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Various Types of Convergence of Sequences of Subharmonic Functions

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Let $\upsilon_n(x)$ be a sequence of subharmonic functions in a domain $G \subset \mathbb{R}^m$. The conditions under which the convergence of $\upsilon_n(x)$, as a sequence of generalized functions, implies its convergence in the Lebesgue spaces $L_p(\gamma)$ are studied. The results similar to ours have been obtained earlier by Hörmander and also by Ghisin and Chouigui. Hörmander investigated the case where the measure γ is some restriction of the m-dimensional Lebesgue measure. Grishin and Chouigui considered the case $m = 2$. In this paper we consider the case $m > 2$ and general measures γ.

Key words: subharmonic function, Radon measure.

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1. Introduction

We study the connection between the convergence of sequences of subharmonic functions in the sense of the theory of generalized functions and other types of convergence. There is no need to justify the relevance of the paper due to its classical problematics.

The descriptions similar to the one below were obtained by Hörmander ([1, Proposition 16.1.2]) and by Grishin and Chouigui ([2, Theorem 11]). In this paper, the cases not studied earlier are considered.

The convergence of a sequence υ_n to υ in the space $L_p(\gamma)$ means that

$$\int |\upsilon_n(x) - \upsilon(x)|^p d\gamma(x) \to 0 \quad (n \to \infty).$$

In the present paper, we find the sufficient conditions on a measure γ with the support compactly embedded in G which guarantee the convergence of sequences of subharmonic functions in the space $L_p(\gamma)$ provided that they converge in the sense of the theory of generalized functions. It turns out that in assertions of this type the condition that υ_n are subharmonic functions can be replaced by the condition that υ_n are δ-subharmonic functions if in addition the sequence μ_n of the Riesz measures of υ_n is required to be weakly bounded.

We have already mentioned several well-known results obtained on the subject. The two-dimensional case is considered in [2], here we consider the case where the dimension of the space $m \geq 3$. In the theory of subharmonic functions in $\mathbb{R}^m$, an important role is played by the kernel

$$h_m(x-y) = \|x-y\|^{2-m}.$$

In this paper, we consider this kernel as a map from the space $\mathbb{R}^m$ into the space $L_p(\mathbb{R}^m, d\gamma(x))$, where γ is a positive measure in $\mathbb{R}^m$. In this case, we can write

$$h_m(x-y) : \mathbb{R}^m_y \to L_p(\gamma).$$

We now briefly describe the content of the paper. It consists of three sections. Section 1 is Introduction. Section 2 contains preliminary results. Here we give the statements of those well-known results that are used in the subsequent proofs of theorems. The main results of the paper are contained in Section 3. We consider the sequences υ_n of subharmonic and δ-subharmonic functions. We obtain the conditions under which the convergence of a sequence υ_n, as a sequence of generalized functions, implies its convergence in the spaces with integral metric over a measure γ.

2. Preliminary Results

Recall that Schwartz's generalized function is a linear continuous functional over the space of test functions $\mathcal{D}(G)$ ($\mathcal{D}$, if $G = \mathbb{R}^m$) consisting of compactly supported infinitely differentiable functions in G with the standard definition of convergence. The space of generalized functions is denoted by $\mathcal{D}'(G)$. We denote by Φ continuous functions with compact support in $\mathbb{R}^m$. In the space Φ, the notion of convergence is introduced as follows. The sequence of the functions φ_n converges to the function φ in the space Φ if there exists a compactum containing the supports of all functions φ_n such that φ_n converges uniformly to φ in the space $\mathbb{R}^m$.

We define a Radon measure as the difference of two locally finite Borel measures $\mu = \mu_1 - \mu_2$. For the Radon measure μ, the domain of definition consists of all Borel sets $E \subset \mathbb{R}^m$ except those E for which the equalities

$$\mu_1(E) = \mu_2(E) = +\infty$$

hold. Thus, in general a Radon measure is not a Borel measure. Its domain of definition does not contain a wide class of Borel sets. It is important for us that the domain contains all Borel sets with compact closure.

