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The nature of superfuidity and Bose-Einstein condensation: from liquid ^{4}He to dilute ultracold atomic gases

(**Review Article**)

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We present a brief overview of crucial historical stages in creation of superfluidity theory and of the current state of the microscopic theory of superfluid ^{4}He. We pay special attention to the role of Bose-Einstein condensates (BECs) in understanding of physical mechanisms of superfluidity and identification of quantum mechanical structure of ^{4}He superfluid component below λ-point, in particular — the possibility that at least two types of condensates may appear and coexist simultaneously in superfluid ^{4}He. In this context we discuss the properties of the binary mixtures of BECs and types of excitations, which may appear due to intercomponent interaction in such binary mixtures of condensates. We also discuss current status of investigations of persistent currents in toroidal optical traps and present an outlook of our recent findings on this subject.

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Contents

1. Introduction

The phenomenon of superfluidity in liquid ^{4}He, which is a Bose-liquid, and in liquid ^{3}He, which is a Fermi-liquid, as well the phenomenon of superconductivity in the electron Fermi-liquid in metals, are fundamental properties of quantum liquids. These phenomena are unique because they are manifestation of quantum laws at the macroscopic level.

The essence of the superfluidity and superconductivity phenomena is the emergence of a certain macroscopic quantity — the complex order parameter, which is the wave function of bosons or Cooper pairs of fermions that occupy the same quantum state. Thus, the so-called coherent condensate appears.

In the Bose-systems such condensate appear due to a direct grouping of bosons in the ground state and in the Fermi-systems — due to the formation of Cooper pairs, which are also bosons. Obviously, the phenomena of superfluidity and superconductivity may occur in Bose-systems or, more precisely, in systems, which obey Bose-statistics. Therefore, it seems very attractive to explain superfluidity and superconductivity on the basis of the phenomenon of Bose-Einstein condensation (BEC) — the macroscopic accumulation of bosons in the ground state of the perfect ideal (noninteracting!) gas, which was described first by Einstein [1] in 1925 on the basis of the combination of Bose-statistics and classical expression for the density of states in phase space.

Really, very shortly after discovering the phenomenon of superfluidity (by Kapitsa [2], Allen and Misener [3]) but before creation of the phenomenological theory of superfluidity (by Landau [4,5]) London[6,7] developed the concept of macroscopic occupation of the ground state and showed that it implies long-range coherence properties of the Bose-Einstein condensate and together with Tisza [8] in 1938 suggested that transition to superfluidity in liquid ^{4}He might be an example of BEC. Basis for their arguments was following: atoms of ^{4}He are bosons and transition of normal liquid He (He I) to superfluid phase (He II) takes place at the temperature T_λ = 2.17 K; if atoms of liquid ^{4}He are treated as a perfect Bose-gas, its temperature of BEC would be very close to T_b = 3.14 K. The main idea of London and Tisza was that the condensation in the perfect Bose-gas corresponds to a macroscopic occupation of the ground state, which is related to collective coherent properties of the condensate in superfluid ^{4}He. Moreover, first microscopic theory of superfluidity, which was proposed by Bogolyubov [9], is based on the fundamental hypothesis of the existence in the superfluid systems of a BEC in one specific mode!

As it is well known, notion of the Bose-Einstein condensate of bosons has to be reexamined when we consider interacting systems. Therefore, it is appropriate to clarify this notion, since atoms in liquid helium are an example of the interacting system. Intuitively we understand that BEC is a fraction of particles which does not move — frozen in momentum space with $q = 0$. Onsager and Penrose [10,11] in 1956 were the first who tried to work out a definition of condensate in the case of interacting system and they have proposed to identify condensation with an off-diagonal long-range order, related to asymptotic of single-particle density matrix. In this approach it was shown that mean-value of particle operator for the mode $q = 0$ can still be used as a characterization of a BEC and were obtained 8% as an estimate of the fraction of particles in liquid ^{4}He that have $q = 0$ at $T = 0$ K. Hohenberg and Platzman [12] proposed to use deep-inelastic neutron scattering for experimental observation of the Bose-Einstein condensate in ^{4}He, then Cowley and Woods [13] in 1968 observed about 17% of condensate at $T = 1.1$ K. After that Dubna–Obninsk groups improved experiments and for the first time not only reexamined the fraction of the condensate $(3.6 \pm 1.4)\%$ at $T = 1.2$ K [14], but also measured its temperature dependence and obtained $(2.2 \pm 0.2)\%$ for the condensate at $T = 0$ K and $(2.24 \pm 0.04)\%$ for the condensate at critical temperature [15].

After these pioneering attempts, different groups repeated and improved experiments in this direction, and from the middle of 90th of the last century up to nowadays it is universally agreed by “the superfluid helium communitee” that the best estimate of the Bose-Einstein condensate fraction is about 9% at $T = 0$ K [16–18].

The experimental evidence of the existence of Bose-Einstein condensate in the superfluid liquid ^{4}He bolsters one of the main hypotheses of London and Tisza, who tried to show that the superfluidity is closely related to the motion of the BEC, which moving as whole.

So, if we want to clarify the physical mechanisms of the superfluidity and superconductivity phenomena (from the modern point of view superconductivity is nothing but superfluidity, which occurs in a charged system), we should understand how these phenomena are connected with the phenomenon of macroscopic accumulation of bosons in the ground state of the noninteracting Bose-system, which was described by Einstein, nowadays known as BEC. At this point we touch one of the most fascinating problems of physics — would it be possible for BEC to be present in interacting systems? From this point of view the experimental realization of Bose-Einstein condensates of dilute atomic gases has opened new opportunities for investigation of this question.

The main subject of the present review is the discussion of these issues from the both side of view — theoretical and experimental.

First experimental demonstration of a BEC of alkali-metal atoms of rubidium [19] and sodium [20] has opened new possibilities for exploring superfluidity at a much higher level of control. The low-temperature atomic condensates can be prepared with essentially all atoms being in the state of Bose condensate. Because of specific features the atomic BECs differ significantly from the helium BEC: liquid helium usually is uniform while the trapping potential that confines a vapor BEC yields significantly nonuniform density; unlike spinless ^{4}He atoms, alkali atoms have nonzero hyperfine spins, and various forms of spin-dependent effects are most pronounced in spinor BECs [21]. For the weakly interacting BECs, a relatively simple Gross–Pitaevskii equation (a variant of nonlinear Schrödinger equation) gives basically good description of the atomic condensates and their dynamics at low temperatures. It is remarkable that the strength of interaction can be tuned using Feshbach resonance [22], and different geometries of the trapping potential provide the possibility to study a 2D and even a 1D system. Now atomic BEC is widely used for investigation of a superfluidity allowing for quantitative tests of microscopic theories using the tools and precision of atomic physics experiments.

Many phenomena, previously observed in liquid helium below the λ-point, have found their counterpart with ultracold alkali-metal gases. As it is well known, superfluid liquid are distinguished from normal fluids by their ability to support dissipationless flow. Such persistent currents are intimately related to the existence of quantized vortices, which are localized phase singularities with integer topological charge. The superfluid vortex is an example of a topological defect that is well known in liquid helium and in superconductors [23]. After many efforts such quantized vortices, and also arrays of vortices, were observed in atomic condensates [24–26]. Observation of the first sound [27,28], scissor modes [29] or the critical velocity [30] beyond which the superfluid flow breaks down are examples of the manifestation of this spectacular macroscopic quantum phenomenon in trapped ultracold atomic systems [31].

We do not intend to discuss in this review the vast subject of BEC in atomic gases and refer to the standard monographs and reviews [32–34]. We shall concentrate just on the crucial historical stages of creation of the theory of superfluidity and on the current state of the microscopic theory of superfluity of ^{4}He (it will be done in Sec. 2 and partially in Sec. 3). In Sec. 3 and Sec. 4 we pay special attention to the role of the BECs in understanding of the physical mechanisms of superfluidity and possibility that at least two types of condensates may appear and coexist simultaneously in ^{4}He. In this context in the second part of Sec. 4 we discuss the properties of the binary mixtures of BECs and types of excitations, which may appear due to intercomponent interaction in such binary condensate mixtures. Section 5 will be devoted to conclusions and future challenges, namely, persistent currents in toroidal trapped spinor BECs and vortex ring-on-line structures in BECs. There we will present an outlook of our recent findings on stability of superflow in ring spinor BECs. In the Appendix we will present a derivation of the general form of Gross–Pitaevskii equation for spinor condensates.

2. Theory of superfluidity: historical aspects and current state

As it is well known, superfluidity is the phenomenon of dissipationless mass transfer in macroscopic quantum systems. It was discovered in 1938 by Kapitsa [2] and bit later independently by Allen and Misener [3] during the research of liquid helium. It was established that at temperatures below T_λ = 2.17 K helium can flow through thin capillaries and slits without viscosity.

A phenomenological theory of superfluidity was created in 1941 by Landau [4,5,35]. On the contrary to the hypotheses of London and Tisza, Landau was confident that the physical reason of superfluidity is connected with the spectrum of “elementary excitations” in a liquid helium, while BEC had nothing to do with this effect. Theory of Landau (which was named the two-fluid hydrodynamics) was based on the assumption that below T_λ helium consists of two components — normal and superfluid (should be noted that Tisza was the first, who suggested this idea publicly [8,36] in the frame of his naive theory of superfluidity, and was the first, who gave a correct qualitative explanation of the behavior of superfluid helium, but his articles did not contain any proper two-fluid equations).

Each component has its own field of velocities and density, so the total density is a sum of densities of the components. It is important to note that superfluid helium is not a mixture of two different substances, so it is impossible to separate normal and superfluid components from each other. It is more precise to talk about a coexistence of two motions — normal and superfluid. Two-fluid model assume that normal motion has all the usual properties of the motion of a viscous liquid, whereas the superfluid motion is responsible for the phenomenon of superfluidity and its associated effects.

So, Landau phenomenology supposes that in quantum Bose-liquid (like ^{4}He, which at a low temperatures is a weakly excited macroscopic quantum system) collective (coherent) behavior dominates over the individual movement of atoms.

Theory of Landau allowed to achieve a high level of understanding of the properties of nonrelativistic superfluid state in a Bose-liquid ^{4}He. For instance, existence of different types of sound waves in superfluid ^{4}He were predicted on the basic of two-fluid hydrodynamics [37] and then observed in the experiment [38,39]. Moreover, calculations of thermal conductivity and viscosity of superfluid helium, obtained according to this theory, are in good agreement with experiment. Also, Landau two-fluid hydrodynamics allows to give a very nice description of the properties of ^{4}He–^{3}He solutions [40].

One of the greatest achievements of Landau's phenomenological approach is that, basing on the temperature dependence of heat capacity of superfluid ^{4}He, Bose-statistics and on the basis of his superfluidity criteria for the quantum liquids, he predicted the form of the elementary excitation spectrum of superfluid helium.

Very important that properties of superfluid liquids can be entirely described by this spectrum of collective excitations, which has two branches: the “phonon” — for long-wavelength excitations and the “roton” — for the relatively short-wavelength collective excitations. The form of the energy spectrum of elementary excitations in superfluid helium, which characterized by the linear dispersion relation at low momenta and so-called roton minimum at $q \neq 0$ was confirmed later by Woods and Henshaw [41] in the experiments on scattering of slow neutrons in liquid helium. Since then many other precise experiments confirmed and refined the shape of this curve [42–47]. Now we are well established: at long wavelength, the quasiparticles are phonons with linear relation $\omega = c|\mathbf{q}|$ between energy ω and momentum $\mathbf{q}$ and c is the (first) sound velocity; at larger momenta the superfluid ^{4}He quasiparticle spectrum is given by the well-known maxon–roton dispersion relation (see Fig.1).

