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Doping-dependent superconducting properties of two-dimensional metals with different types of interparticle coupling

(Review Article)

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The superconducting properties of a two-dimensional metallic system with arbitrary carrier density and both local and various types of the indirect boson-exchange attractive interaction in the cases of *s*- and *d*-wave pairing are analyzed and reviewed at $T = 0$. In particular, the possibility of a crossover from the Bose–Einstein condensation regime to Bardeen–Cooper–Schrieffer-like superconductivity with growing carrier density and changing coupling in the case of different pairing channels is discussed. Gaussian fluctuations of the order parameter are taken into account, and the carrier density dependence of the gap magnitude is studied. The role of the form of the interparticle attractive interaction in the physical behavior of the system is also considered.

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1. Introduction

The doping dependence of the various properties of superconductors (including high-temperature ones (called HTSCs below) raises many theoretical questions. In particular, the question of the consistent description of such a dependence, which has a long history. Probably the first attempt to solve this problem self-consistently was made by Eagles in Ref. 1, in which the author tried to employ for HTSCs his results for the description of the superconducting properties of Zr-doped $SrTiO_3$ [2] (see also Ref. 3, where some experimental data were interpreted as the observation of the Bose–Einstein condensation (BEC) of electronic pairs in this compound at low itinerant carrier densities, and Refs. 4, 5, where it was theoretically demonstrated that there were reasons for such an interpretation). More exactly, the author studied the dependence of the superconducting gap at $T = 0$ and of the mean-field critical temperature on free charge carrier densities in two-dimensional (2D) and three-dimensional (3D) systems with a phonon-like indirect attraction at low densities. The set of coupled equations for the gap (gap equation) and the Fermi energy, or chemical potential (number equation), for these cases was obtained and analyzed. It was estimated that at such densities the diameter of the electronic pairs is smaller than the distance between them, and therefore in fact the superconductivity has to correspond to superfluidity of spatially separated (and in this sense small, or so-called local) pairs. It is important that if one can change (for instance, by doping) the distance between the particles, the transition from one regime to another (with strongly overlapped pairs) becomes in principle possible. Now this phenomenon is well-known as the crossover from BEC to Bardeen–Cooper–Schrieffer (BCS) superconductivity with changing carrier density.

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In Ref. 1 it was also shown that in 3D systems there exists a critical value of the attraction below which there are no local pairs, so that BEC–BCS crossover is impossible. It is noteworthes that the possibility of a superfluidity scenario for superconductivity was proposed by Ogg [6] and Schafroth [7–9] long before the Eagles's paper was published.

Later on the problem of the BEC–BCS crossover was considered by Leggett in [10], where 3D systems with short-range repulsion and finite-range attraction were studied. The author has analyzed the dependence of superconducting properties of the system upon the dimensionless parameter $1/(ak_F)$ (a is the scattering length and $\mathbf{k}_F$ is the Fermi momentum), which defines the itinerant particle number in the crystal. It was shown that in the limiting cases the metallic system consists of bound «bielectronic molecules» in real space if $1/(ak_F) = \infty$ and of Cooper pairs in $\mathbf{k}$ space when $1/(ak_F) = -\infty$. Nozières and Schmitt-Rink generalized these results to the case of finite temperatures and the lattice model with a separable interparticle nonretarded attractive potential V [11]. They have demonstrated that the BEC–BCS crossover is smooth with changing k_F and V, when V is larger than the corresponding critical value for the two-particle (two-fermion) bound state formation. The forementioned results were obtained for the case of s-wave pairing.

The real and unprecedented boom of interest in the crossover phenomena has started in the second half of the 1980s after the discovery of copper oxide HTSCs, materials with an evident and rather unusual dependence of the superconducting properties on the carrier density. It should be stressed that practically all physical (and observable) properties of the HTSC compounds are doping dependent, but below we shall concentrate on the description of the superconductivity and superconducting properties only.

Development of a consistent theory of HTSCs still remains one of the most difficult and one of the most important problems of modern condensed matter physics. Due to the complicated (multicomponent) crystal structure, lowered dimensionality, magnetism, strong electron correlations, inevitable presence of disorder, etc. a generally accepted theory of these compounds has not yet been completed.

During the last decade many models (see, for example, the review [12]) which take into account some (not all) cuprate peculiarities have been proposed to describe various properties of HTSCs, among them the doping dependence of the superconducting gap and the critical temperature. For instance, the 3D BEC–BCS crossover in the s-wave pairing channel was studied in Refs. 12–19 for the model with local (so-called four-fermion, or 4F) attraction, and in Ref. 20 for the model of a nonlocal although separable attractive potential. In Ref. 12, additional example with the on-site repulsion and intersite attraction was also considered. The role of the order parameter fluctuations at $T = 0$ was analyzed in Ref. 21 (3D case) and in Ref. 22 (both 2D and 3D cases). The quasi-2D model with s-pairing at zero temperature was also studied in Ref. 23. It can be also noticed that the role of the number equation in connection with the HTSC problem was emphasized in Refs. 24.

For a pure 2D system the problem of the crossover in the isotropic s-wave pairing channel was discussed at zero T (when the real long-range superconducting order is certainly possible — the Mermin–Wagner–Hohenberg–Coleman–Bogolyubov theorem [25]) in Refs. 12, 15, 18, 20, 26–31 and at finite T (when this order is characterized by the weak algebraic decay of correlations) in Refs. 12, 15, 18–20, 28–30, 32, 33. The metal–superconductor phase boundaries on the $n_f - U$ phase diagram (n_f is the number of fermionic particles, U is the on-site attraction in the «negative-U» Hubbard model) was investigated in Ref. 34 for $T = 0$ by means of the dynamical mean-field theory. The indirect retarded, phonon mediated, interfermion attraction and the features of the corresponding BEC–BCS crossover at $T = 0$ were for the first time studied in papers [35,36].

The 2D crossover for the anisotropic d-wave pair symmetry is more important because just such a pairing is usually considered to be observed in HTSCs. The models which lead to this phenomenon were considered in Refs. 18, 32, 37 at $T = 0$. The case of finite temperatures was studied in Ref. 38, where an effective (Ginzburg–Landau) potential with carrier-density-dependent coefficients was also derived. The doping dependence of the critical temperature T_c in the strongly correlated electron model with electron–phonon interaction was studied in Refs. 39, 40. It was found that the vertex corrections to the electron–phonon coupling in this model lead to strengthening of the d-wave superconductivity. In paper [29] the cases with different pairing symmetries, s, s_{ext}, d_{xy} and $d_{x^2-y^2}$, in the 2D system with nearest neighbor (n.n.) and next nearest neighbor (n.n.n.) attraction in the square lattice were studied at zero and finite temperatures in order to describe the doping dependence of the superconducting gap and critical temperature (see also [41]). The possibility of the BEC–BCS crossover at zero temperature for s-, d- and mixed $s + id$-pairings in the 2D system as a function of coupling constant was considered in Ref. 42. The same problem in the quasi-2D Hubbard model with n.n. attraction at finite T was studied in the pa-

per [43], where the doping dependence of the superconducting properties was also analyzed.

The 2D model with n.n. attraction and also n.n. and with n.n.n. hopping at $T = 0$ was studied in Ref. 44. It was shown that for some relation between n.n. and n.n.n. hopping parameters the system proves always to be in the BEC regime and there is no pairing at low carrier densities when the coupling is weak (see also Ref. 37). However, as was stressed in Refs. 31, 45, such a statement cannot be correct. A more general model with on-site repulsion and n.n. attraction at $T = 0$ in the *s*- and *d*-wave channels was investigated in Ref. 46, where the role of n.n.n. hoping was also studied. The *s*- and *d*-wave crossover at zero temperature in the model with a doping-dependent attractive interaction was considered in Ref. 47. It is interesting to note that the boson–fermion model with electrons and holes and different kinds of the fermion–boson coupling was proposed in Ref. 48 to unify the Bose–Einstein and BCS collective phenomena. It was shown that both the regime of superfluidity with local pairs and the regime of superconductivity with Cooper pairs take place in different sectors of the model parameters.

Even though the *d*-wave pairing symmetry is now considered as the typical and almost imprescriptible property of the HTSC cuprate compounds [49], there is experimental evidence that mixed $s + id$- or even pure *s*-wave pairing can exist in some of these materials at certain doping values. Indeed, it was observed that optimally doped $Pr_{1.855}Ce_{0.145}CuO_{4y}$ at low enough temperatures demonstrates a nodeless gap inconsistent with pure *d*-wave symmetry [50], and the superconducting compound $Sr_{0.9}La_{0.1}CuO_2$ reveals the *s*-wave pairing near the optimal doping [51]. The analysis shows [52] that the dominant bulk symmetry of the order parameter in some cuprates is the extended (or anisotropic) *s*-wave one. It was also observed that the crossover from the *d*-wave in the underdoped and optimally doped regime to the $s + id$-wave pairing in the overdoped regime takes place in $YBa_2Cu_3O_7$ [53]. A crossover from *d*-wave to *s*-wave pairing with doping near optimal x was found for the electron-doped $Pr_{2x}Ce_xCuO_4$ [54], $Pr_{2x}Ce_xCuO_{4y}$ and $La_{2x}Ce_xCuO_{4y}$ [55]. However, the *s*-wave pairing is not always present in underdoped and overdoped cuprates, but the *d*-wave order parameter symmetry in overdoped $TlBa_2Cu_3O_{6+\delta}$ was found in Ref. 56.

