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Analytical solution of Kompaneets equation

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Analytical solution of Kompaneets equation, describing the movement of shock front from the strong point explosion in the medium with the density changing as the hyperbolic tangent, was obtained. Solution allows to restore all shock front and investigate its evolution for arbitrary values of the density change and position of the explosion point. The solution is applied for description of interaction of supernova remnants with molecular clouds.

Key words: ISM: clouds, ISM: supernova remnants, shock waves

INTRODUCTION

Interaction of the supernova remnant (SNR) with inhomogeneities of the interstellar medium (ISM) and especially with molecular clouds (MC) is of great interest. Interaction influences on the shape of the remnant and its evolution. Also in the region of interaction between SNR and MC, due to collisional pumping, the hydroxyl maser emission appears at 1720 MHz [3]. Observation on this frequency allows to detect the interaction of the SNR with MC by the regular way [6]. The gamma emission [1] from SNR interacting with MC has also a great interest due to ability to throw light on the acceleration of the nuclear component of cosmic rays on the shock waves in SNR.

Among various approaches to the solution of the problem the special place is occupied by the Kompaneets equation (KE) [4] which gives a good qualitative description of the phenomenon. Analytical solution of KE allows to restore shock front and investigate its evolution. In the given short message the continuation of our previous work [2] with analytical expressions for the shock front movement are presented.

KOMPANEETS EQUATION FOR INTERACTION OF SUPERNOVA REMNANT AND CLOUD

For the description of interaction of a shock wave with a molecular cloud we chose the following density distribution:

$$\rho(z) = \rho_0 \cdot (a - b \cdot \tanh(z/z_*)), \qquad (1)$$

where z_* is the scale of inhomogeneity of the medium between MC and ISM. At $z \to +\infty$ $\rho(z)$ tends to small but finite density of the interstellar medium $\rho_{ISM} \equiv \rho(+\infty) = \rho_0 \cdot (a - b)$. At $z \to -\infty$ $\rho(z)$ tends to the density of the molecular cloud $\rho_{MC} \equiv \rho(-\infty) = \rho_0 \cdot (a + b)$. Parameter $\gamma^2 = \dfrac{a-b}{a+b}$ describes the density changes.

Chosen density distribution allows to solve the KE equation analytically and to investigate behaviour of the shock front (SF). On the other hand, it allows adequate SNR penetration analysis.

KE for the SF in non-uniform medium with plane stratification along z-axis is [4]:

$$\left(\frac{\partial r}{\partial y}\right)^2 - \frac{1}{\varphi(z)} \cdot \left[\left(\frac{\partial r}{\partial z}\right)^2 + 1\right] = 0, \qquad (2)$$

here $r = r(z, y)$ describes the SF form in the cylindrical coordinates as a function of z coordinate and "Kompaneets time" y which equals:

$$y = \int\limits_0^t dt \cdot \sqrt{\frac{E_0 \lambda (\Gamma^2 - 1)}{2\rho_0 V(t)}}, \qquad (3)$$

where E_0 is the energy of explosion at the moment $t = 0$ in medium with density ρ_0 in point of explosion, $V(t)$ is the volume limited by the SF, Γ is an adiabatic index, λ is the dimensionless factor of an order of unit, considering proportionality of the pressure behind the front to energy density of explosion $E_0/V(t)$. Function $\varphi \equiv \rho(z)/\rho_0$ describes the medium density distribution. The basic assumption based on (1) includes constant pressure behind the SF at adiabatic stage of SNR evolution.

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As it is known [4], the general integral of the equation (2) can be found by method of envelope construction of the partial solutions of KE obtained by the method of separation of variables. It looks like:

$$r = \pm z_* \int_{z_0/z_*}^{z/z_*} dx \sqrt{\xi^2 \cdot \varphi(z) - 1} + \xi y + \mu, \quad (4)$$

where ξ is a separation constant, z_0 is the point of explosion. The integration constant μ will be considered as a function of ξ: $\mu = \mu(\xi)$. The condition of envelope construction $dr/d\xi = 0$ converts ξ into a function $\xi = \xi(z, y)$ which should be found from the equations:

$$y = \pm z_* \int_{z_0/z_*}^{z/z_*} dx \frac{\xi \varphi(x)}{\sqrt{\xi^2 \cdot \varphi(x) - 1}} - \mu', \qquad \frac{d\mu}{d\xi} \equiv \mu', \quad (5)$$

with the account of initial conditions. Substituting (5) into (4) we got:

$$r = \pm z_* \int_{z_0/z_*}^{z/z_*} \frac{dx}{\sqrt{\xi^2 \cdot \varphi(x) - 1}} + \mu - \xi \mu'. \quad (6)$$

As initial conditions we demanded that a limiting case ($y \to 0$, $t \to 0$) the solution of (4) is the Sedov solution for a homogeneous medium. In this case $\mu(\xi) = 0$, and SF represents a sphere of radius $R = \lambda \left[E_0 t^2/\rho_0\right]^{1/5}$, that corresponds to $r^2 + (z - z_0)^2 = y^2$.

