0 \end{aligned}$$

with the same function g as in (1).

Throughout this paper the term *measure* instead of *Lebesgue measure* μ is used as well as other concepts such as measurability and integrability are understood as Lebesgue measurability and Lebesgue integrability. By $C[0,T]$ we denote the normed linear space of all continuous functions $x:[0,T]\to\mathbb{R}$ with the norm $\|x\| = \sup\limits_{t\in[0,T]} |x(t)|$.

2. Main result

Theorem 1 *Assume that $g:\mathbb{R}\to\mathbb{R}$ is measurable and for some $n\in\mathbb{N}$, $a_0,a_1,\ldots,a_n\in\mathbb{R}$ and $a>0$*

$$a \leqslant g(u) \leqslant a_0 + \sum_{k=1}^{n} a_k u^k, \quad u\in[0,+\infty).$$

Then there exists $T>0$ such that the problem

$$\begin{aligned} u_{xy}(x,y) &= g(u(x,y)), \text{ a.e. in } [0,T]\times[0,T], && (5)\\ u(x,0) &= u(0,y) = 0, && (6) \end{aligned}$$

has a solution.

The proof of Theorem 1 is based on the following lemma.

Lemma 1 *If $g:\mathbb{R}\to\mathbb{R}$ is measurable and for some $n\in\mathbb{N}$, $a_0,a_1,\ldots,a_n\in\mathbb{R}$ and $a>0$*

$$a \leqslant g(u) \leqslant a_0 + \sum_{k=1}^{n} a_k u^k, \quad u\in[0,+\infty),$$

then there exists $T>0$ such that the problem

$$\begin{aligned} q'(t) &= g\left(\int_0^t \frac{q(\sigma)}{\sigma}\mathrm{d}\sigma\right) \text{ a.e. in } [0,T], && (7)\\ q(0) &= 0 && (8) \end{aligned}$$

has a solution.

Proof of Lemma 1. Define

$$b = a_0 + 1$$

and take $T>0$ such that for all $t\in[0,T]$

$$a_0 + a_1 bt + \ldots + a_n b^n t^n \leqslant b.$$

Let

$$Z = \{x\in C[0,T] : x(0)=0,\ a(t-s)\leqslant x(t)-x(s)\leqslant b(t-s),\ 0\leqslant s<t\leqslant T\}.$$

Obviously Z is closed and convex. Moreover, for each $x \in Z$ and all $t, s \in [0, T]$ we have

$$0 \leqslant x(t) \leqslant bT$$

and

$$|x(t) - x(s)| \leqslant b\,|t - s|\,,$$

which implies that Z is compact.

We claim that $A : Z \to Z$, defined by

$$(Aq)(t) = \int_0^t g\left(\int_0^z \frac{q(\sigma)}{\sigma}\mathrm{d}\sigma\right)\mathrm{d}z,\ \ t \in [0, T]\,,$$

is continuous.

Fix $q \in Z$ and define $h : [0, T] \to \mathbb{R}$ by

$$h(z) = \int_0^z \frac{q(\sigma)}{\sigma}\mathrm{d}\sigma.$$

The function h is continuous, strictly increasing and for each $t, s \in [0, T]$, $t > s$, satisfies

$$a(t - s) \leqslant h(t) - h(s) = \int_s^t \frac{q(\sigma)}{\sigma}\mathrm{d}\sigma \leqslant b(t - s)\,.$$

Thus $h \in Z$, and for each $u, v \in h([0, T])$ we have

$$\left|h^{-1}(u) - h^{-1}(v)\right| \leqslant \frac{1}{a}\left|h\left(h^{-1}(u)\right) - h\left(h^{-1}(v)\right)\right| = \frac{1}{a}\,|u - v|\,, \tag{9}$$

which in turn implies that h^{-1} is absolutely continuous and strictly monotonic on a closed interval $h([0, T])$. Consequently, for each open interval $P \subset h([0, T])$,

$$(g \circ h)^{-1}(P) = h^{-1}\left(g^{-1}(P)\right)$$

is measurable. Thus $g(h(\cdot))$ is measurable and Aq is well defined.