The results cited above evidently show that the investigation of the doping dependence of the physical properties of superconductors is now an important field in solid state physics. The corresponding questions have not yet been reviewed in the literature and demand some generalization. Below an attempt will be made to analyze the behavior of properties of superconducting systems with different interparticle potentials in the *s*- and *d*-wave pairing channels as functions of particle densities and interaction strength. For the sake of simplicity we restrict here our discussion to the case $T = 0$. We briefly survey the main superconducting properties of different systems. We should kindly apologize because the results of many authors could not be included in our paper, since it is very difficult and in fact impossible to cover all of them in such wide and rapidly growing field as high-temperature superconductivity, even within a separate, special and rather narrow topic.

2. The model and the main equations

The most general and, at the same time, the simplest Hamiltonian, which is usually studied in the theory of superconductivity, can be written as

$$H = -\sum_{\mathbf{n},\mathbf{m},\sigma} t_{\mathbf{nm}} c^{\dagger}_{\mathbf{n}\sigma} c_{\mathbf{m}\sigma} - \mu \sum_{\mathbf{n},\sigma} c^{\dagger}_{\mathbf{n}\sigma} c_{\mathbf{n}\sigma} -$$
$$- V_0 \sum_{\mathbf{n}} c^{\dagger}_{\mathbf{n}\uparrow} c^{\dagger}_{\mathbf{n}\downarrow} c_{\mathbf{n}\downarrow} c_{\mathbf{n}\uparrow} - \sum_{\mathbf{n},\mathbf{m}} V_{\mathbf{nm}} c^{\dagger}_{\mathbf{n}\uparrow} c^{\dagger}_{\mathbf{m}\downarrow} c_{\mathbf{m}\downarrow} c_{\mathbf{n}\uparrow} +$$
$$+ \sum_{\mathbf{n},\mathbf{q}} g_{\mathbf{n}}(\mathbf{q}) c^{\dagger}_{\mathbf{n}\sigma} c_{\mathbf{n}\sigma} X_{\mathbf{n}}(\mathbf{q}) +$$
$$+ \frac{1}{2} \sum_{\mathbf{n},\mathbf{q}} \left[\frac{\mathbf{p}_{\mathbf{n}}{}^2(\mathbf{q})}{m(\mathbf{q})} + m(\mathbf{q}) \omega^2_{\mathbf{n}}(\mathbf{q}) X^2_{\mathbf{n}}(\mathbf{q}) \right], \qquad (1)$$

where $c_{\mathbf{n}\sigma} \equiv c_{\sigma}(\mathbf{n}, \tau)$ is the fermionic field operator with spin $\sigma = \uparrow, \downarrow$ at the lattice site $\mathbf{n}$ and at the time τ, $t_{\mathbf{nm}}$ describes the n.n., n.n.n. and other hopping processes; μ is the chemical potential of the system; the nonretarded interparticle interaction is modeled by the terms proportional to V_0 (the on-site attraction if $V_0 > 0$ and repulsion otherwise) and $V_{\mathbf{nm}}$ (n.n. or n.n.n. interaction). The last two terms in (1) describe an additional retarded fermion–boson interaction and the free boson parts of the Hamiltonian, where $\mathbf{q}$ is a boson mode with the coordinate $X_{\mathbf{n}}(\mathbf{q})$, momentum $\mathbf{p}_{\mathbf{n}}(\mathbf{q})$, and frequency $\omega_{\mathbf{n}}(\mathbf{q})$ and $g_{\mathbf{n}}(\mathbf{q})$ is the fermion–boson coupling. One can easily pass to the continuum version of this Hamiltonian by replacing the n.n. hopping operator $t_{\mathbf{nm}}$ by $t_{\mathbf{nm}} \to \delta_{\mathbf{nm}} t(1 - (a^2/(2d))\nabla^2)$ (here a is the intercite distance, $d(= 2, 3)$ is the dimensionality of the system), introducing the cutoff radius in the interaction terms, etc.

In the case of the *d*-dimensional square lattice, the free fermion dispersion relation in momentum space has the following form, when the n.n. hopping takes place

$$\xi(\mathbf{k}) = -2t\sum_{j=1}^{d}\cos ak_j - \mu, \qquad (2)$$

where $\mathbf{k}$ is a d-dimensional wave vector.

As is well known, it is convenient to calculate the thermodynamic potential by using the path integral approach for studying the properties of a quantum many-particle system. This method is not necessary in the case of the mean-field solution, but it is extremely useful (see below) when the fluctuations are studied. Probably one of the first to apply the path integral methodics for the description of superconductivity was Svidzinsky [57].

The partition function of the system is

$$Z = \int D\psi^{\dagger} D\psi\ \mathrm{e}^{-S} \qquad (3)$$

with the action

$$S = \int_0^{\beta} d\tau \left[\sum_{\mathbf{n},\sigma}\psi^{\dagger}_{\mathbf{n}\sigma}(\tau)\partial_{\tau}\psi_{\mathbf{n}\sigma}(\tau) + H(\tau)\right],\ \beta = 1/T. \qquad (4)$$

To study the superconducting properties of the system, one should make the Hubbard–Stratonovich transformation with so-called bilocal fields $\Phi_{\mathbf{nm}}(\tau_1,\tau_2)$ and $\Phi^{\dagger}_{\mathbf{nm}}(\tau_1,\tau_2)$ [57] (it is necessary to pay attention that in the textbook [57] the functional integration was developed for the situation when spatially inhomogeneous order parameter of the system depends upon one space variable only, which excluded a d-wave symmetry):

$$\exp\left[\psi^{\dagger}_{\mathbf{n}\uparrow}(\tau_1)\psi^{\dagger}_{\mathbf{m}\downarrow}(\tau_2)V_{\mathbf{nm}}(\tau_1,\tau_2)\psi_{\mathbf{m}\downarrow}(\tau_2)\psi_{\mathbf{n}\uparrow}(\tau_1)\right] =$$

$$= \int D\Phi^{\dagger}D\Phi \exp\left[-\int_0^{\beta}d\tau_1\int_0^{\beta}d\tau_2\left(\frac{|\Phi_{\mathbf{nm}}(\tau_1,\tau_2)|^2}{V_{\mathbf{nm}}(\tau_1,\tau_2)} - \right.\right.$$

$$- \Phi^{\dagger}_{\mathbf{nm}}(\tau_1,\tau_2)\psi_{\mathbf{n}\downarrow}(\tau_1)\psi_{\mathbf{m}\uparrow}(\tau_2) -$$

$$\left.\left. - \psi^{\dagger}_{\mathbf{n}\uparrow}(\tau_1)\psi^{\dagger}_{\mathbf{m}\downarrow}(\tau_2)\Phi_{\mathbf{nm}}(\tau_1,\tau_2)\right)\right], \qquad (5)$$

where $V_{\mathbf{nm}}(\tau_1,\tau_2)$ includes the effective interparticle attraction due to boson coupling after integrations over the boson fields $X_{\mathbf{n}}(\mathbf{q})$.

Let us introduce the Nambu spinor

$$\Psi_{\mathbf{n}}(\tau) = \begin{pmatrix}\psi_{\mathbf{n}\uparrow}(\tau)\\ \psi^{\dagger}_{\mathbf{n}\downarrow}(\tau)\end{pmatrix},$$

$$\Psi^{\dagger}_{\mathbf{n}}(\tau) = (\psi^{\dagger}_{\mathbf{n}\uparrow}(\tau), \psi_{\mathbf{n}\downarrow}(\tau)).$$

In these terms the partition function can be formally written as

$$Z = \int D\Psi^{\dagger}D\Psi D\Phi^{\dagger}D\Phi\ \mathrm{e}^{-S(\Psi^{\dagger},\Psi,\Phi^{\dagger},\Phi)},$$

where

$$S(\Psi,\Psi^{\dagger},\Phi,\Phi^{\dagger}) =$$

$$= \int_0^{\beta}d\tau_1\int_0^{\beta}d\tau_2\sum_{\mathbf{n},\mathbf{m}}\left\{\frac{|\Phi_{\mathbf{nm}}(\tau_1,\tau_2)|^2}{V_{\mathbf{nm}}(\tau_1,\tau_2)} - \delta(\tau_1-\tau_2)\Psi^{\dagger}_{\mathbf{n}}(\tau_1)\times\right.$$

$$\times\left[-\delta_{\mathbf{nm}}\partial_{\tau_2} - \hat{\tau}_z(t_{\mathbf{nm}} - \delta_{\mathbf{nm}}\mu)\right]\Psi_{\mathbf{m}}(\tau_2) -$$

$$- \Phi^{\dagger}_{\mathbf{nm}}(\tau_1,\tau_2)\Psi^{\dagger}_{\mathbf{n}}(\tau_1)\hat{\tau}_-\Psi_{\mathbf{m}}(\tau_2) -$$

$$\left. - \Psi^{\dagger}_{\mathbf{n}}(\tau_1)\hat{\tau}_+\Psi_{\mathbf{m}}(\tau_2)\Phi_{\mathbf{nm}}(\tau_1,\tau_2)\right\}, \qquad (7)$$

where $\hat{\tau}_{\pm} = \frac{1}{2}(\hat{\tau}_x \pm \hat{\tau}_y)$ and $\hat{\tau}_z$ are the Pauli matrices.