However, there are some data points that cause difficulties in explanation of these observations in the frame of Landau two-fluid hydrodynamics. First difficulty is that application of the Landau criterion of superfluidity to the spectrum of elementary excitations gives $v_c \sim 60$ m/s for the critical velocity. Whereas it was well established experimentally that superfluidity in capillaries disappears when velocity is of the order of few cantimeter/second and its very sensitive to the diameter of the channel. Furthermore, whereas superfluidity will be destroyed for the tempera-

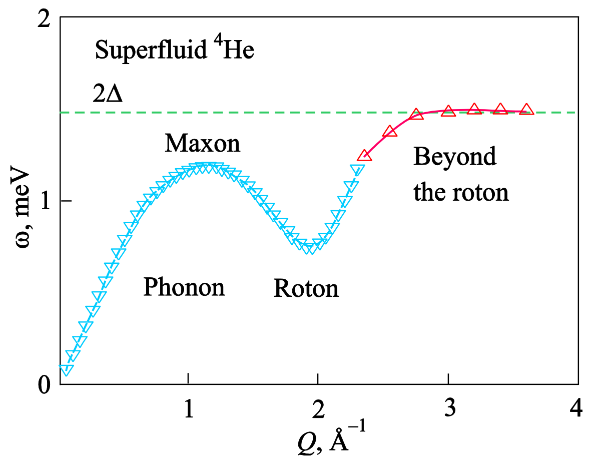

Fig. 1. Phonon–roton dispersion curve [42,48].

tures $T > T_\lambda$, Landau criterion still gives $v_c > 0$. Next contradiction is that in the frame of Landau theory is impossible to find the answer why the spectrum of elementary excitations does not change drastically when temperature crosses from $T = 0$ to $T_\lambda = 2.17\,\mathrm{K}$.

But it is expected since Landau two-fluid theory of superfluidity gives only phenomenological explanation of the phenomenon. Really, Landau phenomenological theory for the strongly interacting Bose-systems asserts that low-lying excitations may be conveniently represented by the noninteracting quasiparticles. It seems to be incredible and mysterious that this assumption describes many experimental observations successfully. We should understand how this extremely successful phenomenological description can be understood from an underlying field theory. Of course, this is a task for microscopic theory of superfluidity of Bose-liquids to provide a underground scene of this phenomenon.

However, theoretical investigation of the properties of the superfluid phase of ^{4}He at microscopic level meets with a set of fundamental problems. Most of them connected with strong interaction between bosons and complex quantum-mechanical structure of the effective coherent condensate. This condensate in the same way as single-particle BEC in case of ideal Bose-gas forms the basis of the superfluid component.

The big important steps towards a field theory of strongly interacting bosons were made in the 1950's by Bogoliubov [9,49–52], Beliaev [53,54], Hugenholtz and Pines [55] and later by Gavoret and Nozières [56].

The first microscopic theory of superfluidity, which stemming from a model of a weakly nonideal Bose-gas, was proposed by Bogoliubov and the concept of the BEC is the important ingredient of the Bogoliubov theory.

Using the fact that at very low temperature crystalline order of ^{4}He atoms suppresses their individual displacements in favor of their coherent collective movements (which are nothing more than sound waves) Bogoliubov showed that for low-energy excitations (in harmonic approximation) spectrum of corresponding hamiltonian of the microscopic system of atoms can be calculated exactly.

Bogoliubov pointed an attention to the fact that despite a long coherence in the condensate (in other words, existence of nonzero off-diagonal elements in condensate density matrix) the elementary excitations in the Bose-gas exists because of the mobility of individual atoms in the condensate ("quasiparticles" are real particles in this case). Thus, accepting important role of BEC in understanding of superfluidity phenomena, Bogoliubov proposed to find solution of quantum-mechanical problem of bosons interaction that can give an energy spectrum, which would correspond to Landau spectrum. It means that interaction between BEC particles can transform separate excitations in the Bose-gas into collective excitations, which are observed in superfluid helium as spectrum of elementary excitations.

The main advantage of Bogoliubov theory is the departure from standard perturbative methods, which based on series expansions over a small interaction constant. Existence of intensive BEC, which density is close to the total density of the superfluid system is the basic idea of this theory. As a consequence, for Bose-particles with zero momentum and energy one can neglect the noncommutativity of the creation and annihilation operators and employing the so-called linear canonical Bogoliubov transformations diagonalize the initial Hamiltonian of the system and find an expression for the renormalized momentum of quasiparticles. Note, that main Bogoliubov's suggestion — the approximation of operators by *c*-numbers, it is a fundamental hypothesis about the existence of BEC in one specific mode in the He II.

While Bogoliubov's theory was a great step forward in the understanding of the low-lying spectrum of interacting bosons, the theory is certainly *not applicable to strongly interacting systems like superfluid* ^{4}He. However, the theory paved the way for many important developments, e.g., the concept of symmetry breaking [57], which plays such a prominent role in modern theoretical physics.

Especially should be noted that gauge symmetry breaking plays a very important role in the understanding of the infrared structure of the Bose-system. A very lucid discussion of the concept of symmetry breaking and its implications is given by Anderson in Ref. 58.

Bogoliubov theory has been improved further (see, for instance, Ref. 59) and in the most of these improved models of superfluidity, using the arbitrary selection of parameters, one can achieve a good agreement between theoretical and experimental spectrum of elementary excitations in superfluid ^{4}He for a certain range of the momentum. But such agreement is more coincidental, because, as it was mentioned above, and also it was shown in the experimental studies (physically different types!) [18,60], a part of single-particle BEC in the superfluid ^{4}He is small and ranges from 2 to 10% in any case, which is the opposite to a condition of weak nonideality of the Bose-gas in the Bogoliubov theory.

Therefore, in order to give an adequate microscopic description of the Bose-liquid superfluid properties the most promising is many-body field theory approach and the Greens function method, which for the first time was used in the papers of Beliaev [53,54] and widely developed in the future. Beliaev's approach forms the basis of the systematic application of the quantum field theory methods to boson systems with condensate, including anomalous propagators, which representing two particles going into or out of the condensate. The Green function method let one find the energy of the system, its equation of state and the spectrum of quasiparticles for multiparticle Bose-systems. It allows to obtain a system of equations (Dyson–Beliaev equations), which express the normal $\tilde{G}_{11}$ and anomalous $\tilde{G}_{12}$ single-particle boson Green functions in terms of the corresponding self-energy parts $\tilde{\Sigma}_{11}$ and $\tilde{\Sigma}_{12}$. Since the BEC acts as a particle reservoir from which particles can be created or into which particles may be lost, it was necessary to introduce “anomalous" Green's functions in order to describe such processes. Therefore, the Dyson equation for the Green's functions

$$G = G_0 + G_0 \Sigma G \tag{1}$$

is turned into a 2×2 matrix equation. The diagonal elements in this equation correspond to the conventional Green's functions, and the off-diagonal elements in G and Σ are the anomalous Green's functions and self energies, respectively.

But the main difficulty of the microscopic description of the superfluid state of a Bose-liquid with a nonzero BEC is the fact that direct application of perturbation theory leads to divergences and nonanalyticities at small energies $\varepsilon\to0$ and momenta $\mathbf{q}\to0$ and, as a consequence, to erroneous results in the calculations of various physical quantities.

Thus, for example, for a Bose-system with weak interaction, when the ratio of the mean potential energy $V(q_0)q_0^3$ (q_0 being a typical momentum transfer) to the corresponding kinetic energy $q_0^2/2m$ of the bosons is small, the zeroth-approximation polarization operator $\Pi(\mathbf{q},\omega)$ and the density–density response function $\tilde{\Pi}(\mathbf{q},\omega)$ calculated to the first order in the small interaction parameter $\xi = mq_0V(q_0)\ll 1,$ are logarithmically divergent at $q\to0,$ $\omega\to0,$ whereas the exact values $\Pi(0,0)$ and $\tilde{\Pi}(0,0)$ are finite:

$$\Pi(0,0) = -\frac{\partial n}{\partial\mu} = -\frac{n}{mc^2}; \qquad \tilde{\Pi}(0,0) = \frac{n}{m(c_B^2 - c^2)}. \tag{2}$$

Here n is the total concentration of bosons, μ is the chemical potential of the quasiparticles, $c_B = \sqrt{nV_0/m}$ is the sound velocity in the Bogoliubov approximation for a weakly nonideal Bose-gas, $V_0 \equiv V(0)$ is the zero Fourier component of the potential, and c is the speed of sound in the $\mathbf{q}\to0$ limit for the spectrum of elementary excitations $\varepsilon(q)\simeq c|\mathbf{q}|$ in the Beliaev theory.

Handling of these divergences and contradictions in a satisfactory manner is necessary in order to understand the infrared response of a Bose-system and requires a suitable renormalization procedure. Unfortunately, this proves to be a rather difficult problem.

Beliaev theory was reanalyzed and extended by Hugenholtz and Pines [55] in 1959. Using gauge invariance arguments and a careful analysis of the perturbation series they showed that the quasiparticle spectrum of superfluid helium is gapless. In particular, they obtained that

$$\Sigma_{11}(0) - \Sigma_{12}(0) = \mu \tag{3}$$

(Hugenholtz–Pines theorem). Really, inserting this theorem into Eq. (1) it is easy to see that the spectrum is gapless. It is now tempting to obtain the infrared behavior of the Green's functions directly from the Eq. (1). Assuming the self-energy parts Σ_{11} and Σ_{12} to be analytic at small momenta one obtains

$$G_{11}(\omega,\mathbf{q}) = \frac{\Sigma_{11}(0)-\mu}{B(\omega^2 - c^2\mathbf{q}^2)} = -\frac{\Sigma_{12}(0)}{B(\omega^2 - c^2\mathbf{q}^2)} \tag{4}$$

where B and c are constants involving derivatives of the self-energies at $q=0.$ Within this approach one also finds that $\Sigma_{12}(0)\neq0$. The argument just presented is given in more detail in Ref. 61, and it indeed leads to a linear spectrum. However, it assumes analyticity of the self-energies and Green's functions at zero momentum, which appears to be erroneous as was recently shown in [62].

The quantum-field theory for Bose-systems as developed by Beliaev and Hugenholtz and Pines was further thoroughly reexamined in 1963 by Gavoret and Nozières [56]. They showed that the singular character of interaction in the Bose-systems with condensate remarked by Bogoliubov manifests itself in divergences of perturbation theory at small momenta (infrared divergences). They showed that infrared divergences cancel out in all physical quantities and the energy gap in elementary excitation spectrum vanishes to all orders of perturbation theory. In other words, Gavoret and Nozières established the phonon character of the spectrum up to all orders in perturbation theory. Furthermore, they successfully related the sound parameter of the field theoretical propagator with the macroscopic sound velocity c given by $c^2 = dp/d\rho$ where p is the pressure and ρ is the mass density of the Bose-system. The theory of Gavoret and Nozières effectively sums up perturbation theory to infinite order, but it does not solve the problem with infrared divergences. It yields an anomalous self-energy $\Sigma_{12}(0)\neq0.$

A first satisfactory attempt to handle the infrared divergences of the bosonic field theory was undertaken by A. Nepomnyashchii and Yu. Nepomnyashchii (NN) [63].

Their calculations are rather involved and entail a partial summation of the perturbation series. If this resummation of diagrams is done correctly then infrared divergences disappear from the theory. As an important consequence of this diagrammatic analysis one obtains that the long-wavelength behavior of the anomalous self-energy is actually nonanalytic at $q = 0$,

$$\Sigma_{12}(\omega \to 0, \mathbf{q} \to 0) \sim \frac{1}{\ln(q_0/q)}. \qquad (5)$$

Here, $q = (\omega/c, \mathbf{q})$ and $1/q_0$ is a length of the order of the interparticle distance. Equation (5) is a very important result, which makes the Green's functions G_{11} and G_{12} also behave nonanalytically at $(\omega, \mathbf{q}) = 0.$ Obviously, Eq. (5) leads to $\Sigma_{12}(0) = 0,$ which contradicts Eq. (4). NN confirmed, that the spectrum remains acoustic despite the nonanalytic behavior of the correlation functions.