Observe that $Aq \in Z$, because $(Aq)(0) = 0$ and for all $s, t \in [0, T]$, $t > s$, we have

$$a(t - s) \leqslant (Aq)(t) - (Aq)(s) = \int_s^t g\left(\int_0^z \frac{q(\sigma)}{\sigma}\mathrm{d}\sigma\right)\mathrm{d}z \leqslant b(t - s)$$

because for all $z \in [0, T]$

$$g(h(z)) \leqslant \sum_{k=0}^{n} a_k \left(\int_0^z \frac{q(\sigma)}{\sigma}\mathrm{d}\sigma\right)^k \leqslant \sum_{k=0}^{n} a_k (bz)^k \leqslant \sum_{k=0}^{n} a_k b^k T^k \leqslant b.$$

Fix $\varepsilon > 0$ and consider any sequence $q_n \in Z$, $n \in \mathbb{N}$, convergent (uniformly) to $q \in Z$. For each $n \in \mathbb{N}$, define $h_n : [0, T] \to \mathbb{R}$,

$$h_n(z) = \int_0^z \frac{q_n(\sigma)}{\sigma}\mathrm{d}\sigma.$$

By Lusin's theorem there exists a compact set $K \subset [0, bT]$, such that $g_{|K} : K \to [a, b]$ is continuous and

$$\mu([0, bT] \setminus K) < \frac{a\varepsilon}{8b}\,.$$

Since $g_{|K}$ is uniformly continuous, there exists $\delta > 0$ such that

$$|u-v| < \delta,\ u,v \in K \quad \text{implies} \quad |g(u)-g(v)| < \frac{\varepsilon}{2T}.$$

For $z \in \left[0, \frac{\delta}{2b+1}\right]$ we have

$$|h_n(z)-h(z)| \leqslant \int\limits_0^z \frac{|q_n(\sigma)-q(\sigma)|}{\sigma}\mathrm{d}\sigma \leqslant \int\limits_0^{\frac{\delta}{2b+1}} \frac{|q_n(\sigma)-q(\sigma)|}{\sigma}\mathrm{d}\sigma \leqslant \frac{2b\delta}{2b+1} < \delta.$$

If $z \in \left[\frac{\delta}{2b+1}, T\right]$, then

$$\begin{aligned} |h_n(z)-h(z)| &\leqslant \int\limits_0^z \frac{|q_n(\sigma)-q(\sigma)|}{\sigma}\mathrm{d}\sigma = \int\limits_0^{\frac{\delta}{2b+1}} \frac{|q_n(\sigma)-q(\sigma)|}{\sigma}\mathrm{d}\sigma + \int\limits_{\frac{\delta}{2b+1}}^z \frac{|q_n(\sigma)-q(\sigma)|}{\sigma}\mathrm{d}\sigma \\ &\leqslant \frac{2b\delta}{2b+1} + \int\limits_{\frac{\delta}{2b+1}}^z \frac{\|q_n-q\|}{\sigma}\mathrm{d}\sigma \leqslant \frac{2b\delta}{2b+1} + \|q_n-q\| \int\limits_{\frac{\delta}{2b+1}}^T \frac{\mathrm{d}\sigma}{\sigma} \\ &\leqslant \frac{2b\delta}{2b+1} + \|q_n-q\| \ln\frac{T(2b+1)}{\delta}. \end{aligned}$$

Since $\lim\limits_{n\to\infty} \|q_n-q\| = 0$, there exists n_0, such that for each $n > n_0$

$$\|q_n-q\| < \frac{\delta}{2b+1}\left(\ln\frac{T(2b+1)}{\delta}\right)^{-1}.$$

Therefore, for each $n > n_0$ and each $z \in [0,T]$ we have

$$|h_n(z)-h(z)| \leqslant \sup_{z\in[0,T]} \left| \int\limits_0^z \frac{q_n(\sigma)}{\sigma}\mathrm{d}\sigma - \int\limits_0^z \frac{q(\sigma)}{\sigma}\mathrm{d}\sigma \right| < \delta.$$

Consequently, if $n > n_0$ and $z \in [0,T]$, then

$$|g(h_n(z)) - g(h(z))| < \frac{\varepsilon}{2T},$$

provided that $h_n(z)$ and $h(z)$ belong to K.