The latter action is diagonal over the fermionic fields, and therefore the integration over $\Psi^{\dagger}$ and Ψ can be performed exactly. In this case the partition function becomes

$$Z = \int D\Phi D\Phi^{*}\exp\left(-\beta\Omega[G]\right),$$

where $\Omega[G]$ is the thermodynamic potential, which in the «leading order» is

$$\beta\Omega[G] = \int_0^{\beta}d\tau_1\int_0^{\beta}d\tau_2\sum_{\mathbf{n},\mathbf{m}}\frac{|\Phi_{\mathbf{nm}}(\tau_1,\tau_2)|^2}{V_{\mathbf{nm}}(\tau_1,\tau_2)} -$$

$$- \mathrm{Tr}\ \mathrm{Ln}\ G^{-1} + \mathrm{Tr}\ \mathrm{Ln}\ G_0^{-1}. \qquad (8)$$

The Nambu spinor Green function G satisfies the following equation:

$$\left[-\delta(\tau_1-\tau_3)\delta_{\mathbf{nl}}\partial_{\tau_3}\hat{I} + \delta(\tau_1-\tau_3)(t_{\mathbf{nl}} - \delta_{\mathbf{nl}}\mu)\hat{\tau}_z + \right.$$

$$\left. + \hat{\tau}_+\Phi_{\mathbf{nl}}(\tau_1,\tau_3) + \hat{\tau}_-\Phi^{*}_{\mathbf{nl}}(\tau_1,\tau_3)\right]G_{\mathbf{lm}}(\tau_3,\tau_2) =$$

$$= \delta(\tau_1-\tau_2)\delta_{\mathbf{nm}} \qquad (9)$$

with anti-periodic boundary conditions for fermions

$$G_{\mathbf{nm}}(\tau_1 - \tau_2 + \beta) = -G_{\mathbf{nm}}(\tau_1 - \tau_2).$$

The thermodynamic potential (8) is the most general form of the superconducting effective action, or Ginzburg–Landau potential, with a nonlocal retarded interparticle interaction. As was already mentioned, it will be used below to study the fluctuation effects.

The minimization of the thermodynamic potential with respect to the order parameter and the chemical potential leads to the following system of the coupled equations:

$$\frac{\delta\Omega}{\delta\Phi_{\mathbf{nm}}(\tau_1,\tau_2)} = 0; \qquad (10)$$

$$\frac{\partial\Omega}{\partial\mu} = -N_f, \tag{11}$$

or

$$\Phi_{\mathbf{nm}}(\tau_1,\tau_2) = V_{\mathbf{nl}}(\tau_1,\tau_3)\ \mathrm{Tr}\ \hat{\tau}_+ G_{\mathbf{lm}}(\tau_3,\tau_2); \tag{12}$$

$$n_f = -\mathrm{Tr}\ \hat{\tau}_z G_{\mathbf{nn}}(\tau,\tau), \tag{13}$$

where $n_f = N_f/v$ is the free fermion density in the system (v is the volume of the system).

In general, it is very difficult to find the Green's function $G_{\mathbf{lm}}(\tau_1,\tau_2)$, and therefore some simplifications must be applied. In particular, we shall consider the case of the space and time invariance, $V_{\mathbf{nm}}(\tau_1,\tau_2) = V_{\mathbf{n-m}}(\tau_1-\tau_2)$. In this case the Green's function has the following form in momentum space:

$$G(i\omega_n,\mathbf{k}) = \\ = -\frac{i\omega_n + \hat{\tau}_z\xi(\mathbf{k}) - \Phi(i\omega_n,\mathbf{k})\hat{\tau}_+ - \Phi^*(i\omega_n,\mathbf{k})\hat{\tau}_-}{\omega_n^2 + \xi^2(\mathbf{k}) + |\Phi(i\omega_n,\mathbf{k})|^2},$$

and the system of the equations (12), (13) acquires the following form:

$$\Phi(i\omega_n,\mathbf{k}) = \int\frac{d^dp}{(2\pi)^d}\sum_m \frac{\Phi(\omega_m,\mathbf{p})}{\omega_m^2 - \xi^2(\mathbf{p}) - |\Phi(\omega_m,\mathbf{p})|^2}\times \\ \times\left[V(\mathbf{p},\mathbf{k}) + g_{f-b}^2\int\frac{d^dk}{(2\pi)^d}\frac{\omega^2(\mathbf{p}-\mathbf{k})}{(\omega_m-\omega_n)^2 - \omega^2(\mathbf{p}-\mathbf{k})}\right]; \tag{14}$$

$$n_f = \int\frac{d^dp}{(2\pi)^d}\left[1 - \sum_m\frac{\xi(\mathbf{p})}{\omega_m^2 - \xi^2(\mathbf{p}) - |\Phi(\omega_m,\mathbf{p})|^2}\right]. \tag{15}$$

In Eq. (14) $V(\mathbf{p},\mathbf{k})$ is the Fourier transform of the nonretarded interaction, and the term proportional to g_{f-b} corresponds to the interparticle attraction through the boson field; $\omega_n = \pi T(2n+1)$ is the Matsubara frequency. The interaction term in (14) is written in general form; it describes, for example, a local nonretarded interaction, when $V(\mathbf{p}-\mathbf{k}) = \mathrm{const}$, $g_{f-b} = 0$; a nonlocal (nonretarded) interaction, when $V(\mathbf{p}-\mathbf{k}) \neq \mathrm{const}$, but $g_{f-b} = 0$; a local retarded interaction, when $V(\mathbf{p}-\mathbf{k}) = \mathrm{const}$, $g_{f-b} \neq 0$, and $\omega(\mathbf{p}-\mathbf{k}) = \mathrm{const}$, etc. The system (14), (15) will be analyzed in the next Section for various forms of the interparticle potential $V(\mathbf{p},\mathbf{k})$ and boson spectrum $\omega(\mathbf{k})$.

3. The solutions

3.1. The model with local nonretarded attraction

The problem of the crossover from small to large fermion density in the model with local attraction was considered in [1,12–19] for the 3D case and in [1,12,18,26,31] for the 2D case. The corresponding results can be reduced to the following.

For the simplest case of local nonretarded attraction, the interaction parameters in equations (14), (15) have the following form: $V(\mathbf{p},\mathbf{k}) \equiv V = \mathrm{const}$, $g_{f-b} = 0$. Therefore, the gap in this case is momentum- and frequency-independent:

$$\Phi(\omega_n,\mathbf{k}) = \Delta = \mathrm{const}.$$

The summation in (14), (15) over frequency can be easily performed, and one gets the standard system of equations:

$$1 = V\int\frac{d^dk}{(2\pi)^d}\frac{1}{2\sqrt{\xi^2(\mathbf{k})+\Delta^2}}\tanh\left(\frac{\sqrt{\xi^2(\mathbf{k})+\Delta^2}}{2T}\right); \tag{16}$$

$$n_f = \int\frac{d^dk}{(2\pi)^d}\left[1 - \frac{\xi(\mathbf{k})}{\sqrt{\xi^2(\mathbf{k})+\Delta^2}}\tanh\left(\frac{\sqrt{\xi^2(\mathbf{k})+\Delta^2}}{2T}\right)\right], \tag{17}$$

which at $T = 0$ have a simple form:

$$1 = V\int\frac{d^dk}{(2\pi)^d}\frac{1}{2\sqrt{\xi^2(\mathbf{k})+\Delta^2}}; \tag{18}$$

$$n_f = \int\frac{d^dk}{(2\pi)^d}\left[1 - \frac{\xi(\mathbf{k})}{\sqrt{\xi^2(\mathbf{k})+\Delta^2}}\right]. \tag{19}$$

Since the gap is momentum-independent, only the isotropic s-wave pairing regime is possible in the model with the on-site attraction. Equation (19) is crucial if one wants to investigate the doping dependence of superconducting properties. As a rule, it is not taken into account in the standard theory of superconductivity (or the BCS theory).

The s-wave pairing regime in the case of a quadratic dispersion law

$$\xi(\mathbf{k}) = \frac{\mathbf{k}^2}{2m} - \mu$$

and an indirect pairing with boson («Debye») energy cutoff $\theta(\omega_D - |\xi(\mathbf{k}) - \mu|)$ was considered in the mean-field approximation for the 2D and 3D cases and at low carrier densities in Ref. 1 (see also [4,5]). In fact, in such a situation the integration over mo-

mentum $\int d^d k/(2\pi)^d$ can be replaced by the integration over energy $\int \rho(\epsilon)\, d\epsilon$, where $\rho(\epsilon)$ is the density of states (DOS); it is constant in the case $d = 2$ and $\sim \sqrt{\epsilon}$ in the case $d = 3$.

