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On the Growth of the Cauchy–Szegő Transform in the Unit Ball

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The growth of analytic and harmonic functions in the unit ball B_n represented by the Cauchy–Stieltjes or Poisson–Stieltjes integral is studied. A description of the growth is given in terms of smoothness of the Stieltjes measure.

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1. Introduction

It is reasonable to expect for a function analytic in a bounded domain and continuous up to the boundary to be smooth on the boundary if its derivative grows slowly, and conversely. For the unit disk this was established by Hardy–Littlewood ([8], [6, Chap. 5]).

We say that a complex-valued function $f(e^{i\theta})$, $\theta \in \mathbb{R}$ is of the class Λ^*_α $(0 < \alpha \le 1)$ if $\omega^*(t) = O(t^\alpha)$ as $t \to 0$ where $\omega^*(t)$ is the modulus of continuity of $f(e^{i\theta})$, i.e.,

$$\omega^*(t) = \sup_{|e^{i\theta_1} - e^{i\theta_2}| \le t} |f(e^{i\theta_1}) - f(e^{i\theta_2})|.$$

Theorem A ([6, Theorem 5.1]). *Let $f(z)$ be a function analytic in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Then $f(z)$ is continuous in $\overline{\mathbb{D}}$ and $f(e^{i\theta}) \in \Lambda^*_\alpha$ $(0 < \alpha \le 1)$ if and only if*

$$f'(z) = O\left(\frac{1}{(1 - |z|)^{1-\alpha}}\right).$$

Let $z, w \in \mathbb{C}^n$, $n \in \mathbb{N}$, $\langle z, w\rangle = \sum_{i=1}^n z_i \overline{w_i}$, $|z| = \langle z, z\rangle^{\frac{1}{2}}$. We denote by $B_n = \{z \in \mathbb{C}^n : |z| < 1\}$ the unit ball in $\mathbb{C}^n$ and by $S_n = \{z \in \mathbb{C}^n : |z| = 1\}$ the unit sphere.

For a complex-valued function f on S_n and a Borel measure μ on S_n, we denote the *Cauchy integral*

$$C[f](z) = \int_{S_n} \frac{f(\xi) dm_{2n-1}(\xi)}{(1 - \langle z, \xi\rangle)^n}, \ z \in B_n,$$

where m_{2n-1} is the normalized Lebesgue measure on S_n, $m_{2n-1}(S_n) = 1$, and

$$C[\mu](z) = \int_{S_n} \frac{d\mu(\xi)}{(1 - \langle z, \xi\rangle)^n}, \ z \in B_n \tag{1}$$

the *Cauchy–Stieltjes integral*. Similarly, we denote by

$$P[f](z) = \int_{S_n} \frac{(1 - |z|^2)^n}{|1 - \langle z, \xi\rangle|^{2n}} f(\xi) dm_{2n-1}(\xi), \ z \in B_n,$$

$$P[\mu](z) = \int_{S_n} \frac{(1 - |z|^2)^n}{|1 - \langle z, \xi\rangle|^{2n}} d\mu(\xi), \ z \in B_n$$

the *Poisson* and *Poisson–Stieltjes integrals*, respectively.

Let f be a holomorphic function in B_n and $f = \sum_{k=0}^{\infty} F_k$ be the homogeneous decomposition of f, then $(\mathcal{R}f)(z) = \sum_{k=0}^{\infty} kF_k(z)$, $z \in B_n$ is the radial derivative. The following theorems were proved by W. Rudin for several complex variables.