The method applied by NN in order to remove the infrared divergences from the bosonic field theory is certainly not very transparent. It would be desirable to be able to construct a perturbation theory where infrared divergences are eliminated from the outset. Such a perturbation expansion has been suggested by Popov [64,65] starting from a functional integral approach. This approach yields the same perturbation expansion as the conventional field theoretical approach but suggests a more convenient method to eliminate infrared divergences. Popov's method is based on a separation of the bosonic fields into “fast” and “slow” components with respect to a certain momentum q_0. Integrating out the “fast” fields, Popov was able to construct an effective action for the “slow” fields only. Representing the “slow” fields by their amplitude and phase one obtains an effective hydrodynamic action. The diagram technique obtained from this action is free of infrared divergences. It is then straightforward to calculate the infrared structure of the various correlation functions. Popov and Serednyakov [66] were able to obtain Eq. (5), which was first derived by NN, from the effective hydrodynamic action. It implies that the Green's functions obtained in Bogoliubov's theory are not correct despite the fact that an acoustic spectrum is obtained.

Should be noted that Bogoliubov theory as well as further developments by Beliaev, Hugenholtz, Pines, Gavoret, Noziéres, NN were restricted to the zero temperature. The first finite-temperature calculations was attempted in 1957 by Lee and Huang [67], for a gas of hard spheres. It was shown that for $T \neq 0$ K the thermally induced depletion of condensate take place, so it may spoil the validity of the Bogoliubov theory. To avoid the “mismatching” one has to treat the condensate in some consistent fashion and it was Popov [64,65], who proposed in 1965 a generalization of the Bogoliubov theory for $T \neq 0$ K that gives the elementary excitation spectrum similar to that for $T = 0$ K but now with temperature-dependent condensate.

While the method employed by Popov contains the essential ideas of modern renormalization group theory, it still contains a number of phenomenological elements. In particular, the sharp separation of the fields into “slow” and “fast” components at a given momentum q_0 appears to be somewhat artificial and, furthermore, the parameter q_0 is not really well defined. A full-fledged renormalization group analysis of the infrared behavior of the Green's functions of a Bose-system was undertaken only recently by Pistolesi *et al.* [68,69] in the frame of renormalization-group approach to the infrared behavior of a zero-temperature Bose-system. Within this theory it was explicitly shown that the effective hydrodynamic action proposed by Popov is indeed the correct infrared fixed point of the renormalization group flow which starts at the “bare” action of strongly interacting bosons. In order to expose the effects of the broken gauge symmetry on the Green's functions, the Bose fields were separated into longitudinal and transverse components. The gauge symmetry is broken in the longitudinal component only. Using this formulation it is particularly easy to set up Ward identities, which relate various vertices to each other. To show this, in [62] Popov’s theory was developed further. Namely, using an analogous separation of the fields into longitudinal and transverse components and Ward identities were then used in order to obtain the vertices for the calculation of the density–density and current–current correlation functions. So, in this work was shown that if a general form of the action were written down in terms of “running” couplings, and were found that all the couplings, that are present in the “bare” interaction (but not in the hydrodynamic action) flow to zero or are irrelevant. This analysis confirms all results obtained in the NN and Popov approaches.

Should be emphasized that vanishing of the anomalous self-energy $\Sigma_{12}(0)$ at zero momentum has a definite physical origin and is not just a peculiar mathematical result. In the framework of broken symmetry, it is consistent with the general picture proposed by Patashinskii and Pokrovskii [70] where divergences, which arise in transverse correlation functions connected with a Goldstone mode (zero mass phonon), drive a divergence in the longitudinal propagators due to the continuously broken symmetry. From this point of view, the divergence of the Green's functions at zero momentum due to the vanishing of Σ_{12} is an immediate consequence of the Goldstone mode.

Of course, need to note that besides the quantum field-theoretical methods, there are various other methods that attempt to provide a basis for Landau's quasiparticle concept. A successful picture of superfluid ^{4}He has been developed using correlated basis functions or similar approaches [71,72]. A very good quantitative description of the response of superfluid ^{4}He at long and intermediate wavelength is obtained using numerical quantum Monte Carlo (for a review see Ceperley Ref. 73). The hydrodynamic formulation by Hohenberg and Martin [74] describes the infrared response without the problem of spurious infrared divergences, but it does not obtain the important result that $\Sigma_{12}(0) = 0$. Finally, a very complete picture of the excitations of Bose-systems is obtained using a method based on seminal work by Feynman [75] and Feenberg [76], which is also at the root of a recent theory of superfluid ^{4}He by Vakarchuk [77].

However, all these methods do not solve the problem of an *ab initio* calculation of the quasiparticle spectrum in the superfluid ^{4}He Bose-liquid. Nowadays could be concluded that, despite above mentioned big progress and achievements at the same time, the physical origin of superfluidity on the microscopic level still remains obscure. The analysis of experimental and theoretical publications indicates that investigations of the unique phenomenon of superfluidity of liquid helium are far from being completed (see for example [78–82]).

In fact, there are some big contradictions between the theory and the experiment and next questions remain unsolved.

1. As was mentioned above, *applicability of the Landau criterion of superfluidity for the determination of the critical velocity dissipationless flow in superfluid helium remains unclear*.

2. *The origin of the roton minimum in the spectrum of elementary excitations* — an *ab initio* computation of the spectrum of elementary excitations in the superfluid ^{4}He Bose-liquid remains an actual problem nowadays, despite certain recent successes in that direction, like an excellent agreement with experimental data in the region of the roton minimum obtained by the Monte Carlo method making use of the so-called "shadow wave function" [83] or by the correlation basic function method [71] employing modern interatomic potentials for ^{4}He. However, the physical reason behind the appearance of the roton minimum in the quasiparticle spectrum still remains unclear.

3. *Nonlinear dependence of the velocity of the first sound at small value of momentum*. Numerous precise data, obtained from the scattering of the cold neutron [84–86], clearly indicates that so-called phonon part of the quasiparticle spectrum decays.

As was shown in series of works [87–90], this character of the spectrum of elementary excitations in superfluid helium leads to the interaction and mutual transformation of low-frequency and high-frequency phonons, and also leads to the nonlinear dependence of the velocity of the first sound from the momentum. Investigation of this problem is still continue.

4. Also remains open more general *question about the interaction of original atoms in superfluid* ^{4}He *and emergent collective fluid volume elements (so-called fluid particles in the Lagrangian description*). Nowadays searches for the most suitable collective degrees of freedom in superfluids are still continue. And so the question how the formation and stability of the volume element of ^{4}He as a continuous medium can be explained from first principles remains a subject of active studies (more details see [91,92]).

5. The quantum mechanical structure of the superfluid component of the ^{4}He Bose-liquid below the λ-point, at $T < T_\lambda = 2.17$ K. As was mentioned above, according to numerous precise experimental data on neutron inelastic scattering [93–95] and experiments on quantum evaporation of ^{4}He atoms [18], the maximal density ρ_0 of the single-particle Bose-Einstein condensate in the ^{4}He Bose-liquid even at very low temperatures $T \ll T_\lambda$ does not exceed 9% of the total density ρ of liquid ^{4}He, whereas the density of the superfluid component $\rho_s \to \rho$ at $T \to 0$ K.

Such low density of the BEC (suppressed BEC) is an indication of the fact that the quantum structure of the part of the superfluid condensate in He II carrying the "excess" density $(\rho_s - \rho_0) \gg \rho_0$, which calls for a more thorough investigation.

3. Self-consistent microscopic theory of superfluidity ^{4}He for the case $T = 0$ K

For the case $T = 0$ K part of above mentioned contradictions between the theory and the experiment were partially resolved in [96–98]. In this papers authors discussed both the quantum structure of the superfluid state in a Bose-liquid at $T = 0$ K and the self-consistent calculation of the spectrum $E(q)$ of elementary excitations in the framework of renormalized field perturbation theory [63–66]. The approach is based on the Pashitskii and Yu. Nepomnyashii [99] microscopical model of superfluidity of a Bose-liquid with a suppressed BEC and intensive pair coherent condensate (PCC), which can appear due to sufficiently strong effective attraction between bosons in some domains of momentum space and is analogous to the Cooper condensate in a Fermi liquid with attraction between fermions near the Fermi surface. Should be noted that in the frame of Bogolubov approach the model of superfluidity with two types of condensates were considered earlier (see, for instance, article of Shevchenko [100]).

In our approach as a small parameter we use the ratio of the single-particle BEC density to the total Bose-liquid density $(n_0/n) \ll 1$, unlike in the Bogoliubov theory of a nonideal Bose-gas, where the small parameter is the ratio of the number of supracondensate excitations to the density of the intensive BEC $(n - n_0)/n_0 \ll 1$. Because of this, superfluid state within presented model can be described by a "truncated" self-consistent system of Dyson–Beliaev equations for the normal and anomalous self-energy parts $\tilde{\Sigma}_{ij}(\mathbf{k}, \omega)$, where the diagrams of second and higher orders in the BEC density was neglected. For this renormalized field perturbation theory [64], which is built on combined field variables [65,66] were used. In this case, the superfluid component ρ_s is a superposition of the "weak" single-particle BEC and an intensive "Cooperlike" PCC with coinciding phases (signs) of the corresponding order parameters.

Without any doubts, choice of the pair interaction potential, which will be used during the calculation of quasiparticle spectrum, is very important step, and that step should be discussed more carefully. To describe interaction of helium atoms in real space, various semiempirical potentials are conventionally used. All of them describe strong repulsion at small distances and weak Van der Waals attraction at large distances [101–103]. However, those model potentials fail to take into account the fact that at distances less than the quantum radius of the helium electron shell $r_0 = 1.22$ Å, the Coulomb repulsion between the nuclei $(Ze)^2/r$ (partially screened by bound electrons) sets in. The following simple approximation for the ^{4}He interatomic potential, diverging as r^{-1} at $r \to 0$, could be suggested:

$$V(r) = \begin{cases} \dfrac{4e^2}{r}(1+\beta r)\exp(-r/\alpha), & r \le r_c , \\ \varepsilon\left[\left(\dfrac{\sigma}{r}\right)^{12} - \left(\dfrac{\sigma}{r}\right)^{6}\right], & r > r_c . \end{cases} \quad (6)$$

Such a potential has a finite Fourier component with an oscillating sign-changing momentum dependence (Fig. 2, dashed curve).

The Fourier component will look like any other potential of the form (6) in which the interaction at $r > r_c$ is determined by any of the modern ^{4}He potentials [101–103].

However, those Fourier components are analytically very complicated and it is technically very difficult to use in the actual calculations. To be able to go forward, while retaining the crucial features of the interaction, one should employ a model potential, characterized by the same sign-changing Fourier component as the one of Eq. (6), but with a simpler analytic expression. For example, it is possible to choose the simple finite repulsive potential of the "semi-transparent spheres" model, $V(r) = (V_0/4\pi a^3)\theta(a-r)$ (θ is the step function), whose Fourier component

$$V(p) = V_0 \frac{\sin(pa) - pa\cos(pa)}{(pa)^3} \quad (7)$$

is an oscillating sign-changing function of momentum transfer p (Fig. 2, curve *1*). It is necessary to emphasize that the existence of negative values of the Fourier component $V(p) < 0$ is not directly associated with Van der Waals forces. These oscillations arise even in the absence of attraction in real space, and are an implication of quantum mechanical diffraction effects of mutual scattering of the particles.