We consider three types of convergence in the set of Radon measures. Suppose that μ_n is a sequence of Radon measures in the space $\mathbb{R}^m$ which, as a sequence of generalized functions, converges to a generalized function μ, that is, $(\mu_n, \varphi) \to (\mu, \varphi)$ for any function $\varphi \in \mathcal{D}$. We denote this type of convergence by the formula

$$\mu = \mathcal{T}_1 \lim_{n\to\infty} \mu_n.$$

If there exists a Radon measure μ such that the relation $(\mu_n, \varphi) \to (\mu, \varphi)$ holds for any function $\varphi \in \Phi$, then we say that the sequence μ_n converges widely to μ, and in this case we can write

$$\mu = \mathcal{T} \lim_{n\to\infty} \mu_n.$$

Let K be a compactum. If there exists a measure μ on K such that the relation $(\mu_n, \varphi) \to (\mu, \varphi)$ holds for any function $\varphi \in C(K)$, then we say that the sequence μ_n converges weakly to μ and we can write

$$\mu = \mathrm{w} \lim_{n\to\infty} \mu_n.$$

When we speak about the w-convergence, it is assumed that the compactum K is uniquely determined by the context. Note that the w-convergence coincides with the weak convergence in the space $C^*(K)$.

Let us recall some definitions and results from the theory of integrals and measures.

A subset $\mathfrak{M}_1$ of Radon measures is said to be weakly bounded if the set $A(\varphi) = \{|(\mu, \varphi)| : \mu \in \mathfrak{M}_1\}$ is finite for any function $\varphi \in \Phi$.

A subset $\mathfrak{M}_1$ of Radon measures is said to be $\mathcal{T}$-compact if any sequence μ_n in $\mathfrak{M}_1$ contains a widely convergent subsequence.

Let μ be a positive measure, and E be a Borel set. The set E is said to be Jordan measurable with respect to the measure μ if $\mu(\partial E) = 0$.

The following theorems hold (see ([3], Introduction); ([4], Proposition and its Corollary 15 in 1 of Ch. 3 therein)).

Theorem 1. *A set $\mathfrak{M}_1$ is weakly bounded if and only if the set $B(K) = \{|\mu|(K) : \mu \in \mathfrak{M}_1\}$ is finite for any compactum $K \subset \mathbb{R}^m$.*

Theorem 2. *A set $\mathfrak{M}_1$ is $\mathcal{T}$-compact if and only if it is weakly bounded.*

We study the connection between the various types of convergence of sequences of Radon measures. It is obvious that the $\mathcal{T}$-convergence implies the $\mathcal{T}_1$-convergence. The converse assertion is false. Indeed, let $\mu_n = \sqrt{n}(\delta_{1/n} - \delta_0)$, where δ_{x_0} is the Dirac measure on the real axis concentrated at the point x_0. We have $\mathcal{T}_1 \lim_{n\to\infty} \mu_n = 0$. Nonetheless, the sequence μ_n is $\mathcal{T}$-divergent. However, the following assertion holds

Theorem 3. *Let μ_n be a sequence of positive Radon measures. Then $\mathcal{T}$- and $\mathcal{T}_1$- convergences are equivalent.*

This theorem implies the following assertion

Theorem 4. *If a sequence μ_n of Radon measures is $\mathcal{T}_1$-convergent and weakly bounded, then it is $\mathcal{T}$-convergent.*

Theorem 5. *Suppose that $\mu = \mathcal{T} \lim \mu_n$ $(n \to \infty)$ and the measures μ_n are positive. Let K be a compactum that is Jordan measurable with respect to the measure μ. Then $\mu = \mathrm{w} \lim \mu_n$ $(n \to \infty)$ on the compactum K.*

Theorem 6. *Suppose that a sequence of Borel measures μ_n on a compactum K, regarded as a sequence of elements of the space $C^*(K)$, converges weakly to zero. Let M be a compact set in $C(K)$. Then*

$$\sup\{|(\mu_n, \varphi)| : \varphi \in M\} \to 0 \quad (n \to \infty).$$

P r o o f. It follows from the weak convergence and the uniform boundedness principle that there exists a number a_1 such that the inequality $|\mu_n|(K) \leq a_1$ holds for all n. Let $\varepsilon > 0$ be an arbitrary number, and $\varphi_1, \cdots, \varphi_N$ be an ε-net in M. Since μ_n converges weakly to zero, there exists N_1 such that for $n > N_1$ we have the inequalities $|(\mu_n, \varphi_k)| < \varepsilon, k = 1, \ldots, N$. Let φ be an arbitrary element of M. There exists k such that $\|\varphi - \varphi_k\| \leq \varepsilon$. Then for $n > N$ we have the inequality

$$|(\mu_n, \varphi)| \leq |(\mu_n, \varphi_k)| + a_1\|\varphi - \varphi_k\| \leq (1 + a_1)\varepsilon,$$

which completes the proof.