Theorem B ([10]). *Let $0 < \alpha < 1$ and f be a measurable complex function such that $|f|$ is integrable with respect to the measure m_{2n-1} on S_n. Then $|f(e^{i\theta}\xi) - f(e^{it}\xi)| \le |e^{i\theta} - e^{it}|^{\alpha}$, $\xi \in S_n, \theta, t \in \mathbb{R}$, implies that*

$$|(\mathcal{R}C[f])(z)| \le A_{\alpha}(1 - |z|)^{\alpha-1}, \ z \in B_n.$$

Theorem C ([10]). *Let $0 < \alpha < 1$ and f be holomorphic in B_n. Then*

$$|(\mathcal{R}C[f])(z)| \le (1 - |z|)^{\alpha-1}, \ z \in B_n,$$

implies that f has a continuous extension to $\overline{B}_n$ which satisfies the Lipschitz condition of order α.

Some results in this direction that concern the unit polydisk can be found in [4, 5, 7]. In particular, necessary and sufficient conditions of the growth of Poisson–Stieltjes integral in terms of Stieltjes measure were described in [3, 4]. Some properties of harmonic functions of Lipschitz type spaces and their generalizations are described in [1, 2]. In particular, a multidimensional counterpart of Theorem A for harmonic functions in B_n was proved by S. Krantz in [9]. Note that, in general, the functions represented by the Poisson–Stieltjes integral or the Cauchy–Stieltjes integral can not be represented by the Poisson integral or the Cauchy integral, respectively.

We are interested in the description of the growth of analytic and harmonic functions in the unit ball B_n represented by the Cauchy–Stieltjes or the Poisson–Stieltjes integral. The case of differentiable measures (with respect to m_{2n-1}) is well known (see, e.g., [10, Chap. 3] and [11, Chap. 7]). We find sharp estimates for the growth of the Cauchy integral in the unit ball in $\mathbb{C}^n$ in terms of smoothness of the Stieltjes measure.

Denote by

$$d(z,\zeta) = \sqrt{|1-\langle z,\zeta\rangle|},\ z,\zeta \in \overline{B}_n,$$

the anisotropic metric on S_n ([10, Sec. 5.1]) and by

$$\omega(\delta,\mu) = \sup_{z_0\in S_n} |\mu|(\{\xi \in S_n : d(\xi,z_0) \le \delta\})$$

the modulus of continuity, where $|\mu|$ is the total variation of a complex-valued Borel measure μ on S_n.

Theorem 1. *Let μ be a complex-valued Borel measure on S_n, $p \in (0,n]$. Then*

$$\exists c > 0\ \omega(\delta,\mu) \le c\delta^{2(n-p)}, \quad 0 < \delta \le \sqrt{2},$$

implies that

$$C[\mu](z) = O\left(\frac{1}{(1-|z|)^p}\right), \quad z \in B_n.$$

The examples in Sec. 3 show that the estimate is sharp up to a constant factor. In order to prove Theorem 1, we use the standard approach [10]. The same method allows us to prove a criterion for the Poisson integral.

Theorem 2. *Let μ be a positive Borel measure on S_n, $p \in (0,n)$. Then*

$$\exists c > 0\ \omega(\delta,\mu) \le c\delta^{2(n-p)}, 0 < \delta < 1 \Leftrightarrow P[\mu](z) = O\left(\frac{1}{(1-|z|)^p}\right), z \in B_n.$$

R e m a r k. Theorems 1 and 2 could be easily generalized for the integrals with kernels of the form

$$\frac{1}{(1-\langle z,\xi\rangle)^{n+s}},\ \frac{1}{|1-\langle z,\xi\rangle|^{n+s}},\ s\in\mathbb{R},$$

respectively, for an appropriate choice of p.