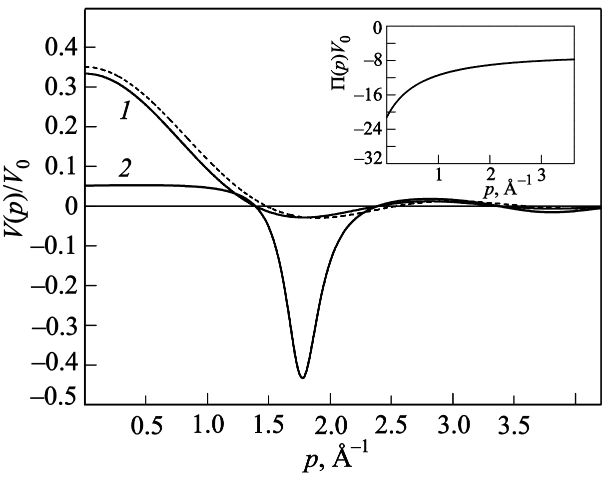

Fig. 2. The Fourier components of: potential (6) (dashed curve); model potential (7) (curve *1*); the corresponding renormalized potential (10), with account for the momentum dependence (11) of the polarization operator Π (inset) on the "mass shell" (curve *2*).

Should be also noted that the same behavior is characteristic for the Fourier components of more realistic potentials that diverge not faster than r^{-2} at $r \to 0$ and possess an inflection points in the radial dependence.

For the calculation of the spectrum of elementary excitations in the superfluid ^{4}He the system of Dyson–Beliaev equations were used. These equations allow one to express the normal $\tilde{G}_{11}$ and anomalous $\tilde{G}_{12}$ renormalized single-particle boson Green functions in terms of the respective self-energy parts $\tilde{\Sigma}_{11}$ and $\tilde{\Sigma}_{12}$. As was shown in Ref. 99 for a Bose-liquid with sufficiently strong interaction between particles when BEC is strongly suppressed, one can, defining $\tilde{\Sigma}_{ik}(\mathbf{p},\varepsilon)$ in the form of a sequence of irreducible diagrams that contain condensate lines, restrict oneself, with good precision, to the lowest terms in the expansion over the small BEC density $(n_0 \ll n)$. As a result for a Bose-liquid, leaving the terms of the first order in small parameter $n_0/n \ll 1$, one gets "truncated" system of equations for $\tilde{\Sigma}_{ik}$:

$$\tilde{\Sigma}_{11}(\mathbf{q},\varepsilon) = n_0\Lambda(\mathbf{q},\varepsilon)\tilde{V}(\mathbf{q},\varepsilon) + n_1 V(0) + \tilde{\Psi}_{11}(\mathbf{q},\varepsilon); \quad (8)$$

$$\tilde{\Sigma}_{12}(\mathbf{q},\varepsilon) = n_0\Lambda(\mathbf{q},\varepsilon)\tilde{V}(\mathbf{q},\varepsilon) + \tilde{\Psi}_{12}(\mathbf{q},\varepsilon), \quad (9)$$

where

$$\tilde{V}(\mathbf{q},\varepsilon) = V(q)[1 - V(q)\Pi(\mathbf{q},\varepsilon)]^{-1}. \quad (10)$$

Here $V(q)$ the Fourier component of the input pair interaction potential; $\tilde{V}(\mathbf{q},\varepsilon)$ is the renormalized ("screened"), due to multiparticle collective effects, Fourier component of the nonlocal interaction; $\Pi(\mathbf{q},\varepsilon)$ is the boson polarization operator that takes into accounting the multiparticle collective effects:

$$\Pi(\mathbf{q},\varepsilon) = i\int \frac{d^3\mathbf{k}}{(2\pi)^3}\int \frac{d\omega}{2\pi}\Gamma(\mathbf{q},\varepsilon,\mathbf{k},\omega)\{G_{11}(\mathbf{k},\omega)\times$$
$$\times G_{11}(\mathbf{k}+\mathbf{q},\ \varepsilon+\omega) + G_{12}(\mathbf{k},\omega)G_{12}(\mathbf{k}+\mathbf{q},\ \varepsilon+\omega)\}; \quad (11)$$

$\Gamma(\mathbf{q},\varepsilon,\mathbf{k},\omega)$ is the vertex part, which describes multiparticle correlations; $\Lambda(\mathbf{q},\varepsilon) = \Gamma(\mathbf{q},\varepsilon,0,0) = \Gamma(0,0,\mathbf{q},\varepsilon)$, and n_1 is the number of supracondensate particles $(n_1 \gg n_0)$, which is determined from the condition of conservation of the total number of particles.

In addition, as it was also shown by Pashitskii and Y. Nepomnyaschii in [99], only residues at the poles of single-particle Green functions were taken into account, whereas, the contributions of the poles of the functions $\Gamma(\mathbf{q},\varepsilon,\mathbf{k},\omega)$ and $\tilde{V}(\mathbf{q},\varepsilon)$, which do not coincide with the poles of $\tilde{G}_{ij}(\mathbf{q},\varepsilon)$, were neglected.

As a result, the functions $\tilde{\Psi}_{ij}(\mathbf{q},\varepsilon)$ on the "mass shell" $\varepsilon = E(q)$ have the following form (at $T = 0$ K):

$$\tilde{\Psi}_{11}(\mathbf{q},E(q)) = \frac{1}{2}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\Gamma(\mathbf{q},E(q);\mathbf{k},E(k))\tilde{V}(\mathbf{q}-\mathbf{k},E(q)-E(k))\left[\frac{A(\mathbf{k},E(k))}{E(k)}-1\right]; \tag{12}$$

$$\tilde{\Psi}_{12}(\mathbf{q},E(q)) = -\frac{1}{2}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\tilde{V}(\mathbf{q}-\mathbf{k},E(q)-E(k))\Gamma(\mathbf{q},E(q);\,\mathbf{k},E(k))\frac{n_0\Lambda(\mathbf{k},E(k))\tilde{V}(\mathbf{k},E(k))+\tilde{\Psi}_{12}(\mathbf{k},E(k))}{E(k)}, \tag{13}$$

where

$$E(q) = \frac{1}{2}[\tilde{\Psi}_{11}(\mathbf{q},E(q))-\tilde{\Psi}_{11}(-\mathbf{q},-E(q))]+\{A^2(\mathbf{q},E(q))-[n_0\Lambda(\mathbf{q},E(q))\tilde{V}(\mathbf{q},E(q))+\tilde{\Psi}_{12}(\mathbf{q},E(q))]^2\}^{1/2}, \tag{14}$$

$$A(\mathbf{q},E(q)) = n_0\Lambda(\mathbf{q},E(q))\tilde{V}(\mathbf{q},E(q))+\tilde{\Psi}_{12}(0,0)-\tilde{\Psi}_{11}(0,0)+\frac{1}{2}[\tilde{\Psi}_{11}(\mathbf{q},E(q))+\tilde{\Psi}_{11}(-\mathbf{q},-E(q))]+\frac{q^2}{2m}. \tag{15}$$

In this case, the total quasiparticle concentration is determined by the relation

$$n = n_0 + \frac{1}{2}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\left[\frac{A(\mathbf{k},E(k))}{E(k)}-1\right]. \tag{16}$$

From Eqs. (14) and (15) it follows that the quasiparticle spectrum, because of the analyticity of the functions $\tilde{\Psi}_{ij}(\mathbf{q},\varepsilon)$, is acoustic at $p \to 0$, and its structure at $q \neq 0$ depends essentially on the character of the renormalized interaction of pair of bosons.

Note that in the absence of a BEC ($n_0 = 0$), Eq. (13) becomes homogeneous and degenerate with respect to the phase of the function $\tilde{\Psi}_{12}(\mathbf{p})$. It is then become analogous to the Bethe–Goldstone equation for a pair of particles in momentum space

$$\Psi(\mathbf{q}) = -\int \frac{d^3\mathbf{k}}{(2\pi)^3}V(\mathbf{q}-\mathbf{k})\frac{\Psi(\mathbf{k})}{2E(k)-\Omega},$$

with zero binding energy $\Omega = 0$, which has a nontrivial solution only in the case of attraction $V(q) < 0$. This analogy allows one to treat $\tilde{\Psi}_{12}(\mathbf{q})$ at $n_0 = 0$ as a PCC order parameter [99], which describes boson pair condensation in momentum space (identical to the Cooper condensate of fermion pairs). Equation (13) being degenerate over the phase of $\tilde{\Psi}_{12}(\mathbf{q})$ at $n_0 \to 0$ allows one to meet the condition of stability of the phonon spectrum $c_1^2 = \tilde{\Psi}_{12}(0)/\tilde{m}^* > 0$ by choosing the appropriate sign of the pair order parameter $\tilde{\Psi}_{12}(0) > 0$. Since at $T = 0$ K the density ρ_s of the superfluid component, on the one hand, coincides with the total density $\rho = mn$ of the Bose-liquid and, on the other hand, is proportional to $\tilde{\Sigma}_{12}(0)$, which plays the role of the superfluid order parameter, one gets the following relations:

$$\rho_s \equiv \rho_0 + \tilde{\rho}_s = \beta m\frac{\tilde{\Sigma}_{12}(0)}{\Lambda(0)\tilde{V}(0)} = \beta m[n_0(1-\gamma)+\Psi] \tag{17}$$

where

$$\gamma \equiv \frac{1}{(2\pi)^2\Lambda(0)\tilde{V}(0)}\int_0^\infty \frac{k^2dk}{E(k)}[\Lambda(k)\tilde{V}(k)]^2,$$

$$\Psi \equiv -\frac{1}{(2\pi)^2\Lambda(0)\tilde{V}(0)}\int_0^\infty \frac{k^2dk}{E(k)}\Lambda(k)\tilde{V}(k)\tilde{\Psi}_{12}(k),$$

and β is a certain dimensionless constant.

Since the density of the single-particle BEC is equal to $\rho_0 = mn_0$, we obtain $\beta = (1-\gamma)^{-1}$. This means that the density of the "Cooperlike" PCC is

$$\tilde{\rho}_s = mn_1 = \frac{m\Psi}{(1-\gamma)}, \tag{18}$$

the concentration $n_1 = n - n_0$ can be determined from relation (16), and for liquid ^{4}He at $T \to 0$ K, in accordance with the experimental data, it should be approximately 90% of the full concentration $n = 2.17\cdot 10^{22}$ cm^{-3}. Thus, the superfluid component of the Bose-liquid at $T = 0$ K in this model is an effective coherent condensate [99], which is a superposition of the weak single-particle BEC and the intensive PCC.

The key point in the behavior of the Fourier component of the screened potential $\tilde{V}(\mathbf{q},E(q))$ is that, as long as the quasiparticle spectrum $E(q)$ satisfies the condition of stability with respect to decay into a pair of quasiparticles — $E(q) < E(k) + E(q-k)$, the real part of the polarization operator is negative: $\Re\Pi(\mathbf{q},E(q)) < 0$ on the "mass shell" (Fig. 2, inset). As a result strong suppression of repulsion in the region where $V(q) > 0$ and strong enhancement of attraction in the region where $V(q) < 0$ are take place (compare curves *1* and *2* on Fig. 2).

In order to calculate quasiparticle spectrum $E(q)$, one has to calculate the polarization operator (11) and the renormalized retarded interaction (10) on the "mass shell" $\omega = E(q)$ as well as $\omega = E(q) \pm E(k)$. At the same time is necessary to solve the nonlinear integral Eqs. (12) and (13) for the functions $\tilde{\Psi}_{ij}(\mathbf{q}, \pm E(q))$. The only parameter varied in order to ensure the best coincidence of $E(q)$ with the experimental ^{4}He quasiparticle spectrum $E_{\text{exp}}(q)$ was the amplitude V_0 of the initial potential (7) (for $V_0/a^3 = 1552$ K at $a = 2.44$ Å). The BEC concentration was given, in accordance with the experimental data, as $n_0 = 9\%n = 1.95\cdot 10^{21}$ cm^{-3}. Figure 2, curve *2* depicts the momentum dependence of the renormalized retarded interaction (10). On Fig. 3 solid line is the theoretical quasiparticle spectrum $E(q)$ (14), dots are the experimental spectrum [93–95], points beyond the roton minimum shown as stars ($T = 0.6$ K) [47]. Note that the phase velocity of quasiparticles $[E(q)/q]|_{q\to 0}$, obtained within this model, coincides with the speed of hydrodynamical sound $c_1 \simeq 236$ m/s in liquid ^{4}He. Satisfactory agreement of $E(q)$ with $E_{\text{exp}}(q)$ at $q \le 3.5$ Å^{-1} is evident.