It is very easy to solve the system (18) and (19) in the 2D case. When $\varepsilon_F << W$ (W is the free fermion bandwidth), the solution has a simple form:

$$\Delta \simeq \sqrt{2W\epsilon_F}\, e^{-2\pi/mV} = \sqrt{2|\varepsilon_b|\epsilon_F};$$

$$\mu \simeq \epsilon_F - |\varepsilon_b|/2,$$

where $\varepsilon_b = -2W\, e^{-4\pi/mV}$ is the two-fermion bound state energy. Obviously, the crossover from superfluidity to superconductivity with doping occurs in the 2D case at any coupling constant; then there exists the value of ϵ_F when $\mu = 0$ for any V.

This is not true in the 3D case, when the BEC–BCS crossover takes place only when the coupling constant is larger than some critical value V_{cr}. This distinction follows from the above-mentioned difference in the DOS in the gap equation. In the 2D case, when $\rho(\epsilon) = \text{const}$, the gap equation has the solution $\Delta = \sqrt{2W|\varepsilon_b|}$ at $\mu = 0$ and any coupling constant.

For the 3D system the gap equation has the following form:

$$1 = V \int \frac{k^2 dk}{2\pi^2} \frac{1}{2\sqrt{k^4 + \Delta^2}} \qquad (20)$$

at $\mu = 0$. The integral over k on the right-hand side has the maximal value $\sqrt{2mW}/(4\pi^2)$ at $\Delta = 0$. Therefore, a simple estimation for V_{cr} is given by the relation $1 \simeq V_{cr}\sqrt{2mW}/(4\pi^2)$, or $V_{cr} \simeq 4\pi^2/\sqrt{2mW}$.

In the case of the momentum cutoff

$$\int \frac{d^d k}{(2\pi)^d}\, \theta(\omega_D - |\xi(\mathbf{k}) - \mu|)\,,$$

the approximate solution can be also easily obtained in the 2D case:

$$\Delta \simeq \sqrt{2|\varepsilon_b|\Delta_{BCS}}\; \theta(\omega_D - \epsilon_F) + \Delta_{BCS}\theta(\epsilon_F - \omega_D);$$

$$\mu = \epsilon_F - \frac{|\varepsilon_b|}{2},$$

where $\Delta_{BCS} = 2\omega_D\, e^{-2\pi/(mV_{\text{eff}})}$ is the BCS expression for the gap, and the bound state energy in this case also exists: $\varepsilon_b = -2\omega_D\, e^{-4\pi/(mV_{\text{eff}})}$, where V_{eff} is proportional to the fermion–boson coupling constant g_{f-b}. The gap is a growing function with doping in this case, and asymptotically approaches its maximal value $\Delta = \Delta_{BCS}$ when $\epsilon_F >> \omega_D$. In other words, the dependence of the gap value on the carrier density has no maximum, which evidently means that such a dependence with saturation is not identical to the «increasing–decreasing» dependence of the gap upon the effective coupling constant V_{eff}.

In the 3D case, the DOS in equations (18) and (19) can be substituted by the DOS at the Fermi level, and the solution can be obtained from the 2D result with substitution $mV/(2\pi) \to k_F mV/(4\pi^2)$. It is possible to estimate the critical value of the coupling constant when the crossover takes place: $V_{cr} = 2\pi^2/\sqrt{m\omega_D}$. In both cases, with and without momentum cutoff, the BEC–BCS transition from superfluidity to superconductivity is smooth and is not a phase transition; the gap value grows continuously with doping.

3.2. The models with the nonlocal nonretarded interaction

It is important to study a more realistic case of the nonlocal attraction in presence of a short-range Coulomb repulsion. In order to study the superconducting properties of such a model in the channels with different pair angular momentum l, it is convenient (see, for example, Refs. 32, 33, 46), to approximate the interaction potential by a separable function:

$$V_l(\mathbf{k}_1, \mathbf{k}_2) = -\lambda_l w_l(\mathbf{k}_1) w_l(\mathbf{k}_2), \qquad (21)$$

where λ_l is an effective coupling constant, and

$$w_l(\mathbf{k}) = h_l(\mathbf{k}) \cos l\varphi_{\mathbf{k}}; \qquad (22)$$

$$h_l(\mathbf{k}) = \frac{(k/k_1)^l}{(1 + k/k_0)^{l+1/2}}; \qquad (23)$$

$k = |\mathbf{k}|$ is the momentum modulus and $\varphi_{\mathbf{k}}$ is the momentum angle in polar coordinates $\mathbf{k} = k(\cos\varphi_{\mathbf{k}}, \sin\varphi_{\mathbf{k}})$. Parameters k_0 and k_1 put the momentum range in the proper region — the potential is attractive at $r_0 < r < r_1$ and repulsive at $r < r_0$, where $k_0 \sim 1/r_0$ and $k_1 \sim 1/r_1$. The separable form of the interparticle potential (21) is based on the group theory decomposition of the arbitrary potential by, for example, spherical harmonics, and the restriction to the relevant terms only, which in our case correspond to the order-parameter symmetry under consideration.

It is easy to see that the interaction potential (21) has the correct asymptotic behavior at small and large momenta: $V_l(\mathbf{k}_1, \mathbf{k}_2) \sim k_1^l k_2^l$ and $V_l(\mathbf{k}_1, \mathbf{k}_2) \sim 1/\sqrt{k_1 k_2}$, respectively. Since the region of low carrier concentrations, where the crossover can take place, is the most interesting, the correct behavior of the interaction potential at small momenta should be most important. These momenta give main contribution to the integrals in the case of low carrier concentrations (see equations below). We shall study the s- and d-wave channels with $l = 0$ and 2 separately, so we assume that the parameters λ_l for both channels are independent.

In this case the equations for the gap and for the chemical potential have the following form:

$$\Delta_l(\mathbf{k}) = -\lambda_l \int \frac{d\mathbf{p}}{(2\pi)^2} \frac{\Delta_l(\mathbf{p})}{2\sqrt{\varepsilon^2(\mathbf{p}) + \Delta_l^2(\mathbf{p})}} V_l(\mathbf{p},\mathbf{k}); \quad (24)$$

$$n_f = \int \frac{d\mathbf{k}}{(2\pi)^2}\left[1 - \frac{\varepsilon(\mathbf{k})}{\sqrt{\varepsilon^2(\mathbf{k}) + \Delta_l^2(\mathbf{k})}}\right]. \quad (25)$$

The solution of equation (24) has the following form:

$$\Delta_l(\mathbf{k}) = \Delta_l^{(0)} w_l(\mathbf{k}), \quad (26)$$

where $\Delta_l^{(0)}$ does not depend on the momentum $\mathbf{k}$.

As shown in Refs. 32, 33, the BEC–BCS crossover from superfluidity to superconductivity is smooth with doping increasing (see Figs. 1 and 2 below). However, in the d-wave paring channel there exists a critical value of the interaction potential, below which this crossover is impossible. Such a circumstance makes the cases of isotropic s- and anisotropic d-wave pairing essentially different.

The more realistic case in connection to HTSCs was considered in Ref. 47, where the correlation length r_0 was studied at small carrier densities as $r_0 \sim a/\sqrt{n_f}$. This dependence for the length of spin-spin correlations at small carrier densities was found, for example, in $La_{2-x}Sr_xCuO_4$. The magnetic correlation length decreases with carrier density per cell in this material as $3.8\ \text{Å}/\sqrt{n_f}$ [59]. The value of a was considered to be $a = \sqrt{2/\pi}a_0$, where a_0 is the simple square lattice constant. The next equality can be used to estimate the value for a: $(\pi/2)r_0^2 N_f = a_0^2 N_{\text{cell}}$, where in the left-hand side the volume of the 2D system is expressed as a volume (circle of the radius $\sim r_0$) occupied by one particle, multiplied by the total number of particles N_f, N_{cell} is an elementary cell number in the layered system. The free fermion bandwidth W is related to a_0 as $W = \pi^2/(ma_0^2)$. It should be noted, that the relation $r_0 \sim a/\sqrt{n_f}$ at $a = \sqrt{2/\pi}a_0$ is in a good agreement with the experimental data for $La_{2-x}Sr_xCuO_4$ [59], where the magnetically ordered (orthorhombic) plane lattice parameters are equal to 5.354 Å and 5.401 Å, and the corresponding parameter a is $\simeq 3.8$ Å.