Analytical Solution of Kompaneets equation

In the regions adjoining to the leading points of SF, according to results in [7, 5], the function $\mu(\xi)$ can be chosen equal to zero at any y. We passed to dimensionless values: $r \to r/z_*$, $y \to y/z_*$, $z \to z/z_*$. Then the equations for the SF shape $r(z, y)$ are reduced to

$$r(z,\xi) = \pm \int_{z_0}^{z} \frac{dx}{\sqrt{\xi^2 \cdot \varphi(x) - 1}}, \qquad y(z,\xi) = \pm \int_{z_0}^{z} dx \frac{\xi \varphi(x)}{\sqrt{\xi^2 \cdot \varphi(x) - 1}}. \quad (7)$$

Integrals are calculated easily by using inverse hyperbolic functions [2], and the expressions for the radius of the cross-section of the SF in the region moving towards the MC ($z > z_0$) are:

$$r(z,\xi) = \frac{1}{2C_-} \cdot \ln \frac{C_- + k(z)}{C_- - k(z)} \cdot \frac{C_- - k(z_0)}{C_- + k(z_0)} + \frac{1}{2C_+} \cdot \ln \frac{C_+ - k(z)}{C_+ + k(z)} \cdot \frac{C_+ + k(z_0)}{C_+ - k(z_0)}, \quad (8)$$

$$y(z,\xi) = \frac{\xi}{2} \frac{a-b}{C_-} \cdot \ln \frac{C_- + k(z)}{C_- - k(z)} \cdot \frac{C_- - k(z_0)}{C_- + k(z_0)} + \frac{\xi}{2} \frac{a+b}{C_+} \cdot \ln \frac{C_+ - k(z)}{C_+ + k(z)} \cdot \frac{C_+ + k(z_0)}{C_+ - k(z_0)}, \quad (9)$$

where $C_\pm$ and $k(z)$ stand for:

$$C_\pm = \sqrt{(a \pm b)\xi^2 - 1}, \qquad k(z) = \sqrt{\xi^2 (a - b \cdot \tanh z) - 1}. \quad (10)$$

In the region moving to the periphery of the MC ($z < z_0$):

$$r(z,\xi) = \frac{1}{2C_-} \cdot \ln \frac{C_- - k(z)}{C_- + k(z)} \cdot \frac{C_- + k(z_0)}{C_- - k(z_0)} + \frac{1}{2C_+} \cdot \ln \frac{C_+ + k(z)}{C_+ - k(z)} \cdot \frac{C_+ - k(z_0)}{C_+ + k(z_0)}, \quad (11)$$

$$y(z,\xi) = \frac{\xi}{2} \frac{a-b}{C_-} \cdot \ln \frac{C_- - k(z)}{C_- + k(z)} \cdot \frac{C_- + k(z_0)}{C_- - k(z_0)} + \frac{\xi}{2} \frac{a+b}{C_+} \cdot \ln \frac{C_+ + k(z)}{C_+ - k(z)} \cdot \frac{C_+ - k(z_0)}{C_+ + k(z_0)}. \quad (12)$$

However this expressions describe only a part of SF. In the intermediate region the surface of SF is described by solutions (5-6) with $\mu(\xi) \neq 0$ and can be obtained from continuity condition of r and y at $z = z_{extr}$ which corresponds to the plane where the derivative $\frac{\partial r}{\partial z}$ changes its sign and the wave radius is maximum at the given time [7, 5]. It gives

$$\mu(\xi) = -\frac{\xi(a-b)}{\sqrt{(a-b)\xi^2 - 1}} \times \ln \frac{\sqrt{(a-b)\xi^2 - 1} - \sqrt{(a - b\tanh z_0)\xi^2 - 1}}{\sqrt{(a-b)\xi^2 - 1} + \sqrt{(a - b\tanh z_0)\xi^2 - 1}} - \frac{\xi(a+b)}{\sqrt{(a+b)\xi^2 - 1}} \times \ln \frac{\sqrt{(a+b)\xi^2 - 1} + \sqrt{(a - b\tanh z_0)\xi^2 - 1}}{\sqrt{(a+b)\xi^2 - 1} - \sqrt{(a - b\tanh z_0)\xi^2 - 1}}. \quad (13)$$