The self-consistency of the model is confirmed by the following argumentation. On the one hand, theoretical value of the full particle density from equation (16) $n_{\text{th}} = 2.14\cdot 10^{22}$ cm^{-3} is close to the experimental ^{4}He density; on the other hand, the density n_1 of supracondensate particles from Eq. (18) for the indicated parameter values is above 90%*n*, which also agrees with experiment if taking into account that BEC density is determined to be up to about 9%*n*. The main result, obtained within this model of the superfluid state, is the conclusion that the roton minimum in the spectrum $E(q)$ and maximum in the structure

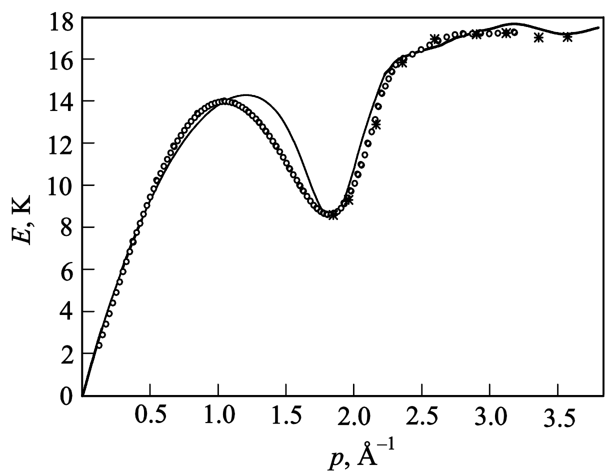

Fig. 3. Theoretical quasiparticle spectrum *E*(*p*) obtained by a self-consistent calculation (solid line); the experimental ^{4}He excitation spectrum (○); the spectrum beyond the roton minimum (∗) [47].

factor $S(q, E(q))$ is associated with the first negative minimum of the Fourier component of the renormalized ("screened") interaction potential.

However, in case $T \ne 0$ contradictions between the theory and the experiment, that we mentioned above, remained unsolved and consistent microscopic superfluid theory of Bose-liquid still not created. It is necessary to emphasize that resolving of these contradictions very closely connected with the problem of quantum mechanical structure of superfluid ^{4}He component below the λ-point.

The answer of this question is crucial for the creation of the consistent microscopic superfluid theory of Bose-liquid for the cases of zero and nonzero temperatures.

In any case it is obvious that phenomenon of BEC plays the key role for the understanding of physical mechanism of superfluidity, and it seems like that at least two types of condensates may appear and coexist simultaneously in ^{4}He.

It is appropriate to mention here that the phenomenon of BEC (macroscopic occupation of the ground state due to the saturation mechanism) is rather subtle and fascinating question. In the next section will be discussed a recent study of the Bogoliubov model for weakly interacting Bose-gas, which shows that under certain conditions this phenomenon manifests at least two kinds of BECs.

4. Conventional and nonconventional Bose-Einstein condensations. Binary mixtures of Bose-Einstein condensates

Shortly after discovering the phenomenon of BEC, London has pointed out that this phenomenon implies long-range coherence properties of the condensate and Uhlenback [104] found that thermodynamic limit is very important for the sharp manifestation of transition into condensed phase. Later, Casimir [105] has pointed out the importance of distinguishing between thermodynamic and coherence properties of BEC in perfect Bose-gas.

Structure of the single-particle density of states (in the thermodynamic limit) implies that for a given temperature *T* there is a critical particle density, which corresponds to maximal total particle density $\rho_c^p(T)$. Under condition that the particle density in the ground state $\rho_0^p(T) = 0$ and increasing of total density beyond the saturation threshold $\rho_c^p(T)$, it leads to macroscopic accumulation of particles in the ground state, i.e., $\rho_0^p(T) = \rho - \rho_c^p(T)$. This saturation mechanism of condensation does not change the average kinetic–energy or pressure, which also reach their maximal values at the critical density $\rho_c^p(T)$.

Girardeau [106] was the first who introduced the concept of generalized (conventional) BEC on the basis of the observation that in one-dimensional model of impenetrable bosons there is a sort of generalized condensation but no macroscopic occupation of the ground state. Theoretical importance of this concept was shown by Van den Berg and Lewis [107–109], who have proposed next classification of Bose condensations: *the condensation is called the type I, when a finite number of single-particle levels are macroscopically occupied; it is of type II, when an infinite number of the levels are macroscopically occupied; it is called the type III, or the nonextensive condensation, when none of the levels are macroscopically occupied*. In work [109] was given an example of these different condensations and were demonstrated that three types of BEC can be realized in the case of the perfect Bose-gas in an anisotropic rectangular box or in a prism of volume V with sides V^a, V^b, V^c, where $a \geq b \geq c$ and $a+b+c=1$. If $c < 0.5$, then for sufficiently large density ρ the Bose-Einstein condensation of type I appears; if $c = 0.5$ one gets a condensation of type II, whereas for $c > 0.5$ a condensation of type III could be caused. Also, in [108] it was shown that the type III condensation can be caused in the perfect Bose-gas by a weak external potential or by a specific choice of the boundary conditions and geometry. In [110,111] it was given another example of nonextensive condensation for repulsive interacting bosons, which are included in an isotropic box, and which are spread out the conventional BEC of type I.

Also Bose-condensates could be divided on classes by their mechanisms of formation. "*Conventional Bose-Einstein condensate*" — the condensation is due to saturation of the total particle density (originally were discovered by Einstein in the Bose-gas without interactions).

Since bosons are very sensitive to attraction, there is another kind of condensation induced by this interaction. Important results were obtained in [79,111], where existence of the phenomenon of condensation that was induced by interaction has been proved. It may concern the model of Huang–Yang–Luttinger [112] or full diagonal models [113], since they both contain attractive interactions, and also in the case of the Bogoliubov weakly imperfect Bose-gas [114]. This condensate is called *nonconventional Bose-condensate*. This kind of condensation might appear when the total particle density (or chemical potential) becomes larger than some critical value, but this is an attractive interaction (and not simply Bose-statistics) which defines the magnitude of the condensate and its behavior.

The difference between conventional and nonconventional condensations consists in the difference of the mechanism of their formation. The conventional BEC is a consequence of the balance between entropy and kinetic energy, whereas the nonconventional condensation follows from the balance between entropy and interaction energy.

Especially important that, as it has been shown in [111], the nonconventional condensation does not exclude the appearance of the BEC when the total density of particles grows and exceeds some saturation limit. To escape the collapse an attractive interaction in a boson system should be stabilized by a repulsion, therefore, *conventional and nonconventional condensation may coexist*. The possibility of emergence of two kinds of condensates in two stages was discussed at the first time in [107,111] in the framework of a so-called "pair Hamiltonian model" and was shown that single-particle and two-particle states could appear simultaneously due to off-diagonal interaction terms. In weakly imperfect Bose-gas these condensates can occur in two stages. For an interval $[\mu_0, 0]$ of negative chemical potentials up to $\mu = 0$, one has a macroscopic occupation $\rho_0(\mu)$ of the mode with momentum $k = 0$ due to effective attraction of bosons in this mode [115]. Since the weak interacting Bose-gas can exist only for $\mu \leq 0$ and $\rho_0(\mu)$ as well as the total density $\rho(\mu)$ attain their maximal values at $\mu = 0$ for $\rho > \rho(\mu = 0)$ one gets (due to the well-known saturation mechanism) a kind of condensation, which occurs despite of effective two-bosons repulsion in the weakly interacting Bose-gas for $k \neq 0$. Moreover, in [116] was demonstrated that in so-called "perturbed mean-field model with a Gaussian interaction kernel" there is no Bose condensation for negative chemical potentials, but condensation appears for $\mu \in [0, \mu_+]$ and then again disappears for $\mu \in [\mu_+, \tilde{\mu}_+]$, where $\tilde{\mu}_+ \geq 2\mu_+$.

So, binary mixtures of BECs are interacting quantum systems of the macroscopic scale, which exhibit rich physics not accessible for a single-component degenerate quantum gas. The key difference between multi-component and single-component BECs is the intercomponent interaction. In view of their unique properties binary mixtures of BECs open up intriguing possibilities for a number of important physical applications, including quantum simulation [117], quantum interferometry [118], and precision measurements [119,120]. As it will be shown later, intercomponent interaction in binary mixtures of Bose-Einstein condensates leads to appearance of different kinds of excitations in such system. Especially interesting that some of these excitations are very similar to the excitations in superfluid ^{4}He.

Recently, fundamental 2D soliton–soliton pairs were investigated in two-component BECs with attractive intracomponent interactions [121]. General properties of vector solitons and their stability were studied variationally and numerically for both attractive and repulsive intercomponent interactions and there were found different types of soliton–soliton pairs including phase-separated pairs, where one component is pushed outwards and forms a ring-like shell and the other component is compressed due to repulsive intercomponent interactions. It turns out that for some values of the chemical potentials μ_1, μ_2 phase-separated steady-states coexist with collocated states characterized by bell-shaped density distributions in both components.

In paper [121] was performed a linear stability analysis of small azimuthal perturbations, which was checked by an extensive series of numerical simulations. For attractive intercomponent interactions matter-wave bright vector solitons were demonstrated to be stable throughout the existence domain. For BEC components, which repel each other, various unstable evolution scenarios including collapse and azimuthal symmetry-breaking instabilities were observed. The instabilities, as a rule, lead either to separation of the condensed phases and then a collapse of the stronger supercritical $(N > N_{cr})$ component or a periodic relative motion of the subcritical $(N < N_{cr})$ solitonic components backwards and forwards near the bottom of trapping potential. Nevertheless, there are conditions where complete stabilization of vector solitons is observed even in the case of repulsive intercomponent interactions.

Consideration of a mixture of two BECs with atoms of equal masses $M_1 = M_2$ and scattering lengths $a_{11} = a_{22}$, $a_{21} = a_{12}$ in axial symmetric harmonic trap $V_{\text{ext}}(\mathbf{r}) = M\omega_\perp^2(x^2+y^2)/2 + M\omega_z^2 z^2/2$ allows to investigate it in nearly two-dimensional case $\omega_z \gg \omega_\perp$. At the limit $T \to 0$ K in mean-field approximation the system can be described by set of Gross–Pitaevskiy equations [122]:

$$i\frac{\partial\tilde{\Psi}_j}{\partial t} + (\Delta_\perp - r^2 + |\tilde{\Psi}_j|^2 + b_{j,3-j}|\tilde{\Psi}_{3-j}|^2)\tilde{\Psi}_j = 0, \quad (19)$$

where $j = 1, 2$, $r = \sqrt{x^2+y^2}$, $\Delta_\perp = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the 2D Laplacian. Here $(x,y) \to (x,y)/l_\perp$, $t \to \omega_\perp t/2$, $\tilde{\Psi}_j \to \tilde{\Psi}_j/\sqrt{C_j}$ is the dimensionless variables, where

$$l_\perp = \sqrt{\frac{\hbar}{M\omega_\perp}}, \quad C_j = \hbar\omega_\perp\frac{\sqrt{\pi l_z^2/2}}{|g_{jj}|}, \quad g_{jj} = \frac{4\pi\hbar^2 a_{jj}}{M}.$$

Dimensionless coupling parameter is defined as: $\sigma = -g_{12}/|g_{11}| = -g_{21}/|g_{22}|$, where $g_{12} = g_{21} = 4\pi\hbar^2 a_{12}/M$.

Let us discuss in more details the case when internal interactions are attractive while intercomponent can be repulsive or attractive.