Fix $n > n_0$ and define

$$F = h^{-1}(K) \cap h_n^{-1}(K).$$

We have

$$[0,T]\setminus F = [0,T]\setminus\left(h^{-1}(K)\cap h_n^{-1}(K)\right) = h^{-1}([0,bT]\setminus K) \cup h_n^{-1}([0,bT]\setminus K).$$

Therefore, using (9), we get

$$\mu([0,T]\setminus F) \leqslant \mu\left(h^{-1}([0,bT]\setminus K)\right) + \mu\left(h_n^{-1}([0,bT]\setminus K)\right) = \int\limits_{h^{-1}([0,bT]\setminus K)} \mathrm{d}z + \int\limits_{h_n^{-1}([0,bT]\setminus K)} \mathrm{d}z$$

$$= \int\limits_{[0,bT]\setminus K} \left(h^{-1}\right)'(z)\,\mathrm{d}z + \int\limits_{[0,bT]\setminus K} \left(h_n^{-1}\right)'(z)\,\mathrm{d}z \leqslant \frac{\mu([0,bT]\setminus K)}{a} + \frac{\mu([0,bT]\setminus K)}{a} \leqslant \frac{\varepsilon}{4b}.$$

Finally, we obtain

$$\begin{aligned}
\|Aq_n - Aq\| &\leqslant \sup_{t\in[0,T]} \int_0^t |g(h_n(z)) - g(h(z))|\,\mathrm{d}z = \int_0^T |g(h_n(z)) - g(h(z))|\,\mathrm{d}z \\
&= \int_F |g(h_n(z)) - g(h(z))|\,\mathrm{d}z + \int_{[0,T]\setminus F} |g(h_n(z)) - g(h(z))|\,\mathrm{d}z \\
&\leqslant \mu(F)\cdot\frac{\varepsilon}{2T} + \frac{\varepsilon}{4b}\cdot 2b \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Thus $A : Z \to Z$ is continuous.

It follows from Schauder's fixed point theorem that A has a fixed point in Z. Thus the problem (7)–(8) has a solution. ∎

Proof of Theorem 1. Let $T > 0$ be defined in the same way as in Lemma 1 and assume that $q : [0,T] \to \mathbb{R}$ is a solution to the problem (7)–(8). Define $v : [0,T] \to \mathbb{R}$

$$v(t) = \int_0^t \frac{q(\sigma)}{\sigma}\mathrm{d}\sigma,$$

and $u : [0,T] \times [0,T] \to \mathbb{R}$

$$u(x,y) = v(xy) = \int_0^{xy} \frac{q(\sigma)}{\sigma}\mathrm{d}\sigma.$$

Obviously, u is a C^1 function and for almost all $(x,y) \in [0,T] \times [0,T]$

$$u_{xy}(x,y) = v'(xy) + xy\cdot v''(xy) = q'(xy) = g(u(x,y)).$$

Moreover, for all $(x,y) \in (0,T) \times (0,T)$

$$u(x,0) = u(0,y) = v(0) = 0.$$

Thus u is a solution to the problem (5)–(6). ∎

References

1. Binding P., J. Differential Equations, 1979, **31**, 183–199.
2. Deimling K., Ann. Mat. Pure Appl., 1970, **(4) 86**, 217–260.
3. Guglielmino F., Boll. Un. Mat. Ital., 1958, **(3) 13**, 308–318 (in Italian).
4. Pikuta P., Rzymowski W., J. Math. Anal. Appl., 2003, **277**, 122–129.
5. Pikuta P., J. Appl. Anal., 2006, **12**, No. 1, 147–152.

Локальні розв'язки проблеми Дарбу з правою частиною, що має розриви

П.Пікута

Інститут математики, Університет Марії Кюрі-Склодовської, Люблін, Польща

Отримано 31 січня 2008 р.

Доведено існування локального розв'язку проблеми Дарбу $u_{xy}(x,y) = g(u(x,y))$, $u(x,0) = u(0,y) = 0$, де g є вимірна за Лебегом функція, що росте не швидше, ніж поліном.

Ключові слова: *проблема Дарбу, диференціальні рівняння з розривами*

PACS: *02.30.Jr*