Theorem 7. *Suppose that a sequence of harmonic functions u_n in a domain $G \subset \mathbb{R}^m$, regarded as a sequence of generalized functions, converges to a generalized function u. Then u is a regular generalized function which is representable by some harmonic function u in the domain G. Furthermore, the sequence u_n converges uniformly to u on each compactum K in G.*

This is a special case of Theorem 4.4.2 in [5].

We now list the requisite properties of the kernel $h_m(x - y)$. The following assertion holds

Theorem 8. *Suppose that a is a continuous function with compact support in $\mathbb{R}^m$, and let*

$$b(y) = \int a(x) h_m(x - y) dx.$$

Then b is a continuous function in $\mathbb{R}^m$.

P r o o f. Note that the change of the variables $\tilde{x} = x - y$ in the integral gives the equation

$$b(y) = \int a(\tilde{x} + y) h_m(\tilde{x}) d\tilde{x}.$$

Let $\delta > 0$ be an arbitrary number. We assume that the inequality $\|y_1 - y_0\| \le \delta$ holds. We have

$$|b(y_1) - b(y_0)| \le \int |a(x + y_1) - a(x + y_0)| h_m(x) dx$$
$$\le M \sup |a(x + y_1) - a(x + y_0)|,$$

and the result follows easily.

Theorem 9. *Let $p \ge 1$ be an arbitrary fixed number. Suppose that γ is a positive finite Borel measure such that*

$$\sup \left\{ \int\limits_{B(y,\delta)} |h_m(x - y)|^p d\gamma(x) : y \in \mathbb{R}^m \right\} \to 0, \quad (\delta \to 0). \tag{1}$$

Then the function $h_m(x - y) : \mathbb{R}^m_y \to L_p(\gamma)$ is uniformly continuous with respect to the variable y in the space $\mathbb{R}^m$.

For this theorem see [6].

3. Main Results

Theorem 10. *Suppose that υ_n is a sequence of subharmonic functions in a domain $G \subset \mathbb{R}^m$ which converges as a sequence of generalized functions to a generalized function* w*. Then*

1) *the generalized function* w *is a regular generalized function which is represented by a subharmonic function* w *in G;*

2) *if μ is the Riesz measure of the function* w*, and μ_n are the Riesz measures of υ_n, then $\mu = \mathcal{T} \lim \mu_n$ $(n \to \infty)$;*

3) *the raising principle holds, that is, $x_n \to x \in G$ $(n \to \infty)$ implies*

$$\overline{\lim_{n\to\infty}} \upsilon_n(x_n) \le \mathrm{w}(x);$$

4) *the set of the points $x \in G$, for which the inequality*

$$\overline{\lim_{n\to\infty}} \upsilon_n(x) < \mathrm{w}(x),$$

holds, has capacity zero;

5) *if* $u^*(x) := \overline{\lim}_{y\to x} u(y)$, *then for all points* $x \in G$ *the equality*

$$\Big(\overline{\lim_{n\to\infty}}\, \upsilon_n(x)\Big)^* = \mathrm{w}(x)$$

holds;

6) *if* β *is a positive Borel measure in* G *such that* $b(y) = \int h_m(x-y)d\beta(x)$ *is continuous and the function* $\int h_m(x-y)d|\beta|(x)$ *is locally bounded, then*

$$\lim_{n\to\infty} \int \upsilon_n(x)d\beta(x) = \int \mathrm{w}(x)d\beta(x);$$

7) *if* γ *is a positive finite Borel measure with compact support in* G *such that the function*

$$h_m(x-y) : \mathbb{R}^m_y \to L_p(\gamma)$$

is uniformly continuous, then

$$\int |\upsilon_n(x) - \mathrm{w}(x)|^p d\gamma(x) \to 0 \quad (n \to \infty).$$