Stationary soliton solutions were given as follows:

$$\tilde{\Psi}_j(\mathbf{r},t) = \psi_j(r)\,\mathrm{e}^{-i\mu_j t}, \quad (20)$$

where $j = 1, 2$, $r = \sqrt{x^2+y^2}$, μ_j are chemical potentials, and real functions $\psi_j(r)$ satisfy the set of equations:

$$\mu_j\psi_j + \psi_j'' + \frac{1}{r}\psi_j' - r^2\psi_j + [\psi_j^2 + \sigma\psi_{3-j}^2]\psi_j = 0. \quad (21)$$

At fixed strength of cross-interaction σ we obtain two-parameter family (with parameters μ_1 and μ_2) of vector soliton solutions.

To find region of existence of soliton–soliton pairs and study some general properties we investigate solutions of stationary equations. Equations (21) were solved numerically. To gain a better insight into the properties of the vector solitons one has to perform also the variational analysis of the stationary vector fundamental solitons. Typical radial profiles were found numerically and are shown in Fig. 4.

Stability of the stationary solutions was tested by three methods. Some results were obtained by means of variation analysis and were found that soliton–soliton pair is stable with respect to radial-symmetric collapse if both of components have number of particles $N < N_{cr}$, where N_{cr} is number of particles on Townes' soliton. Stability

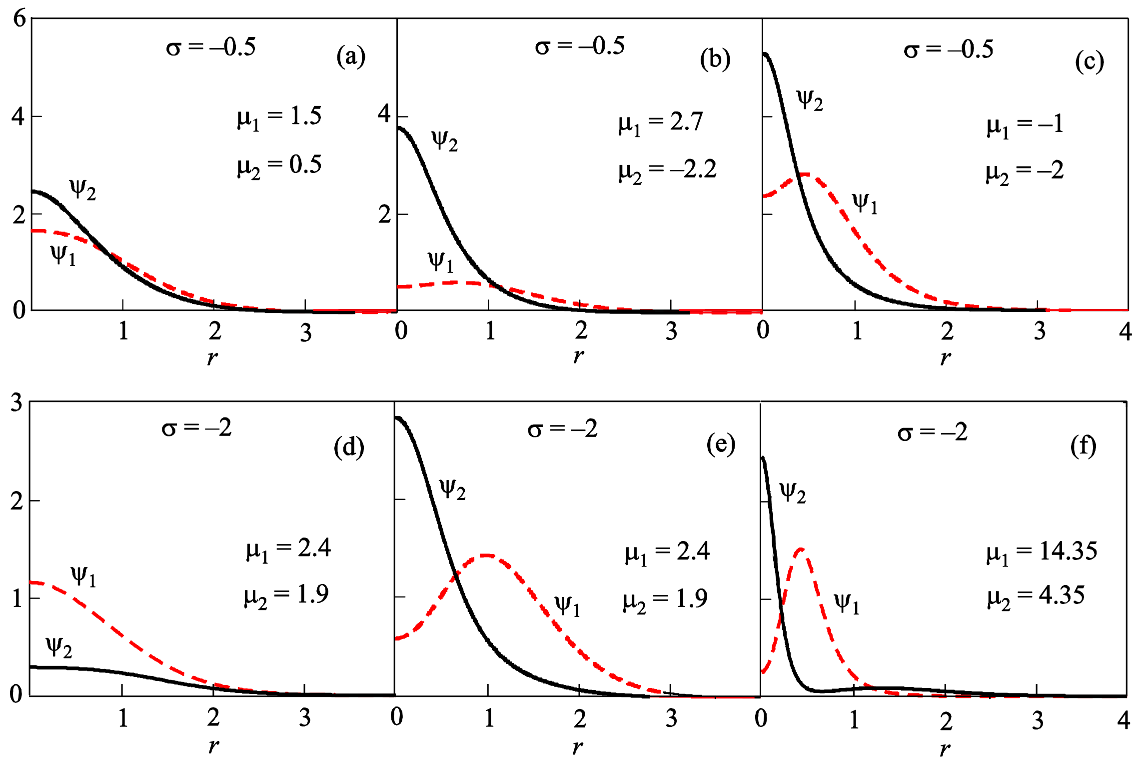

Fig. 4. (Color online) Examples of radial profiles ψ_1 (dashed curves) and ψ_2 (solid curves) for $\sigma = -0.5$ and $\sigma = -2.0$ found numerically.

with respect to different azimuthal modes has been investigated by linear analysis. Both of these results were tested by direct numerical simulations, for which we used split-step Fourier transform method. Different scenarios of unstable and robust evolution have been observed. Examples of unstable evolution are shown in Figs. 5 (b) and (d).

To summarize, the region of existence and stability of soliton–soliton pairs was found and were shown that for attractive cross-interaction region of stability coincides with region of existence. In case of repulsive intercomponent interaction region of stability is situated near existence boundary (Figs. 5 (a) and (c)).

5. Conclusions and future challenges

The main scope of the present review is devoted to the role of the BEC in the superfluidity phenomena. We present a review of large volume of recent literature that has been developed over the last decades in this forefront of research, interfacing between quantum and nonlinear physics. It is worth to close this paper briefly recalling the most important ideas that laid in the groundwork of the progress in understanding of the nature of superfluidity.

Opposite to the Landau's statement that superfluidity phenomena and BEC have nothing in common, Bogoliubov, Onsager, Penrose, Feynman, and Yang clearly elucidated the very close relation between these phenomena. Numerous experimental evidences of existence of the BEC in the liquid superfluid ^{4}He reaffirms this theoretical assumption and now it is well established that the best estimate of the BEC fraction in the superfluid ^{4}He is about 9% at $T = 0$ K. From the other hand, it is well known that the density of the superfluid component practically coincides with total density of ^{4}He, i.e., $\rho_s \to \rho$ at $T \to 0$ K. Such big discrepancy creates the question about the quantum structure of the superfluid part of He II and indicates the fact that the structure of "excess" density $(\rho_s - \rho_0) \gg \rho_0$ is more complicated.

Taking into account the fact that Van den Berg, Levis, Bru, Zagrebnov, Dorlas and Pulé showed that two different kinds of BEC (conventional and nonconventional) of different physical nature can appear and coexist simultaneously in a weakly nonideal Bose-gases, it is tempting to assume that the superfluid component is created from the superposition of these condensates. This means that the observed quasiparticle spectrum in ^{4}He follows from the

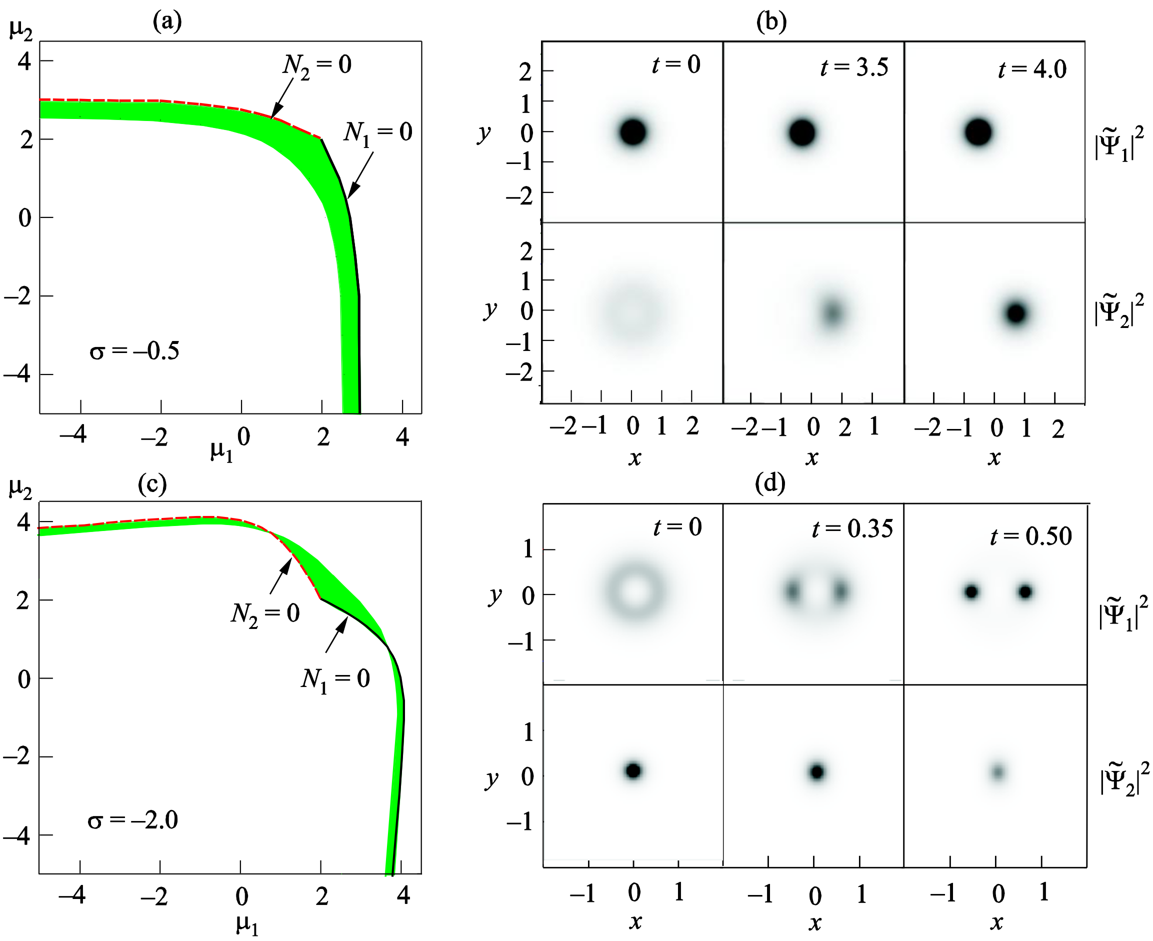

Fig. 5. (Color online) Stability region in the μ_1, μ_2 plane (green) as obtained by numerical simulations: $\sigma = -0.5$ (a), $\sigma = -2.0$ (c). Time evolution of the density distributions $|\tilde{\psi}_1|^2$ (upper rows) and $|\tilde{\psi}_2|^2$ (lower rows) of perturbed vector solitons for $\sigma = -0.5$, $\mu_1 = -1.5$, $\mu_2 = 2$ (b), and $\sigma = -0.8$, $\mu_1 = -5$, $\mu_2 = -5$ (d).

interaction of these condensates. For the case of zero temperature such an assumption enables one to create self-consistent microscopic theory of superfluidity, which is expected to explain practically all observed properties superfluid helium. However, for the nonzero temperature such theory does not exist yet, despite a big efforts. As though it may sound paradoxical, now a successful solution to this problem lies rather in the plane of the experimental than theoretical. It is important to understand which kind of the BEC is observed in superfluid helium (pure condensate, which, again, is no more than a tenth part of the superfluid component) and what is the nature of superfluid component. What is the building block of the superfluid component? As the matter of fact, it is very challenging problem from experimental point of view, however after experimental realization of BEC in an atomic Bose-gas (groups of Cornell and Wieman, Ketterle, and of Hulet in 1995) has made it feasible to investigate the relation between superfluidity and BEC on the experimental basis of ultracold atomic gases, as these systems offer an unprecedented level of control on the interaction strength, density and temperature. Really, such close mutual relation was confirmed and clarified by the investigations on ultracold atomic gases: in 1999 superfluidity was demonstrated through vorticity in a dilute Bose-gas (Cornell, Wieman, Dalibard, Ketterle); in 2003–2005 BEC of pairs, and subsequently superfluidity as again characterized through vortices, was realized in Fermi-gases (Zwierlein, Ketterle, Jin). Future experiments will probe such aspects of superfluidity as the appearance of vortices (related to phase coherence) and critical velocities (related to the Landau criterion). Without any doubts the versatility of quantum dilute gases would allow in the nearest future to find out an answer about nature of the different aspects of the phenomenon of superfluidity ^{4}He, especially the quantum mechanical structure of the superfluid component of the ^{4}He below the λ-point, at $T < T_\lambda$. The answer of this question will play key role for the consistent microscopic theory of quantum Bose-liquid has yet to be created.