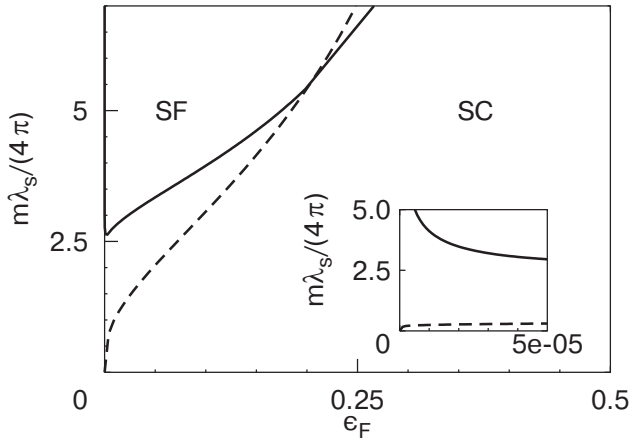

Fig. 1. Crossover line coupling-carrier density for the s-pairing channel (solid line). The dotted line represents the corresponding curve for the case $r_0(n_f) = \text{const}$ at $r_0 = a_0$. The inset shows the doping dependence of the crossover value for coupling at very low charge carrier densities [47]. Here and below all parameters are expressed in units of the bandwidth W.

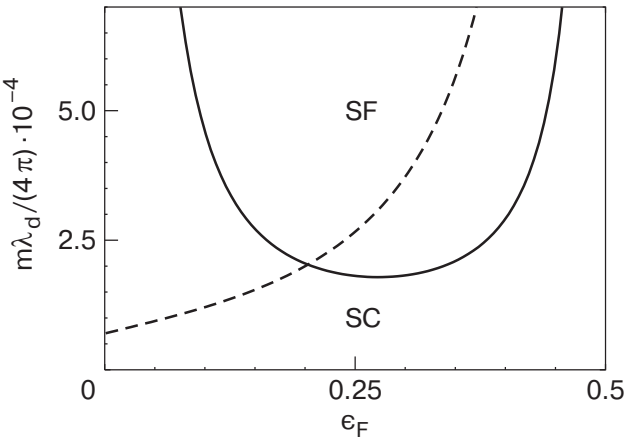

Fig. 2. The crossover line coupling-carrier density for the d-wave case (solid line). The dotted line is the crossover curve for the case $r_0(n_f) = \text{const}$ at $r_0 = a_0$ [47].

It was shown that the critical value of the coupling constant exists even in the s-wave pairing channel for this case (see Figs. 1 and 2).

Let us consider different versions of the previous model. Namely, the correlation radius $r_0(n_f) = a_0\sqrt{2/(\pi n_f)}$ can be introduced in a model with an exponential decay of the attraction:

$$V_b(\mathbf{r},t) = g_{f-b}\frac{e^{-r/r_0}}{r}. \quad (27)$$

Its Fourier transform has the following form:

$$D_b(\mathbf{q}) = \frac{g_{f-b}r_0}{(2\pi)^2\sqrt{1 + r_0^2\mathbf{q}^2}}. \quad (28)$$

In addition, the similar kind of the short-range fermion–fermion repulsion should be taken into account. It is easy to see, that in the 2D case this potential has the following form (see, for example, Ref. 60):

$$D_f(\mathbf{q}) = \frac{g_{f-f}}{q + q_{TF}g(q/2k_F)}, \quad (29)$$

where $g_{f-f} \equiv 2\pi e^2$, $q_{TF} = 4e^2m/\pi = 4/\pi a_B$ is the Thomas–Fermi momentum, and $a_B = 1/(e^2m) \simeq$

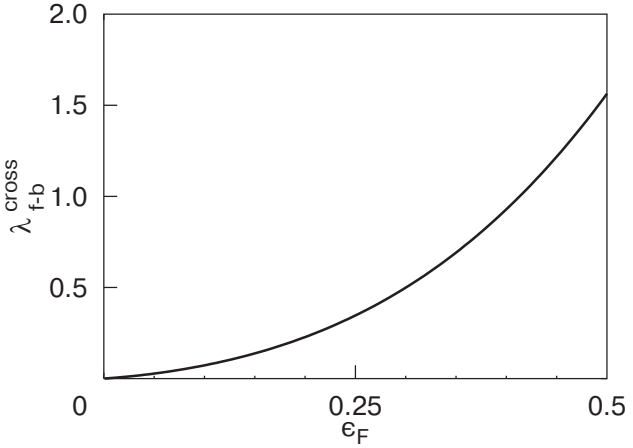

Fig. 3. The crossover interaction value $\lambda_{f-b}^{\text{cross}}$ (defined by the condition $\mu = 0$) as the function of ϵ_F at $\lambda_{f-f} = 0$.

$\simeq 0.529$ Å is the Bohr radius. The function $g(x)$ is defined as

$$g(x) = 1 - \theta(x-1)\sqrt{1 - 1/x^2}. \qquad (30)$$

This model of the *s*-wave pairing channel demonstrates the crossover from superfluidity to superconductivity at any value of the coupling constant, contrary to the previous case (see Fig. 3). The doping dependence of the gap and of the chemical potential at different values of the dimensionless interaction parameters $\lambda_{f-b} = g_{f-b}^2 mr_0/(8\pi^2)$, $\lambda_{f-f} = g_{f-f}^2 mr_0/(4\pi)$ is presented in Figs. 4 and 5. As it follows from these figures, the gap decreases with doping growing at large ϵ_F. This situation is in a qualitative agreement with the experiments for the cuprates. One can consider also an interesting situation of the combined local + nonlocal attraction, when the local (on-site) attraction will tend the Cooper pairs to transform into the local pairs.

The «mixed» case with $h_l(\mathbf{k}) = 1$ and $\lambda = \lambda_s + \lambda_d \cos 2\varphi_{\mathbf{k}}$ was considered in Refs. 41, 42. In particular, it was shown in the paper [41] that the crossover from *d*-wave to *s*-wave superconductivity with the intermediate $s + d$-symmetry takes place with doping in full agreement with the experiments on some cuprate materials [53–55].

The *d*-wave case, when $w_d(\mathbf{k}) = \cos k_x - \cos k_y$ was considered in Refs. 37, 44, 45. For example, the effect of the n.n.n. hopping $t_{n.n.n.}$ on the pairing was

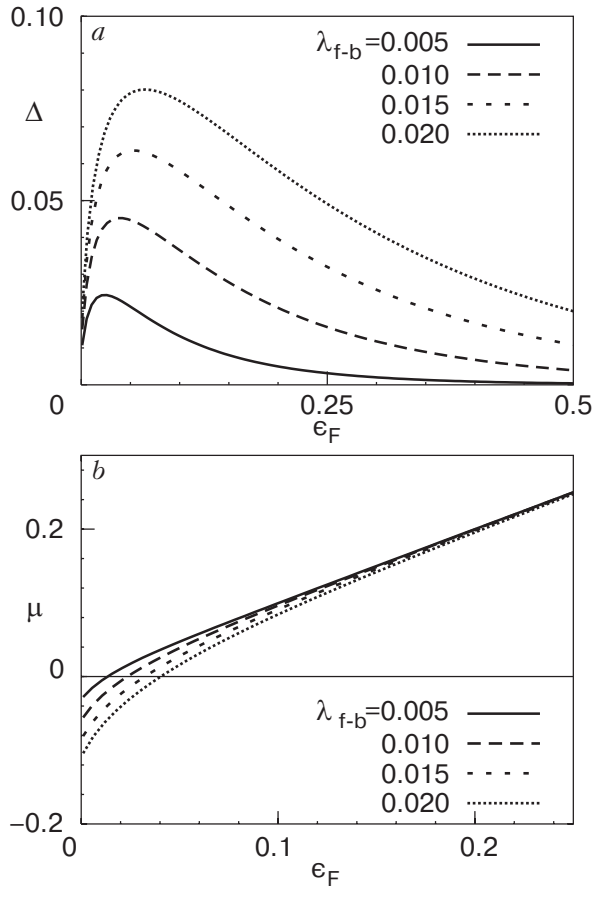

Fig. 4. The gap (*a*) and the chemical potential (*b*) as the functions of ϵ_F at different λ_{f-b} and $\lambda_{f-f} = 0$.

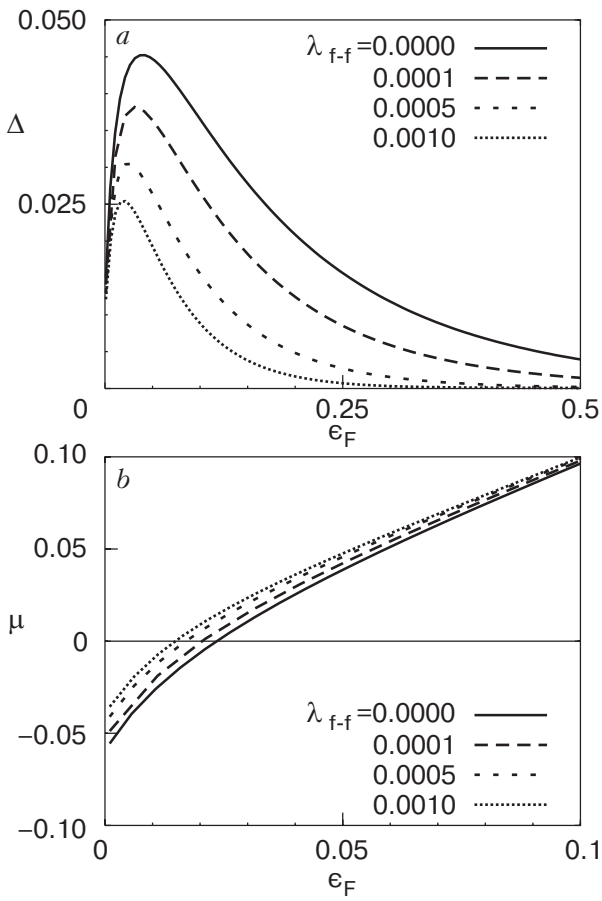

Fig. 5. The gap (*a*) and the chemical potential (*b*) as functions of ϵ_F at different λ_{f-f} and $\lambda_{f-b} = 0.01$.

studied in Refs. 44, 45. As it was shown in latter paper, the crossover with doping takes place at any value of $t_{n.n.n.}$, when the coupling constant is larger than $V_{\rm cr}$.