P r o o f. Let μ_n be the Riesz measure of υ_n. The function υ_n and its Riesz measure are connected by the relation

$$\mu_n = \frac{1}{(m-2)\sigma_{m-1}} \triangle \upsilon_n, \tag{2}$$

where σ_{m-1} is the surface area of the unit sphere in $\mathbb{R}^m$. Since the operation of differentiation is continuous in $\mathcal{D}'(G)$, then the sequence μ_n is $\mathcal{T}_1$-convergent. It follows from Theorem 3 that μ_n is a $\mathcal{T}$-convergent sequence. Let

$$\mu = \mathcal{T} \lim \mu_n \quad (n \to \infty).$$

Let a domain G_1 be compactly embedded in G. This means that the set $K = \overline{G_1}$ is compact and $K \subset G$. Suppose in addition that $\mu(\partial G_1) = 0$. Since $\partial K \subset \partial G_1$, then $\mu(\partial K) = 0$. Then it follows from Theorem 5 that

$$\mu_K = \mathrm{w} \lim_{n\to\infty} (\mu_n)_K, \tag{3}$$

where the subscript K means the restriction of the corresponding measure to the compactum K. The Riesz representation theorem gives

$$\upsilon_n(x) = -\int_K h_m(x-y)d(\mu_n)_K(y) + u_n(x), \tag{4}$$

where u_n is a harmonic function in the set $\overset{\circ}{K}$ and, a fortiori, in the set G_1. Let a be an infinitely differentiable function with compact support in G_1. We multiply equality (4) by a and integrate it. By changing the order of integration, we obtain

$$\int \upsilon_n(x)a(x)dx = \int b_1(y)d(\mu_n)_K(y) + \int u_n(x)a(x)dx, \tag{5}$$

where

$$b_1(y) = -\int h_m(x-y)a(x)dx.$$

By Theorem 8, the function b_1 is continuous in $\mathbb{R}^m$ and also in the compactum K. It now follows from (3) that

$$\lim_{n\to\infty} \int b_1(y)d(\mu_n)_K(y) = \int b_1(y)d\mu_K(y).$$

By the hypothesis of the theorem,

$$\lim_{n\to\infty} \int \upsilon_n(x)a(x)dx = (\mathrm{w}, a(x)).$$

We have thus proved that for any function a in $\mathcal{D}(G_1)$ there exists a limit

$$\lim_{n\to\infty} \int u_n(x)a(x)dx.$$

By Theorem 7, there exists a function u harmonic in G_1 such that the sequence u_n converges uniformly to u on all compacta contained in G_1.

By passing to the limit in equality (5), we obtain

$$\begin{aligned}(\mathrm{w}, a(x)) &= \int b_1(y)d\mu_K(y) + \int u(x)a(x)dx \\ &= \int\limits_{\overset{\circ}{K}} \left(-\int h_m(x-y)d\mu_K(y) + u(x)\right) a(x)dx \\ &= \int\limits_{\overset{\circ}{K}} \left(-\int\limits_{K} h_m(x-y)d\mu(y) + u(x)\right) a(x)dx.\end{aligned}$$

Note that G_1 is an arbitrary domain compactly embedded in G and such that the measure μ does not charge its border. Hence w is a regular generalized function represented in G by a subharmonic function. Assertion 1) of the theorem is proved.

Suppose that μ is the Riesz measure of w, φ is the function in $\mathcal{D}(G)$. Then

$$(\mu,\varphi)=\frac{1}{(m-2)\sigma_{m-1}}(\triangle \mathrm{w},\varphi)=\frac{1}{(m-2)\sigma_{m-1}}\lim_{n\to\infty}(\triangle \upsilon_n,\varphi)=\lim_{n\to\infty}(\mu_n,\varphi).$$

Assertion 2) follows from this equality and Theorem 3.

Assertions 3)–5) of the theorem follow from the well-known facts from the potential theory. For example, assertion 3) is a consequence of Theorem 1.3 in [3]. Assertion 4) is a consequence of Theorem 3.8 in [3], and assertion 5) follows from Remark 2 to Theorem 3.8 in [3].

We now prove assertion 6). By the hypothesis, β is a measure with compact support in the domain G. Therefore there exists a compactum $K_1\subset G$ such that supp $\beta\subset K_1$. Let the compactum $K=\overline{G}_1$ in equality (4) be such that $K_1\subset G_1$. By integrating equality (4) with respect to the measure β, we obtain

$$\int \upsilon_n(x)d\beta(x)=\int b(y)d(\mu_n)_K(y)+\int u_n(x)d\beta(x),$$

where

$$b(y)=-\int h_m(x-y)d\beta(x).$$

The change of order of integration is legitimate since $h_m(x-y)\in L_1(|\beta|\times(\mu_n)_K)$. Since b is a continuous function, we have

$$\lim_{n\to\infty}\int \upsilon_n(x)d\beta(x)=\int b(y)d\mu_K(y)+\int u(x)d\beta(x)$$

$$=\int\left(-\int_K h_m(x-y)d\mu_K(y)+u(x)\right)d\beta(x)=\int \mathrm{w}(x)d\beta(x).$$

Assertion 6) is proved.