And finally:

$$r(z,\xi) = \frac{1}{2C_-}\cdot\ln\frac{C_- - k(z)}{C_- + k(z)}\cdot\frac{C_- - k(z_0)}{C_- + k(z_0)} + \\ + \frac{1}{2C_+}\cdot\ln\frac{C_+ + k(z)}{C_+ - k(z)}\cdot\frac{C_+ + k(z_0)}{C_+ - k(z_0)}, \quad (14)$$

$$y(z,\xi) = \frac{\xi}{2}\frac{a-b}{C_-}\cdot\ln\frac{C_- - k(z)}{C_- + k(z)}\cdot\frac{C_- - k(z_0)}{C_- + k(z_0)} + \\ + \frac{\xi}{2}\frac{a+b}{C_+}\cdot\ln\frac{C_+ + k(z)}{C_+ - k(z)}\cdot\frac{C_+ + k(z_0)}{C_+ - k(z_0)}. \quad (15)$$

It can be seen, that the analytical solution of Kompaneets equation for SF with given density distribution consist of three solutions using which we can restore the form of the whole shock front and investigate its evolution.

SHOCK FRONT EVALUATION

In Fig. 1 the cross-section of the SF is presented, dashed lines correspond to the intermediate region of the SF.

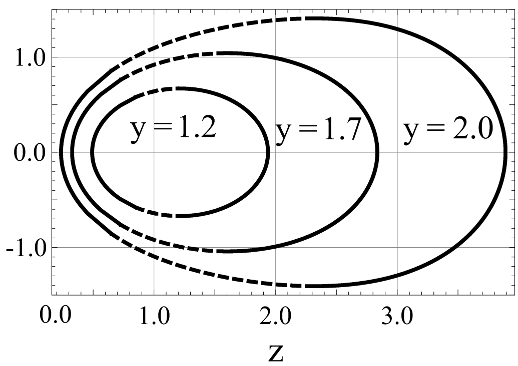

Fig. 1: Cross-section of the SF for the following parameters: $z_0 = 1, \gamma^2 \sim 10^{-3}$.

Similarly SF evolution can be investigated for arbitrary values of the density changes and position of the explosion point (see Fig. 2 and Fig. 3). It is obvious, that the form of the SF significantly changes, depending on chosen values of parameters.

CONCLUSIONS

The analytical solution of the Kompaneets equation describing the evolution of the SF in the nonuniform medium for density distribution in a form of the hyperbolic tangent from ISM to MC was obtained. Solution allows to restore the form of the whole shock front and investigate its evolution for arbitrary values of parameters: density changes, position of the explosion point, time. This solution can be applied for numerous problems such as: detection and investigation of the interaction between SNRs and MCs by observing the line of 1720 MHz hydroxyl maser emission and other maser lines; gamma emission from SNR interacting with MC.

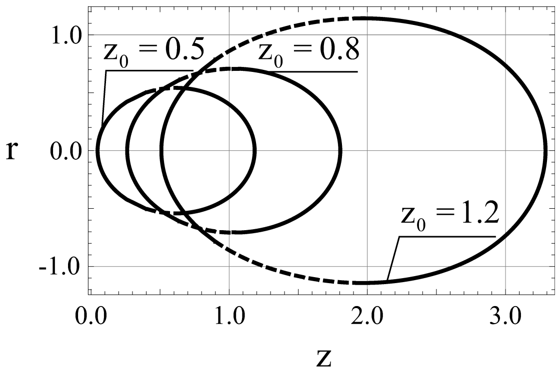

Fig. 2: Cross-section of the SF for the following parameters: $y = 1.5$, $\gamma^2 \sim 10^{-3}$; dashed lines correspond to the intermediate region.

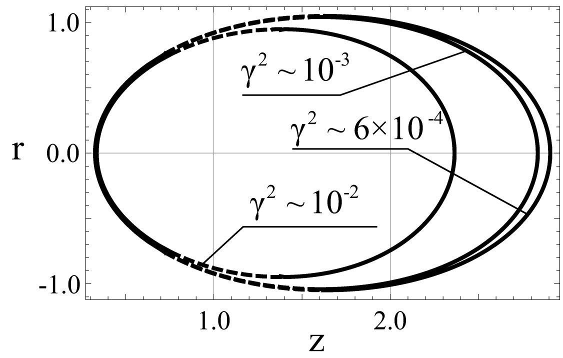

Fig. 3: Cross-section of the SF for the following parameters: $y = 1.7$, $z_0 = 1$; dashed lines correspond to the intermediate region.

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