At last, it was stated in Ref. 29, that the model with small on-site repulsion and n.n. $V_{n.n.}$ and n.n.n. $V_{n.n.n.}$ attraction with $V_{n.n.n.} \sim$ 60–80 meV and $V_{n.n.n.}/V_{n.n.} \simeq$ 1.3–1.5 can describe well the experimental data for the hole-doped oxides.

3.3. The models with retarded interaction

As it is well-known, the retardation effects in the interaction can play very important role in the superconducting properties of the system. For instance, let us consider a boson propagator with the dispersion $\omega = \omega(\mathbf{k})$ (cp. (14)):

$$D(\omega, \mathbf{k}) = \frac{\omega^2(\mathbf{k})}{\omega^2 - \omega^2(\mathbf{k}) + i\delta}\,. \tag{31}$$

It can be noted that in the general case of phonon dispersion one has $\omega(\mathbf{k}) = \sqrt{\omega_0^2 + c_{\rm ph}^2 \mathbf{k}^2}$. We shall describe this rather general case below with the following approximation: $D(\omega, \mathbf{k}) \simeq D(\omega, \mathbf{k}_F)$, so the effective propagator can be written as

$$D(\omega, \mathbf{k}) = \frac{\omega_0^2}{\omega^2 - \omega_0^2 + i\delta}, \tag{32}$$

where $\sqrt{\omega_0^2 + c_{\rm ph}^2 \mathbf{k}_F^2}$ is substituted by new effective frequency ω_0. This approximation corresponds to the case of the optical phonon attraction (in the case when $\omega_0 \neq 0$). Generally speaking, the gap is frequency-dependent in this model. The problem of the crossover in the model with frequency dependent gap was investigated in the Ref. 36.

The set of the equations for the gap and for the chemical potential in this case is:

$$\Phi(\omega) = -ig_{f-b}^2 \times$$
$$\times \int \frac{d^2k d\nu}{(2\pi)^2} \frac{\Phi(\nu)}{\nu^2 - \xi(\mathbf{k})^2 - |\Phi(\nu)|^2 + i\delta} \frac{\omega_0^2}{(\omega - \nu)^2 - \omega_0^2 + i\delta}\,; \tag{33}$$

$$2\epsilon_F = $$
$$= \mathrm{Re} \int_0^\infty d\omega \left[\left(\frac{\omega}{\sqrt{\omega^2 - |\Phi(\omega)|^2}} - 1 \right) \theta(\mu + \sqrt{\omega^2 - |\Phi(\omega)|^2}) + \right.$$
$$\left. + \left(\frac{\omega}{\sqrt{\omega^2 - |\Phi(\omega)|^2}} + 1 \right) \theta(\mu - \sqrt{\omega^2 - |\Phi(\omega)|^2}) \right]. \tag{34}$$

Let us consider briefly how this system of the equations can be analyzed analytically. First of all, it is possible to show that the approximation $\Phi(\omega) = \Delta =$ = const in (34) is rather good. Then, this equation results in

$$\mu = \epsilon_F - \frac{\Delta^2}{2\epsilon_F} \simeq \epsilon_F - \frac{|\varepsilon_b|}{2}\,,$$

where the two-particle bound state energy ε_b depends on the coupling parameter in this case (see below).

After the Wick rotation $\omega \to i\omega$ the gap equation reads:

$$\Phi(\omega) = g_{f-b}^2 \omega_0^2 \times$$
$$\times \int \frac{d^2k d\nu}{(2\pi)^3} \frac{\Phi(\nu)}{\nu^2 + (\mathbf{k}^2/2m - \mu)^2 + \Phi^2(\nu)} \frac{1}{(\omega - \nu)^2 + \omega_0^2}\,. \tag{35}$$

In polar coordinates, after integration over the angle we come to:

$$\Phi(\omega) = \frac{g_{f-b}^2 \omega_0^2}{(2\pi)^2} \int_{-\infty}^{+\infty} d\nu \frac{\Phi(\nu)}{(\omega - \nu)^2 + \omega_0^2} \times$$
$$\times \int_0^\infty \frac{k\, dk}{(\mathbf{k}^2/2m - \mu)^2 + \nu^2 + \Phi^2(\nu)}\,.$$

Since $\Phi(\omega)$ is the even function of ω, it can depend only on ω^2, and we restrict the integration over ν to the positive values:

$$\Phi(\omega) = \frac{\lambda\omega_0^2}{2} \int_0^\infty \frac{d\nu\, \Phi(\nu)}{\sqrt{\nu^2 + \Phi^2(\nu)}} \frac{1}{(\omega - \nu)^2 + \omega_0^2} \times$$
$$\times \left[\frac{\pi}{2} + \arctan \frac{\mu}{\sqrt{\nu^2 + \Phi^2(\nu)}} \right], \tag{36}$$

where the dimensionless coupling constant $\lambda = g_{f-b}^2 m/(2\pi)$ is introduced. The asymptotic behavior for $\Phi(\omega)$ is

$$\Phi(\omega)|_{\omega \to 0} \to \text{const}, \quad \Phi(\omega)|_{\omega \to \infty} \sim \frac{1}{\omega^2}. \tag{37}$$

As next step we use the following approximation in the interaction potential [61,62]:

$$\frac{1}{(\omega - \nu)^2 + \omega_0^2} = \frac{1}{\omega^2 + \omega_0^2} \theta(\omega - \nu) + \frac{1}{\nu^2 + \omega_0^2} \theta(\nu - \omega). \tag{38}$$

Then, the differentiation with respect to ω gives:

$$\Phi'(\omega) = -\frac{\lambda\omega_0^2\nu}{(\nu^2+\omega_0^2)^2} \times$$

$$\times \int_0^{\omega} \frac{d\nu\, \Phi(\nu)}{\sqrt{\nu^2+\Phi^2(\nu)}} \left[1 + \frac{2}{\pi}\arctan\frac{\mu}{\sqrt{\nu^2+\Phi^2(\nu)}}\right]. \quad (39)$$

It is evident that $\Phi' < 0$, i.e. $\Phi_{\max} = \Phi(0) \equiv \Delta$.

After one more differentiation and introduction of a new variable $x = \nu^2/\omega_0^2$, one gets the following differential equation

$$\Phi''(x) + \frac{2}{x+1}\Phi'(x) + \frac{\lambda}{4\sqrt{x}(x+1)^2\sqrt{x+[\Phi(x)/\omega_0]^2}} \times$$

$$\times \left[1 + \frac{2}{\pi}\arctan\frac{\mu/\omega_0}{\sqrt{x+[\Phi(x)/\omega_0]^2}}\right]\Phi(x) = 0 \quad (40)$$

with the boundary conditions

$$\Phi'(x)|_{x=0} = 0; \quad [\Phi(x) + (x+1)\Phi'(x)]|_{x=\infty} = 0\,, \quad (41)$$

which follow directly from expressions (36), (38) and (39).

The analysis of Eq. (40) shows that the approximate solution for the gap in both weak and strong coupling regimes is (see the paper [36] for details): $\Delta(\omega) \simeq \Delta\theta(\omega_0^2 - \omega^2)$, where Δ is the parameter, which depends on the coupling constant and, what is very important, on the carrier density. In the weak coupling regime the parameter Δ has the following coupling and carrier density dependence:

$$\Delta = \sqrt{2|\varepsilon_b|\omega_0}\,\theta(\omega_0 - \epsilon_F) + \Delta_{BCS}\theta(\epsilon_F - \omega_0),$$

where $\Delta_{BCS} = 2\omega_0\, e^{-1/\lambda}$ is in fact the BCS expression for the gap, and the bound (Cooper) state energy in this case is $\varepsilon_b = -2\omega_0\, e^{-2/\lambda}$. In the strong coupling regime $\varepsilon_b \simeq \lambda$ and $\Delta_{BCS} \to (4/3)\lambda\omega_0$ at large carrier densities [35,36].

To summarize, the BEC–BCS crossover with carrier density and coupling constant changing in this effective model with retarded interaction is also smooth, and the gap is not small when the pair frequency is smaller than the boson frequency. Such an approach can be used for studying the crossover in the cases of different symmetries of the order parameter.