2. Proofs of the Theorems

P r o o f of Theorem 1. Denote

$$E_k(z)=\Big\{\xi\in S_n:\Big|1-\langle\frac{z}{|z|},\xi\rangle\Big|<2^{k+1}(1-|z|)\Big\}$$

for $z\in B_n\setminus\{0\}$ and $k\in\{0,1,2,\ldots\}$. Then ($E_{-1}(z):=\varnothing$)

$$\bigcup_{k=0}^{\infty}(E_k(z)\setminus E_{k-1}(z))=S_n.$$

Since $\forall k\in\mathbb{N}\ \forall\xi\in E_k(z)\setminus E_{k-1}(z)\ \forall z:\ 1>|z|>\frac{3}{4}$

$$\begin{aligned}|1-\langle z,\xi\rangle|&\geq||1-|z||-||z|-\langle z,\xi\rangle||=|z|\Big|1-\Big\langle\frac{z}{|z|},\xi\Big\rangle\Big|-(1-|z|)\\&\geq|z|2^k(1-|z|)-(1-|z|)=(|z|2^k-1)(1-|z|)\end{aligned}$$

and $\forall\xi\in E_0(z)$: $|1-\langle z,\xi\rangle|\geq 1-|\langle z,\xi\rangle|\geq 1-|z|$, we have

$$\begin{aligned}|C[\mu](z)|&=\Big|\int_{S_n}\frac{d\mu(\xi)}{(1-\langle z,\xi\rangle)^n}\Big|\\
&=\Big|\sum_{k=1}^{\infty}\int_{E_k(z)\setminus E_{k-1}(z)}\frac{d\mu(\xi)}{(1-\langle z,\xi\rangle)^n}+\int_{E_0(z)}\frac{d\mu(\xi)}{(1-\langle z,\xi\rangle)^n}\Big|\\
&\leq\sum_{k=1}^{\infty}\int_{E_k(z)\setminus E_{k-1}(z)}\frac{|d\mu(\xi)|}{(|z|2^k-1)^n(1-|z|)^n}+\int_{E_0(z)}\frac{|d\mu(\xi)|}{(1-|z|)^n}\\
&\leq(1-|z|)^{-n}\sum_{k=1}^{\infty}(|z|2^k-1)^{-n}|\mu|(E_k(z))+(1-|z|)^{-n}|\mu|(E_0(z))\\
&\leq(1-|z|)^{-n}\sum_{k=1}^{\infty}\frac{\omega(\sqrt{2^{k+1}(1-|z|)},\mu)}{(|z|2^k-1)^n}+(1-|z|)^{-n}\omega(\sqrt{2(1-|z|)},\mu)\end{aligned}$$

$$\leq (1-|z|)^{-n}\sum_{k=1}^{\infty}\left(\frac{3}{4}2^k-1\right)^{-n}c(2^{k+1}(1-|z|))^{n-p}+2^{n-p}c(1-|z|)^{-p}$$
$$<\frac{c}{(1-|z|)^p}\left(2^{n-p}\sum_{k=1}^{\infty}\frac{2^{k(n-p)}}{(3\cdot 2^{k-2}-1)^n}+2^{n-p}\right),\ \frac{3}{4}<|z|<1.$$

Since the last series is convergent, we get the desired result.
Now, let $|z|\leq\frac{3}{4}$. Since $d(1,z)\leq\sqrt{2}$ for all $z\in B_n$,

$$|\mu|(S_n)\leq\omega(\sqrt{2},\mu)\leq c2^{n-p}.$$

Then

$$|C[\mu](z)|\leq\int\limits_{S_n}\left|\frac{d\mu(\xi)}{(1-\langle z,\xi\rangle)^n}\right|\leq\frac{|\mu|(S_n)}{(1-|z|)^n}$$
$$\leq\frac{c2^{n-p}}{(1-|z|)^p\left(\frac{1}{4}\right)^{n-p}}\leq\frac{c8^{n-p}}{(1-|z|)^p}.$$

P r o o f of Theorem 2.($\Leftarrow$) For all $\xi\in E_1(z)$,

$$|1-\langle z,\xi\rangle|\leq\left|1-\left\langle\frac{z}{|z|},\xi\right\rangle\right|+\left|\left\langle\frac{z}{|z|}-z,\xi\right\rangle\right|\leq 4(1-|z|)+\left|\frac{z}{|z|}-z\right|=5(1-|z|).$$