The anniversary review paper [123], devoted to the history of superfluidity in XX century, is ending with the words “... at least at this time, the evidence for superfluidity (in atomic BEC) is still quite circumstantial”. However, nowadays there are many fixed experimental facts that demonstrate superfluid properties of cold atomic gases. In this paper we have reviewed the milestones of this way. But it is definitely not the end of the story. In the future one can expect novel intriguing findings in this fast developing subject. On several examples let us consider an outlook of our findings concerning superfluidity of atomic BECs in toroidal optical traps such as stability of persistent current in spinor BEC and novel type of vortices, which combine both known types of singularities of vortex motion: vortex ring and vortex line.

Persistent currents are a hallmark of superfluidity and superconductivity, and have been studied in liquid helium and solid state systems for decades. Toroidally trapped BECs are attractive both for fundamental studies of superfluidity and for applications in interferometry for precise measurements and atomtronics.

There are different methods to create a toroidal trap: using a Laguerre–Gaussain laser beams [125], combined magnetic trap and standing wave of light [126]. The experimental observation of persistent flow in toroidal BEC is reported in [124,127–129]. The persistent current in 2D spinor $F = 1$ condensates of Na has been investigated in Ref. 130, however spin degree of freedom are not investigated in this work $(g_s = 0)$. The point is that the whole condensate is assumed to be in external magnetic field thus the spin degree of freedom are frozen. The scheme to realize persistent current using optical vortices is proposed in Ref. 131 for spinor BEC and in [132] for single-component BEC. Investigations of rotating toroidal trap is of interest for precise rotation measurement using Sagnac effect. Two-component BECs in a 1D ring trap in a rotating frame are considered in Ref. 133. To generate a ring current it was proposed the stirring mechanism considered in 3D geometry [134]. The remarkable manifestation of non-Abelian magnetic field (the generalization of the nonuniform magnetic field characterized by matrix potential A) is presented the Ref. 135, where two-component 1D BEC are investigated in the presence of exotic magnetic field considering the currents and vortex states. One-component BECs in 1D ring potential are considered in Refs. 136,137. In Ref. 138 superfluid 1D ring in the presence of periodic scattering length modulation along the ring is investigated. The two-component BECs in 1D and 2D toroidal traps are investigated in [139–144] A system of two-component 1D BEC in ring potential is considered also in Ref. 145.

Previous experiments on persistent currents in atomic BECs were limited to spinless, single-component condensates. Extending such studies to multicomponent systems, in particular those involving more spin states, is essential for understanding superfluids with a vectorial order parameter and for applications in atom interferometry. In very recent experimental work [124] the stability of supercurrents in a toroidal two-component gas consisting of ^{87}Rb atoms in two different spin states has been studied. As was pointed out in Ref. 124, none of the existing theories is quantitatively applicable to their experiments, since they are limited to the simplified cases of reduced dimensionality and very weak interactions. Let us review the preliminary results of our investigations of superflow in toroidally trapped spinor Bose-Einstein condensate of ^{87}Rb. We have performed a series of computer simulations of the experiments, presented in [124]. As is seen from Fig. 6 our results turn out to be in agreement with the experiments: the two-component vortices with equal number of atoms in each component decay soon (see Fig. 6(b)), while the pure one-component vortex survives for very long time (see Fig. 6(c)). Our results turn out to be in agreement with experiments. However, this issue deserves a more detailed investigation and will be reported elsewhere.

Atoms of ^{87}Rb have spin $F = 1$ and the state of this condensate is ferromagnetic. In experiment $N = 10^5$ atoms of ^{87}Rb were loaded into optical trap (see Fig. 6(a)). Trap was obtained by "sheet" laser beam with a trapping frequency $\omega_z = 350$ Hz and "tube" laser beam of radius $r_{\min} = 12$ μm. Its depth is about $V_0/h = 1.2$ kHz, where h is Planck's constant. As a tube beam authors used Laguerre–Gauss beam, which carries orbital angular momentum $3\hbar$. Hence they obtained vortex of charge $m = 3$ in effectively two-dimensional ring trap. After this there was obtained state with two components with different spin-projections. Authors found that stability of the current depends on ratio between number of particles in different spin states. This ratio can be presented by spin-polarization $P_z = (N_+ - N_-)/(N_+ + N_-)$. The main result that we test is the presence of critical spin-polarization $P_z \approx$ 0.6–0.7 below which supercurrent rapidly decays.

We simulate the trap by sum of harmonic potential, which models sheet beam, and radial Laguerre–Gauss potential:

$$V(\mathbf{r}) = \frac{M\omega_z^2 z^2}{2} - V_0\left(\frac{r}{r_{\min}}\right)^{2m} \exp\left(-m\left(\frac{r^2}{r_{\min}^2} - 1\right)\right), \tag{22}$$

where $r = \sqrt{x^2 + y^2}$, $r_{\min}$ is the radius of the trap (or coordinate of the minimum), $m = 3$ is the vortex charge, V_0 is the trap depth. This useful form of Laguerre–Gauss potential was obtained in the same way as in [146].

In Appendix we present derivation of GP equations for spinor condensate in general form. Only first two equations remains since $\Psi_0 \equiv 0$ in experiment [124]. We can simplify these equations providing dimensionless variables and using symmetry of the system. We look for solutions in form $\Psi_j(\mathbf{r},t) = \tilde{\Psi}_j(x,y,t)\Upsilon(z,t)$, where

$$\Upsilon(z,t) = (l_z\sqrt{\pi})^{-1/2} \exp\left(-\frac{i}{2}\omega_z t - \frac{1}{2}z^2/l_z^2\right)$$

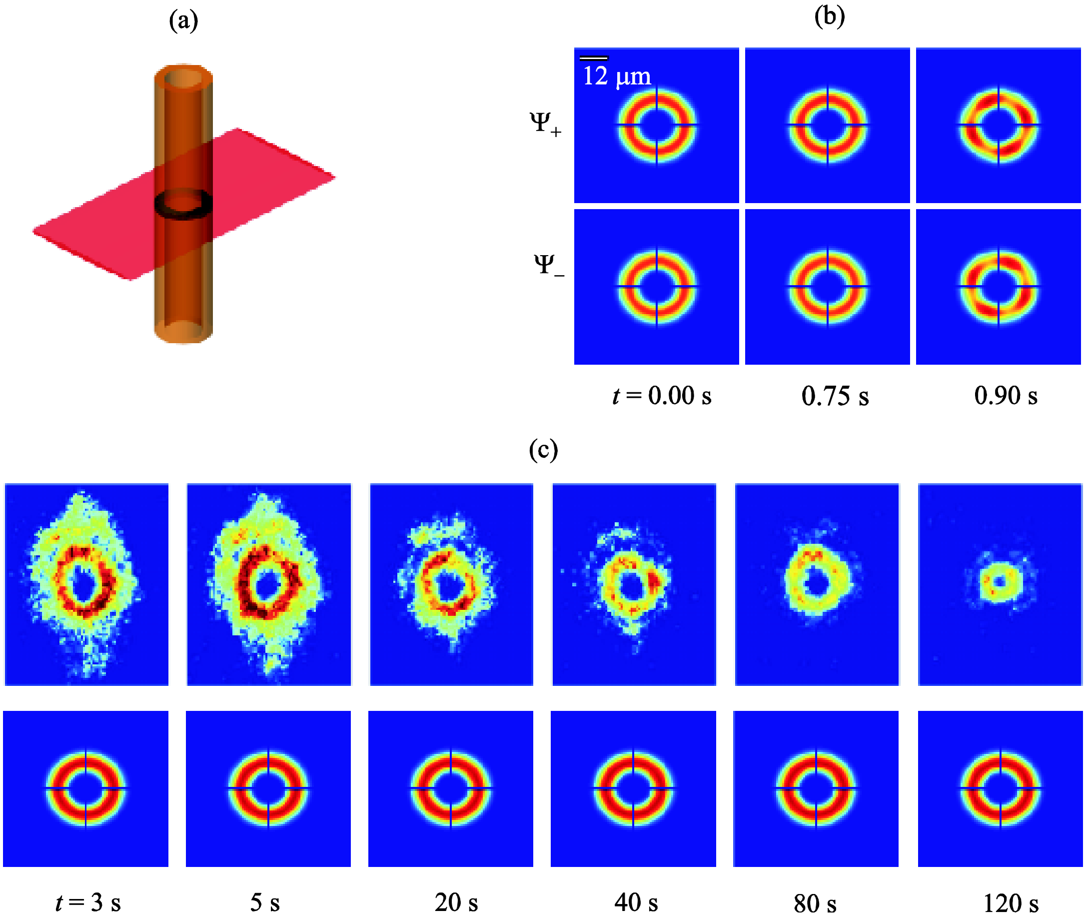

Fig. 6. Schematic illustrations of experimental creation of optical toroidal trap. The trapping potential is created by intersecting a horizontal "sheet" laser beam with a vertical "tube" $LG_{m,0}$ beam with $m = 3$ (a). Snapshots of the density distribution in (x, y) plane: numerical simulation of the unstable evolution of the two-component spinor BEC with equal number of atoms: $N_+ = N_-$ (b); stable single-component superflow with $m = 3$ the upper row is experimental results [124], lower row is numerical simulation (c).

and $l_z = \sqrt{\hbar/(M\omega_z)}$. After integrating out the longitudinal coordinates we obtain two-dimensional system of GPE equations:

$$i\frac{\partial\psi_+}{\partial t} = \{\hat{L} + (\nu_s + \nu_a)|\psi_+|^2 + (\nu_s - \nu_a)|\psi_-|^2\}\psi_+, \quad (23)$$

$$i\frac{\partial\psi_-}{\partial t} = \{\hat{L} + (\nu_s - \nu_a)|\psi_+|^2 + (\nu_s + \nu_a)|\psi_-|^2\}\psi_-, \quad (24)$$

where $\hat{L} = -(\Delta_\perp/2) + V_{\rm ext}(r)$, and dimensionless wave-function and parameters of interaction are $\psi_{\pm,0} = \tilde{\Psi}_{\pm,0}/C$, $C^2 = \sqrt{2\pi}\hbar\omega_\perp l_z/|g_n|$, $\nu_a = g_s/|g_n|$ and $\nu_s = \mathrm{sgn}(g_n)$. Here we provided dimensionless coordinates $r \to r/r_{\min}$, $z \to z/l_z$, and time $t \to \omega_r t$, where $\omega_r = \hbar/(MR^2)$. The external axially-symmetric potential $V(\mathbf{r}) = -V_0 r^{2m}\exp(-m(r^2-1))$ corresponds to Laguerre–Gauss trap.

Dynamical equations (24) have the following integrals of motion: Number of atoms

$$N_\pm = \int_{-\infty}^{+\infty} n_\pm d^2 r,$$

Energy $E = \int_{-\infty}^{+\infty} \mathcal{H} d^2 r$, where

$$\mathcal{H} = \frac{1}{2}|\nabla_\perp\psi_+|^2 + \frac{1}{2}|\nabla_\perp\psi_-|^2 + V_{\rm ext}(r)n +$$

$$+ \frac{v_s}{2}n^2 + \frac{v_a}{2}(n_+^2 + n_-^2 - 2n_+n_-). \quad (25)$$

To define our dimensionless parameters we calculated them using experimental values: effective radial frequency, $\omega_r = \hbar/mR^2 = 4.79$ Hz, depth of the potential, $V_0 \to \to mR^2V_0/\hbar^2 =$ 1574.31, number of particles, $N \to \to mNg_n/\sqrt{2\pi}\hbar^2 l_z =$ 5165.4. To find initial conditions for subsequent dynamics simulation we used imaginary time propagation method. This method allows to find ground states of the system with fixed topology. The imaginary time propagation have been started from the initial guess obtained in Thomas–Fermi approximation.

To illustrate our findings we present two examples of computer simulation of the superflow with parameters fitted to the experimental setup reported in Ref. 124. As is seen from Fig. 6(b) two-component superflow decays fast for the case when number of atoms in each component are equal. However, the single-component persistent current demonstrates a stable evolution up to two minutes as was demonstrated in experiments (compare experimental plots the upper row from Fig. 6(c) and our computer simulations given in the bottom row of Fig. 6(c)).