It is important to mention that the case considered here can be easily generalized on the case when one takes into account the vertex correction to the electron–phonon interaction. This correction is usually small when $\epsilon_F >> \omega_0$ (the Migdal theorem). As it was shown in Ref. 63, this correction is rather small even when $\epsilon_F << \omega_0$. However, in some cases the vertex correction can lead to strong enough enhancement of the superconductivity [40,64,65]. It is also necessary to note, that this correction leads even to the enhancement of the *d*-wave superconductivity in strongly correlated electron system [39,40], despite the phonon interaction due to the symmetry does not allow the *d*-wave pairing in the case when the short range electron repulsion is not considered.

4. The role of the order parameter fluctuations

The fluctuations of the order parameter in the 2D and even in the 3D case at $T = 0$ should be essential. As it is shown in Ref. 22, the Gaussian fluctuation corrections to the *s*-wave order parameter is non-negligible even in the weak-coupling case. On the other hand, the fluctuations of the order parameter phase can lead to increasing of the gap.

In this Section we shall consider how simultaneous order parameter modulus and phase fluctuations in the model with 4F-attraction result in the strong increasing of the order parameter at small carrier densities and to small decreasing of the order parameter when the carrier densities are large.

At zero temperature, as follows from (8), the thermodynamic potential of the system with local attraction has the following form:

$$\Omega = v\left[\frac{|\Phi|^2}{V} - \int \frac{d^2k}{(2\pi)^2}\left[\sqrt{\xi^2(\mathbf{k})+|\Phi|^2} - \xi(\mathbf{k})\right]\right].$$

In other words, it depends on the sum of its real and imaginary parts: $|\Phi|^2 = (\mathrm{Re}\,\Phi)^2 + (\mathrm{Im}\,\Phi)^2$. For studying fluctuations of the order parameter, it is convenient to use new real variables:

$$\overline{\Phi}(x) = \begin{pmatrix}\phi_1(x)\\ \phi_2(x)\end{pmatrix},$$

such that $\overline{\Phi}^2(x) = \phi_1^2(x) + \phi_2^2(x) = |\Phi(x)|^2$. Another possibility is to use the decomposition of the order parameter to its phase and modulus $\Phi(x) = \Delta(x) \times$ $\times \exp[i\theta(x)]$. However, it leads to some difficulties, since one needs to keep the order parameter modulus positive in the functional integration over the fluctuations, therefore we follow another way. The «old» order parameter variables are connected with new ones as

$$\Phi(x) = \phi_1(x) + i\phi_2(x), \quad \Phi^*(x) = \phi_1(x) - i\phi_2(x).$$

We assume that the mean-field value of the field $\overline{\Phi}$ is chosen as

$$\overline{\Phi}_0(x) = \begin{pmatrix}\Delta\\ 0\end{pmatrix}.$$

The order parameter $\overline{\Phi}$ can be written as

$$\overline{\Phi}(x) = \overline{\Phi}_0 + \delta\overline{\Phi}(x) \equiv \begin{pmatrix} \Delta + \delta\phi_1(x) \\ \delta\phi_2(x) \end{pmatrix} \qquad (42)$$

in the case when its fluctuations are considered. We neglect the fluctuations of the carrier density $n_f(\mathbf{r})$, and consider the homogeneous constant value of n_f over the lattice $n_f(\mathbf{r}) = n_f = \text{const}$.

Substitution of (42) into the expression for the thermodynamic potential gives the following correction to the thermodynamic potential to the second order in fluctuations [66]:

$$\delta\Omega = -\int_{-\infty}^{\infty}\frac{d\nu}{2\pi}\int\frac{d^2k}{(2\pi)^2}\,\delta\overline{\Phi}(i\nu,\mathbf{k})\hat{A}(i\nu,\mathbf{k})\delta\overline{\Phi}(-i\nu,-\mathbf{k}), \qquad (43)$$

where $\hat{A}(i\nu,\mathbf{k}) = 1/V + \hat{\chi}(i\nu,\mathbf{k})$ is the 2×2 matrix with the susceptibility components

$$\chi_{jk}(i\nu,\mathbf{k}) = \frac{1}{2}\,\mathrm{Tr}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\times$$

$$\times\int\frac{d^2q}{(2\pi)^2}(-1)^{j+k}G(i\omega_+,\mathbf{q}_+)\hat{\tau}_j G(i\omega_-,\mathbf{q}_+)\hat{\tau}_k,$$

where $\omega_\pm = \omega \pm \nu/2$, $\mathbf{q}_\pm = \mathbf{k} \pm \mathbf{q}/2$, and $j, k \equiv x, y$. The integration over ω can be easily performed:

$$\chi_{11}(i\nu,\mathbf{k}) = -\int\frac{d^2q}{(2\pi)^2}\frac{1}{2}\frac{E_+ + E_-}{\nu^2 + (E_+ + E_-)^2}\times$$

$$\times\left[1 + \frac{\xi_+\xi_- - \Delta^2}{E_+E_-}\right];$$

$$\chi_{12}(i\nu,\mathbf{k}) = -\chi_{21}(i\nu,\mathbf{k}) =$$

$$= -\int\frac{d^2q}{(2\pi)^2}\frac{1}{2}\frac{E_+\xi_- + E_-\xi_+}{[\nu^2 + (E_+ + E_-)^2]E_+E_-};$$

$$\chi_{22}(i\nu,\mathbf{k}) = -\int\frac{d^2q}{(2\pi)^2}\frac{1}{2}\frac{E_+ + E_-}{\nu^2 + (E_+ + E_-)^2}\times$$

$$\times\left[1 + \frac{\xi_+\xi_- + \Delta^2}{E_+E_-}\right],$$

where $E_\pm = \sqrt{\xi_\pm^2 + \Delta^2}$ and $\xi_\pm = (\mathbf{k} \pm \mathbf{q}/2)^2/(2m) - \mu$ is the free fermion dispersion relation. After the integrating out of the fluctuation field $\overline{\Phi}$, the correction to the thermodynamic potential has the following form:

$$\delta\Omega = \frac{1}{2}\int_{-\infty}^{\infty}\frac{d\nu}{2\pi}\int\frac{d^2k}{(2\pi)^2}\ln\left[\{\frac{1}{V} + \chi_{11}(i\nu,\mathbf{k})\}\times\right.$$

$$\left.\times\,\{\frac{1}{V} + \chi_{22}(i\nu,\mathbf{k})\} - \chi_{12}(i\nu,\mathbf{k})\chi_{21}(i\nu,\mathbf{k})\right].$$

The factor $\sim\Delta$ which appears in the measure of the functional integration over Δ in Z due to taking into account the symmetry of the thermodynamic potential with respect to the transformation $\overline{\Phi} \to e^{i\alpha}\overline{\Phi}$ (see, for example, the paper [66]) is omitted in the last expression. This factor can be absorbed in the measure of the functional integral over Δ in the partition function, where the functional integration can be actually performed over the variable Δ^2. Let us note that only the first component $(1/V + \chi_{11}(i\nu,\mathbf{k}))$ under the logarithm in $\delta\Omega$ (see below) will be present if one considers the particular case of the order parameter phase fluctuations.

It is useful to diagonalize the matrix $\hat{A}(i\nu,\mathbf{k})$, in order to find the contributions which come from both the phase and the modulus fluctuations of the order parameter. Obviously, the first component will correspond to the phase fluctuations and the second — to the modulus fluctuations, as follows from the definitions of the field $\overline{\Phi}$ (see (42)). So, one can easy arrive at the following representation:

$$\delta\Omega = \frac{1}{2}\int_{-\infty}^{\infty}\frac{d\nu}{2\pi}\int\frac{d^2k}{(2\pi)^2}\ln\left[\{\frac{1}{V} + \chi_\theta(i\nu,\mathbf{k})\}\{\frac{1}{V} + \chi_\Delta(i\nu,\mathbf{k})\}\right],$$

where

$$\chi_\theta(i\nu,\mathbf{k}) = \frac{1}{2}[\chi_{11}(i\nu,\mathbf{k}) + \chi_{22}(i\nu,\mathbf{k})] -$$

$$-\sqrt{\frac{1}{4}[\chi_{11}(i\nu,\mathbf{k}) - \chi_{22}(i\nu,\mathbf{k})]^2 - \chi_{12}(i\nu,\mathbf{k})\chi_{21}(i\nu,\mathbf{k})}$$

and

$$\chi_\Delta(i\nu,\mathbf{k}) = \frac{1}{2}[\chi_{11}(i\nu,\mathbf{k}) + \chi_{22}(i\nu,\mathbf{k})] +$$

$$+\sqrt{\frac{1}{4}[\chi_{11}(i\nu,\mathbf{k}) - \chi_{22}(i\nu,\mathbf{k})]^2 - \chi_{12}(i\nu,\mathbf{k})\chi_{21}(i\nu,\mathbf{k})}$$

are the effective contributions to the thermodynamic potential from the fluctuations of the order parameter phase and modulus, respectively.