By the assumption $\exists c>0$ such that

$$\frac{c}{(1-|z|)^p}\geq\left|\int\limits_{S_n}\frac{(1-|z|^2)^n}{|1-\langle z,\xi\rangle|^{2n}}d\mu(\xi)\right|$$
$$\geq\left|\int\limits_{E_1(z)}\frac{(1-|z|^2)^n}{|1-\langle z,\xi\rangle|^{2n}}d\mu(\xi)\right|\geq\frac{(1+|z|)^n}{5^{2n}(1-|z|)^n}\mu(E_1(z)).\qquad(2)$$

Since for $d(z,\xi)<\sqrt{3(1-|z|)}$ implies

$$\left|1-\left\langle\frac{z}{|z|},\xi\right\rangle\right|\leq|1-\langle z,\xi\rangle|+\left|\langle z,\xi\rangle-\left\langle\frac{z}{|z|},\xi\right\rangle\right|$$
$$<3(1-|z|)+\left|z-\frac{z}{|z|}\right|=4(1-|z|),$$

we get $E_1(z)\supset\{\xi\in S_n: d(z,\xi)<\sqrt{3(1-|z|)}\}$.

From inequality (2) and the last inclusion it follows that

$$\mu(E_1(z)) \le c_1(1-|z|)^{n-p}, \quad z \in B_n,$$
$$\omega(\sqrt{3(1-|z|)}, \mu) \le c_1(1-|z|)^{n-p}, \quad z \in B_n,$$
$$\omega(\delta, \mu) \le c_1 \delta^{2(n-p)} 3^{p-n}, \quad 0 < \delta \le \sqrt{3},$$

where $c_1 \ge 5^{2n} c/2^n$.

($\Rightarrow$) Using the arguments similar to those of the proof of Theorem 1, we get

$$|P[\mu](z)| = \int\limits_{S_n} \frac{(1-|z|^2)^n d\mu(\xi)}{|1-\langle z,\xi\rangle|^{2n}}$$
$$= \sum_{k=1}^{\infty} \int\limits_{E_k(z)\setminus E_{k-1}(z)} \frac{(1-|z|^2)^n d\mu(\xi)}{|1-\langle z,\xi\rangle|^{2n}} + \int\limits_{E_0(z)} \frac{(1-|z|^2)^n d\mu(\xi)}{|1-\langle z,\xi\rangle|^{2n}}$$
$$\le \sum_{k=1}^{\infty} \int\limits_{E_k(z)\setminus E_{k-1}(z)} \frac{(1-|z|^2)^n d\mu(\xi)}{(|z|2^k-1)^{2n}(1-|z|)^{2n}} + \int\limits_{E_0(z)} \frac{(1-|z|^2)^n d\mu(\xi)}{(1-|z|)^n}$$
$$\le (1-|z|)^{-n} \sum_{k=1}^{\infty} \frac{(1+|z|)^n \mu(E_k(z))}{(|z|2^k-1)^{2n}} + (1-|z|)^{-n}(1+|z|)^n \mu(E_0(z))$$
$$\le (1-|z|)^{-n} \sum_{k=1}^{\infty} \frac{2^n \omega(\sqrt{2^{k+1}(1-|z|)}, \mu)}{(|z|2^k-1)^{2n}} + (1-|z|)^{-n} 2^n \omega(\sqrt{2(1-|z|)}, \mu)$$
$$\le (1-|z|)^{-n} \sum_{k=1}^{\infty} \left(\frac{3}{4}2^k - 1\right)^{-2n} 2^n c(2^{k+1}(1-|z|))^{n-p} + 2^{n-p} 2^n c(1-|z|)^{-p}$$
$$\le \frac{c}{(1-|z|)^p} \left(2^{2n-p} \sum_{k=1}^{\infty} \frac{2^{k(n-p)}}{(3\cdot 2^{k-2}-1)^{2n}} + 2^{2n-p} \right).$$