Quantization of vorticity is a remarkable manifestation of superfluid properties of BECs.

Different kinds of vortex structures have been theoretically predicted and observed experimentally in atomic BECs (see, e.g., [34] and references therein): single vortex lines, vortex-antivortex pairs, vortex arrays, and vortex rings, solitary waves moving along the straight vortex line [147] (which are similar to “hoop” structures known in field theory [148]).

In contrast to vortex line (which only terminate at the superfluid boundary) vortex ring have a closed-loop core. Consequently, a vortex rings in 3D have a lower energy, since their energies do not diverge with system size. Hence, these excitations plays a crucial role in any decay of superflow compared to just a simple vortex line. Several schemes to create a vortex ring in atomic BEC have been proposed: using dynamical instabilities in the condensate to make a dark soliton decay into vortex rings, two-component BECs with different relative velocity, drag of a moving object through the condensate [149], space-dependent Feshbach resonance [150], or by phase imprinting methods [151].

Dynamics of vortex line in 2D trap and vortex ring in spherical trap was addressed in [149]. It was found that a core of the vortex undergoes oscillatory motion around a circle of maximum energy. Due to dissipation the vortex line as well as vortex ring drift to the edge of the condensate and decay eventually. Note that in nonuniform light beam optical vortex also exhibits radial drift and rotation due to background gradients of phase and intensity, respectively [152].

An interesting vortex complex with vortex ring in the first component and vortex line in the second component has been predicted in *two-component* BEC [151]. It usually referred to as a skyrmion. Figure 7 (a) illustrates its structure: the ring vortex core of one component is filled by the superflow of the other. This topological soliton in two-component BEC can be identified as a particle-like skyrmion, closely resembles cosmic vortons. Its topology is defined by the fact that far from the compact skyrmion the density of BEC is not perturbed [153]. Such a structure can be energetically stable even for multiply quantized vortex lines [154]. While for some specific conditions the numerical simulations predict stability of a skyrmion [153], nevertheless it turns out to be rather fragile object. Skyrmion stability appears to be very sensitive to the strength of the intercomponent and intracomponent interactions. Moreover, it requires spatial separation of BEC components, which is not possible in stable regime without additional tuning of the scattering lengths using an optical Feshbach resonance. Thus, experimental observation of the skyrmions remains to be rather challenging issue.

This brings up the following questions: is it possible to stabilize both vortex ring and vortex line in an experimentally available BEC? Probably coexisting vortex line and vortex ring could stabilize each other, and if so, how to

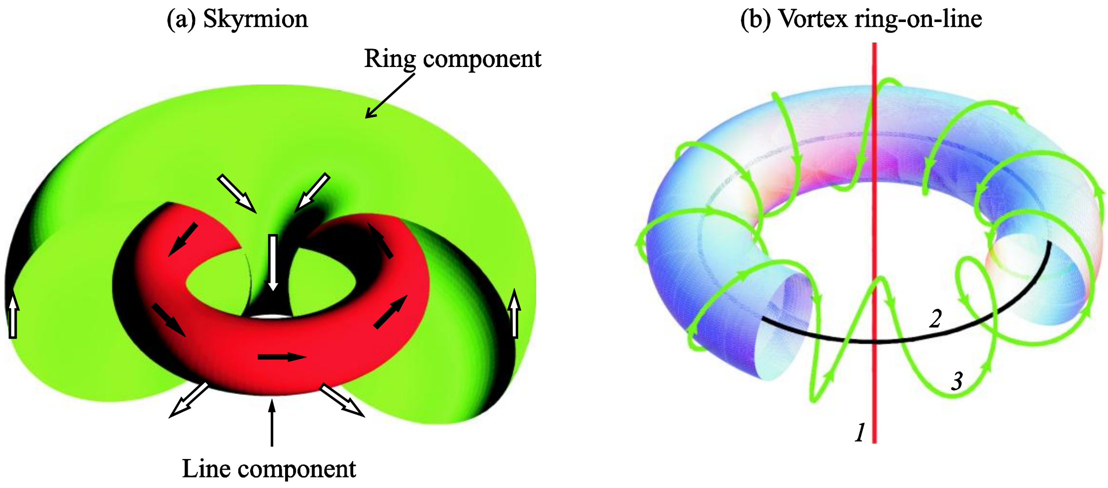

Fig. 7. (Color online) Two-component BEC skyrmion. Schematic illustration of density isosurfaces for the ring component and line component. The circulation directions of the corresponding flows is indicated by arrows (a). Schematic illustration of vortex ring-on-line in toroidal trap. We show a constant surface of the trapping potential. A superflow (green line, *3*) with toroidal and poloidal components creates simultaneously two types of phase singularities: vortex line (red, *1*) surrounded by vortex ring (black, *2*) (b).

create such a vortex complex? It is of interest to find out the conditions for existence of stable vortex ring-on-line (VRL) in toroidal BEC. The VRL appears as the result of simultaneous poloidal and toroidal flows of atoms in a single-component toroidal BEC [see Fig. 7(b)]. It seems reasonable to suggest that a VRL should be completely stable because all precession motions of the both vortex cores are expected to be suppressed. Indeed, the centrifugal barrier caused by toroidal motion around a vortex line prevents a vortex ring from radial shrinking. At the same time, the optical toroidal trap with radial trapping by Laguerre–Gaussian beams not only gives rise to the VRLs by toroidal stirring of the trapped vortex ring, but also it saves the vortex line from a radial drift. The recent experiments [124,129] with persistent currents of BECs in toroidal optical traps demonstrates a stable circulation of the superfluid which corresponds to a multicharge vortex line. The lifetime of the superflow in toroidal trap reached few minutes and it was restricted only by decay of the BEC itself. The life-time of the VRLs is expected to be limited only by dissipative effects. These issues deserves further studies, and this work now in progress and the results will be published elsewhere.

Appendix

Let us consider many-body system of particles with hyperfine spin $F = 1$.

Existence of three components with different spin states occurs to complicating of interaction potential (compared to simple BEC) [33]. The interaction potential for two atoms with $F = 1$ may be written in the form

$$V_{\text{int}}(\mathbf{r}_1 - \mathbf{r}_2) = \delta(\mathbf{r}_1 - \mathbf{r}_2)\frac{4\pi\hbar^2}{m}\sum_{f=0,2} a_f \hat{P}_f , \qquad \text{(A.1)}$$

where a_f is the scattering length and $\hat{P}_f$ is the projection operator on state with total spin equal to *f*. Note, that for identical bosonic atoms with spin $F = 1$ in a state of relative motion only states with total angular momentum $f = 0$ or $f = 2$ can couple together, since the possibility to have unit angular momentum is ruled out by the requirement that wave function has to be symmetric under exchange of two bosons. The interaction via low-energy collision is invariant under rotations, and therefore it is diagonal in the total angular momentum of the two atoms.

We could rewrite this expression as linear combination of identity operator and multiplication of spin operators:

$$V_{\text{int}}(\mathbf{r}_1 - \mathbf{r}_2) = \delta(\mathbf{r}_1 - \mathbf{r}_2)(g_n + g_s \hat{F}_1 \cdot \hat{F}_2). \qquad \text{(A.2)}$$

where we introduce the analogue of Pauli-matrices for spin-1 bosons:

$$\hat{F}_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \hat{F}_y = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}; \hat{F}_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \qquad \text{(A.3)}$$

Let us rewrite the projection operators on spin states with $f = 0$ and $f = 2$ as follows:

$$\hat{P}_0 = \frac{1}{3}(1 - \mathbf{F}_1 \cdot \mathbf{F}_2) , \qquad \text{(A.4)}$$

$$\hat{P}_2 = \frac{1}{3}(2 + \mathbf{F}_1 \cdot \mathbf{F}_2) . \qquad \text{(A.5)}$$

Indeed $\mathbf{F}^2 = (\mathbf{F}_1 + \mathbf{F}_2)^2$, thus $f(f+1) = F_1(F_1+1) + F_2(F_2+1) + 2\mathbf{F}_1 \cdot \mathbf{F}_2$. That is why the eigenvalue γ_f of the scalar product $(\mathbf{F}_1 \cdot \mathbf{F}_2)|f\rangle = \gamma_f |f\rangle$ is determined by the total spin f as follows: $\gamma_0 = -2$, $\gamma_2 = 1$. We obtain the coupling constants as follows:

$$g_n = \frac{4\pi\hbar^2}{m}\frac{a_0 + 2a_2}{3}, \tag{A.6}$$

$$g_s = \frac{4\pi\hbar^2}{m}\frac{a_2 - a_0}{3}. \tag{A.7}$$

Coupling constants describe interaction between densities of components and spin interaction, respectively. Hamiltonian of such system may be written as follows [155]:

$$\hat{H} = \hat{H}^{(1)} + \hat{H}_n^{(2)} + \hat{H}_s^{(2)}, \tag{A.8}$$

where

$$\hat{H}^{(1)} = \int d\mathbf{r}\left\{\sum_i \hat{\Psi}_i^\dagger(\mathbf{r})\hat{h}_i\hat{\Psi}_i(\mathbf{r})\right\} \tag{A.9}$$

$$\hat{H}_n^{(2)} = \frac{g_n}{2}\int d\mathbf{r}\left\{\sum_{i,j}\hat{\Psi}_i^\dagger(\mathbf{r})\hat{\Psi}_j^\dagger(\mathbf{r})\hat{\Psi}_j(\mathbf{r})\hat{\Psi}_i(\mathbf{r})\right\}. \tag{A.10}$$

$$\hat{H}_s^{(2)} = \frac{g_s}{2}\int d\mathbf{r}\times$$

$$\times\left\{\sum_\alpha\sum_{i,j,k,l}\hat{\Psi}_i^\dagger(\mathbf{r})\hat{\Psi}_j^\dagger(\mathbf{r})(\hat{F}_\alpha)_{i,k}(\hat{F}_\alpha)_{j,l}\hat{\Psi}_j(\mathbf{r})\hat{\Psi}_l(\mathbf{r})\right\}, \tag{A.11}$$

where

$$\hat{h}_i = -\frac{\hbar^2\nabla^2}{2m} + V(\mathbf{r}) \tag{A.12}$$

is one-particle Hamiltonian, $V(\mathbf{r})$ is trapping potential and $i, j = 0, \pm 1$ are spin indices. As may be seen, Hamiltonian consist of three components: one-body interaction part (Eq. (A.9)), two-body part, which does not depend on spin and describes interaction of densities (Eq. (A.10)), and two-body part, which characterize spin interaction (Eq. (A.11)).

We can obtain equations of motion using Heisenberg equation $i\hbar\frac{\partial}{\partial t}\hat{\Psi}_i = [\hat{H}, \hat{\Psi}_i]$ and mean-field approximation:

$$i\hbar\frac{\partial}{\partial t}\hat{\Psi}_\pm = \left\{-\frac{\hbar^2}{2m}\Delta + V_{\rm ext}(\mathbf{r}) + g_n n + g_s(n - 2n_\mp)\right\}\Psi_\pm + g_s n_0 \Psi_\mp^*. \tag{A.13}$$

$$i\hbar\frac{\partial}{\partial t}\hat{\Psi}_0 = \left\{-\frac{\hbar^2}{2m}\Delta + V_{\rm ext}(\mathbf{r}) + g_n n + g_s(n - n_0)\right\}\Psi_0 + 2g_s\Psi_+\Psi_-\Psi_0^*, \tag{A.14}$$

where $n = n_+ + n_0 + n_-$, $n_j = |\psi_j|^2$.

The spin-1 BEC in the absence of an external magnetic field has two phases: ferromagnetic (^{87}Rb, $g_s/g_n = -4.66\cdot 10^{-3}$) and polar (^{23}Na, $g_s/g_n = +3.14\cdot 10^{-2}$).

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