The equations for the gap and for the chemical potential (10) and (11) have the following form in the case of Gaussian correction to the thermodynamic potential due to the order parameter fluctuations:

$$\frac{\Delta}{V} = \int\frac{d^2k}{(2\pi)^2}\frac{\Delta}{2\sqrt{\xi^2(\mathbf{k}) + \Delta^2}} +$$

$$+\frac{1}{2}\int_{-\infty}^{\infty}\frac{d\nu}{2\pi}\int\frac{d^2k}{(2\pi)^2}\left[\frac{\partial\chi_\theta(i\nu,\mathbf{k})/\partial\Delta}{1/V + \chi_\theta(i\nu,\mathbf{k})} + \frac{\partial\chi_\Delta(i\nu,\mathbf{k})/\partial\Delta}{1/V + \chi_\Delta(i\nu,\mathbf{k})}\right];$$

$$n_f = \int \frac{d^2k}{(2\pi)^2}\left[1 - \frac{\xi(\mathbf{k})}{\sqrt{\xi^2(\mathbf{k}) + \Delta^2}}\right] +$$

$$+ \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \int \frac{d^2k}{(2\pi)^2}\left[\frac{\partial\chi_\theta(i\nu,\mathbf{k})/\partial\mu}{1/V + \chi_\theta(i\nu,\mathbf{k})} + \frac{\partial\chi_\Delta(i\nu,\mathbf{k})/\partial\mu}{1/V + \chi_\Delta(i\nu,\mathbf{k})}\right].$$

Considering the limit of a weak coupling, the functions χ_θ and χ_Δ in the denominators can be neglected, then the integration over ν can be performed and after some trivial manipulations the following simple enough equation can be obtained (compare with (18) and (19)):

$$\frac{1}{V} = \int \frac{d^2k}{(2\pi)^2} \frac{1}{2\sqrt{\xi^2(\mathbf{k}) + \Delta^2}} + \frac{V}{2} \int \frac{d^2q}{(2\pi)^2} \int \frac{d^2k}{(2\pi)^2} \frac{\xi_+\xi_-}{E_+^3 E_-}; \tag{44}$$

$$n_f = \int \frac{d^2k}{(2\pi)^2}\left[1 - \frac{\xi(\mathbf{k})}{\sqrt{\xi^2(\mathbf{k}) + \Delta^2}}\right] +$$

$$+ V \int \frac{d^2q}{(2\pi)^2} \int \frac{d^2k}{(2\pi)^2} \frac{\Delta^2(\xi_+ - \xi_-)}{E_+^3 E_-}. \tag{45}$$

The substitution $\mathbf{k} \to -\mathbf{k}$ in a part of the terms was made during obtaining equations (44) and (45). It is interesting to note that in the case of the phase fluctuations, the numerator under the integral in the last term of the gap equation will be $\frac{1}{2}\xi_+(\xi_- - \xi_+)$ and the last term in the number equation will be multiplied by $1/2$. It is also a good approximation to put $\mu = \epsilon_F$, since in the weak coupling regime, μ is different from ϵ_F only at extremely low carrier densities. The solution of the equation (44) for the gap parameter as a function of ϵ_F at $\mu = \epsilon_F$ and different values of the coupling constant is presented in Fig. 6. The estimation of the order parameter in the case of phase fluctuation is also presented. As it was shown in Ref. 22, in the 2D case the phase fluctuations lead to an effective increasing of the coupling constant $V \to V(1 + 2/\pi^2)$. The gap can be calculated from the standard mean-field BCS equation. The comparison of different cases shows that the phase fluctuations lead to the gap increasing, while the total fluctuations lead to much stronger increasing of the gap at small carrier densities and to decreasing of the gap when the carrier density is large. The last result is familiar, but the first one is very surprising. The dependence of the gap on coupling at small value of the carrier density is presented in Fig. 7. These results suggest that higher order corrections in fluctuations should be studied for better understanding of the behavior of the system.

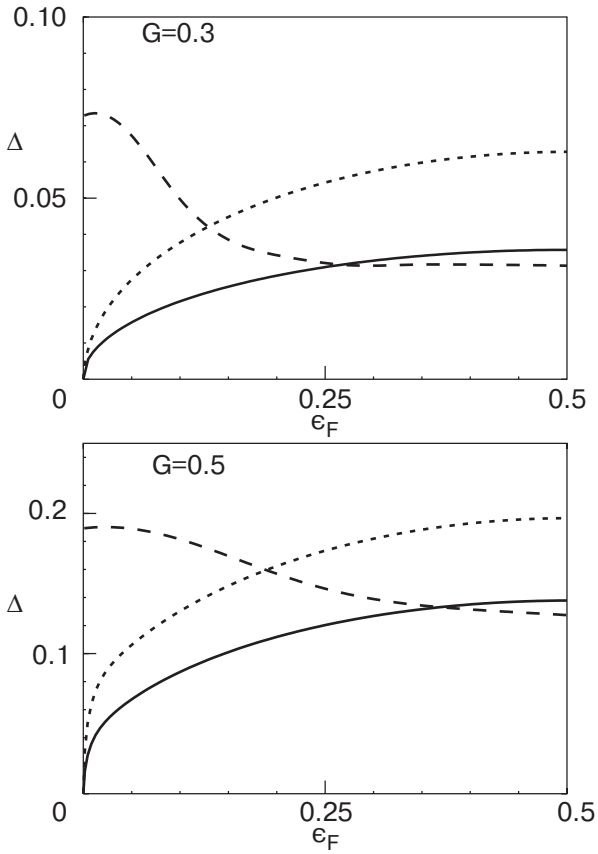

Fig. 6. The dependence of Δ on ϵ_F for the mean-field solution case (solid line) and for the order parameter fluctuations case (dashed line) at different values of the dimensionless coupling parameter $G = mV/(2\pi)$. The dotted line is the estimation from [22] for the case of the order parameter phase fluctuations.

It should be mentioned that the role of the disorder due to the dopants in the fluctuations of the inhomogeneous order parameter was recently studied

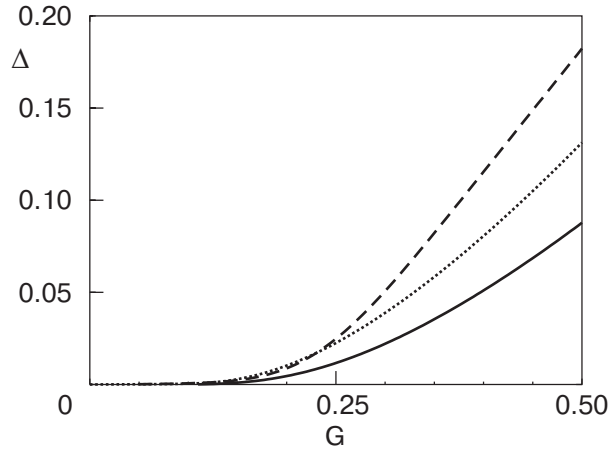

Fig. 7. The dependence of Δ on G for mean field solution case (solid line) and for order parameter fluctuations case (dashed line) at $\epsilon_F = 0.1$. The dotted line is the estimation from [22] for the order parameter phase fluctuations.

by Yu.G. Pogorelov and the present authors in Refs. 67–69. We, however, did not describe here this important issue, since it deserves a special detailed review.

5. Conclusions

In this paper, the BEC – BCS crossover from superfluidity to superconductivity with doping increasing at $T = 0$ in the cases of the *s*-wave and *d*-wave pairing was briefly reviewed. In the 3D case this crossover does not take place at weak coupling constants, and the same situation takes place in the *d*-wave pairing case in two-dimensions, when the interaction does not depend on the doping. When the correlation radius depends on the doping, the minimal value of coupling for the two-particle bound state exists even in the *s*-wave channel. Also the gap can decrease with the doping in this case.

It was also shown, that the fluctuations of the order parameter play an important role even at $T = 0$. The fluctuations of the order parameter phase in weak coupling limit in the case of the *s*-wave pairing regime lead to increasing of superconductivity at any physical carrier density, while the modulus fluctuations lead to much stronger increasing of superconductivity at small carrier densities. At large carrier densities they lead to suppression of the order parameter, as the result, the gap decreases in BCS regime, when both the modulus and the phase fluctuations are taken into account. This means that higher order fluctuation corrections should be investigated in order to develop the self-consistent theory of superconductivity (which can be similar to superfluidity) at low carrier densities.

We would like to mention some direction which can be interesting for future investigations. The study of the problem of the crossover with realistic dispersion relations is not performed even on the mean-field level in many interesting cases. The interplay between disorder and superconductivity, as well strong correlations and superconductivity in so-called bad metals are another important problems. The fluctuations in the *d*-wave pairing channel and in other nonisotropic pairing channels even in the case of Gaussian fluctuations were not studied carefully so far. It is also important to go beyond the Gaussian fluctuations, since the pair susceptibility is diverging in the 2D and in the 3D cases, as it was mentioned in Ref. 22. The role of the interlayer coupling is another problem which is not solved in general at the moment. The solution of the problems mentioned above will lead to better understanding of the superconducting properties of systems with arbitrary carrier density and pairing potential.

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