The convergence of the last series implies the required inequality. If $|z| \le \frac{3}{4}$, using the arguments similar to those of the proof of Theorem 1, we can obtain

$$|P[\mu](z)| \le \int\limits_{S_n} \left| \frac{(1-|z|^2)^n d\mu(\xi)}{|1-\langle z,\xi\rangle|^{2n}} \right| \le \frac{2^n |\mu|(S_n)}{(1-|z|)^n}$$
$$\le \frac{c 2^{n-p} 2^n}{(1-|z|)^p \left(\frac{1}{4}\right)^{n-p}} \le \frac{c 2^{4n-3p}}{(1-|z|)^p}.$$

3. Examples

1. Let in Theorem 1 μ be the Lebesgue measure m_{2n-1} on S_n. Note that $Q_\delta = \{\xi \in S_n : d(\xi, z_0) < \delta\}$ is a "ball" on S_n and $m_{2n-1}(Q_\delta) \asymp \delta^{2n}$, $\delta \to 0$

([10, Ch. 5]). Since "the center" z_0 is of no particular importance, the modulus of continuity

$$\omega(\delta, m_{2n-1}) = \sup_{z_0\in S_n} m_{2n-1}(\{\xi \in S_n : d(\xi, z_0) < \delta\}) \asymp \delta^{2n},\ \delta \to 0,$$

so ([10, Prop. 1.4.10])

$$\int\limits_{S_n} \frac{dm_{2n-1}(\xi)}{|1-\langle z,\xi\rangle|^n} \asymp \ln\frac{1}{1-|z|},\ |z| \uparrow 1.$$

Hence the statement of Theorem 1 is not true for the case $p = 0$.

2. Let $\mu = \delta_{\xi_0}$, i.e.,

$$\mu(A) = \begin{cases} c, & A \ni \xi_0; \\ 0, & \text{otherwise,} \end{cases}$$

where $A \subset S_n,\ \xi_0 \in S_n$. Then

$$\omega(\delta, \mu) = c \in \mathbb{C},\ p = n$$

and

$$C[\mu](t\xi_0) = \int\limits_{S_n} \frac{d\mu(\xi)}{(1-\langle t\xi_0,\xi\rangle)^n} = c\frac{1}{(1-t)^n},\ 0 < t < 1.$$

This example shows the sharpness of Theorem 1 for $p = n$.

3. Let μ be a Borel measure on $S_2 \subset \mathbb{C}^2$ and

$$\mu(\xi) = \begin{cases} k^{-l}, & \xi = (1-k^{-q}, \sqrt{1-(1-k^{-q})^2}) \\ 0, & \text{otherwise,} \end{cases}$$

where $1 < l < 2q+1,\ k = 0, 1, \dots.$

Then $\omega(\delta,\mu) \asymp \delta^{2\frac{l-1}{q}}$, i.e., $\mu \in \Lambda_{\frac{l-1}{q}}$ and $\max\limits_{|z|=r} |C[\mu](z)| \geq c\left(\frac{1}{(1-r)^{2-\frac{l-1}{q}}}\right)$.

Indeed,

$$\omega(\delta,\mu) = \int\limits_{|1-\xi_1|<\delta^2} d\mu(\xi_1,\xi_2) = \sum_{k^{-q}<\delta^2} k^{-l} = \sum_{k>\delta^{-\frac{2}{q}}} k^{-l} \asymp \delta^{\frac{2}{q}(l-1)}.$$

Let $z = re_1$, where $e_1 = (1, 0) \in S_2,\ r \in (0, 1)$,

$$C[\mu](re_1) = \int\limits_{S_2} \frac{d\mu(\xi)}{(1-r\xi_1)^2} = \sum_{k=1}^{\infty} \frac{k^{-l}}{(1-r(1-k^{-q}))^2}$$

$$\geq \sum_{k=\left[\frac{1}{(1-r)^{1/q}}\right]}^{2\left[\frac{1}{(1-r)^{1/q}}\right]} \frac{1}{k^l(1-r+\frac{r}{k^q})^2} \geq \sum_{k=\left[\frac{1}{(1-r)^{1/q}}\right]}^{2\left[\frac{1}{(1-r)^{1/q}}\right]} \frac{(1-r)^{\frac{l}{q}}}{2^l} \frac{1}{((1-r)+r(1-r))^2}$$

$$\geq \frac{1}{2(1-r)^{\frac{1}{q}}} \frac{(1-r)^{\frac{l}{q}}}{2^l} \frac{1}{4(1-r)^2} = \frac{1}{2^{l+3}}(1-r)^{\frac{l-1}{q}-2}.$$

4. The Operator Theory Point of View

Equality (1) is often considered as the Cauchy–Szegő operator acting from the space $M(S_n)$ ($M^+(S_n)$) of Borel (positive) measures on S_n into the class of analytic functions on B_n.

Denote by $H^p_q(B_n)$, $1 \leq p \leq \infty$, $q \geq 0$, the class of analytic functions f on B_n with the norm

$$\|f\|^p_q = \sup_{0<r<1} (1-r)^q \left(\int\limits_{S_n} \|f(r\xi)\|^p dm_{2n-1}(\xi) \right)^{1/p},$$

$$\|f\|^\infty_q = \sup_{0<r<1} (1-r)^q \max_{|\xi|=r} |f(\xi)|.$$

Also denote by h^p_q the class of harmonic functions with the same norm. It is known that there exist the measures $\mu \in M(S_n)$ such that $C[\mu] \notin H^1_0(B_n)$. If we denote by $\Lambda_\alpha(S_n)$ ($\Lambda^+_\alpha(S_n)$) the class of (positive) measures on S_n such that $\|\mu\|_\alpha = \sup\limits_{0<\delta\leq\sqrt{2}} \frac{\omega(\delta,\mu)}{\delta^{2\alpha}} < +\infty$, it then follows from Theorems 1 and 2 that $C[\mu]$ and $P[\mu]$ are bounded operators from $\Lambda_\alpha(S_n) \subset M(S_n)$ and $\Lambda^+_\alpha(S_n) \subset M^+(S_n)$ into $H^\infty_{n-\alpha}(B_n)$ and $h^\infty_{n-\alpha}(B_n)$, respectively. Moreover,

$$\|C\| = \sup_{|\mu|\neq 0} \frac{\|C\mu\|^\infty_{n-\alpha}}{\|\mu\|_\alpha} \leq \max\left\{2^{n-p} \sum_{k=1}^{\infty} \frac{2^{k(n-p)}}{(3\cdot 2^{k-2}-1)^n} + 2^{n-p}, 8^{n-p}\right\}$$

$$\leq \max\left\{2^{n-p} \sum_{k=1}^{\infty} \frac{2^{k(n-p)}}{2^{(k-2)n}} + 2^{n-p}, 8^{n-p}\right\}$$

$$\leq \max\left\{\frac{2^{3n}}{2^p-1} + 2^{n-p}, 8^{n-p}\right\} = \frac{2^{3n}}{2^p-1} + 2^{n-p}$$

and

$$\|P\| = \sup_{|\mu|\neq 0} \frac{\|P\mu\|_{n-\alpha}^{\infty}}{\|\mu\|_{\alpha}} \le \max\left\{2^{2n-p}\sum_{k=1}^{\infty}\frac{2^{k(n-p)}}{(3\cdot 2^{k-2}-1)^{2n}} + 2^{2n-p}, 2^{4n-3p}\right\}$$
$$\le \max\left\{\frac{2^{7n}}{2^{n+p}-1} + 2^{2n-p}, 2^{4n-3p}\right\} = \frac{2^{7n}}{2^{n+p}-1} + 2^{2n-p}.$$

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