We now prove assertion 7). Let

$$J_n=\left(\int|\upsilon_n(x)-\mathrm{w}(x)|^p d\gamma(x)\right)^{\frac{1}{p}},$$

and let n_k be a sequence such that

$$\overline{\lim_{n\to\infty}}\,J_n=\lim_{k\to\infty}J_{n_k}.$$

Let the domain G_1 be compactly embedded in G and such that supp$\gamma\subset G_1$ and $\mu(\partial G_1)=0$. Then, applying Minkowski's inequality, we obtain

$$J_{n_k}\le\left(\int\left|\int h_m(x-y)d((\mu_{n_k})_K-\mu_K)(y)\right|^p d\gamma(x)\right)^{\frac{1}{p}}$$

$$+\left(\int|u_{n_k}(x)-u(x)|^p d\gamma(x)\right)^{\frac{1}{p}}=J_{1k}+J_{2k},$$

where $K = \overline{G}_1$. It is obvious that $J_{2k} \to 0$ $(k \to \infty)$. Let $p > 1$. For J_{1k}, we have the equality

$$J_{1k} = \sup_{\|s\|_q \le 1} \left| \int s(x) \int h_m(x-y) d((\mu_{n_k})_K - \mu_K)(y) d\gamma(x) \right|, \tag{6}$$

where q is found from the equality $1/p + 1/q = 1$, and the norm of the function s is calculated in the space $L_q(\gamma)$. We have

$$\int \left(\int |s(x)| h_m(x-y) d\gamma(x) \right) d(\mu_{n_k})_K(y)$$
$$\le \int \left(\int h_m^p(x-y) d\gamma(x) \right)^{\frac{1}{p}} d(\mu_{n_k})_K(y).$$

It follows from the condition of part 7) of the theorem that the function $\left(\int h_m^p(x-y)d\gamma(x)\right)^{\frac{1}{p}}$ is continuous with respect to the parameter y and thus bounded on the compactum K. Therefore,

$$\int \left(\int |s(x)| h_m(x-y) d\gamma(x) \right) d(\mu_{n_k})_K(y) \le M_1 \mu_{n_k}(K).$$

Similarly,

$$\int \left(\int |s(x)| h_m(x-y) d\gamma(x) \right) d(\mu)_K(y) \le M_1 \mu(K).$$

It follows now from Tonelli's and Fubini's theorems that it is possible to change the order of integration in the integral from (6). Therefore,

$$J_{1k} = \sup_{\|s\|_q \le 1} \left| \int \left(\int s(x) h_m(x-y) d\gamma(x) \right) d((\mu_{n_k})_K - \mu_K)(y) \right|,$$

For $y \in K$, we have the bound

$$\left| \int s(x) h_m(x-y) d\gamma(x) \right| \le \left(\int |h_m(x-y)|^p d\gamma(x) \right)^{\frac{1}{p}} \le M_1.$$

Note that the sequence $\mu_{n_k}(K)$ is bounded. Therefore the family

$$F_s(y) = \int s(x) h_m(x-y) d\gamma(x), \quad \|s\|_q \le 1,$$

is uniformly bounded. The Hölder inequality yields

$$|F_s(y_2) - F_s(y_1)| \le \left(\int |h_m(x-y_2) - h_m(x-y_1)|^p d\gamma(x) \right)^{\frac{1}{p}}.$$

It follows from the restrictions of part 7) of the theorem that the family of functions $F_s(y)$ is equicontinuous. By Arzelàs theorem, the family $F_s(y)$ is compact. Next, Theorem 6 gives $J_{1k} \to 0$ $(k \to \infty)$.

We have considered the case $p > 1$. Now let $p = 1$. Then

$$J_{1k} = \int \left(\int s_k(x) h_m(x-y) d\gamma(x) \right) d\left((\mu_{n_k})_K - \mu_K \right)(y),$$

where

$$s_k(x) = \operatorname{sign} \int h_m(x-y) d\left((\mu_{n_k})_K - \mu_K \right)(y).$$

The sequence

$$F_k(y) = \int s_k(x) h_m(x-y) d\gamma(x)$$

is compact in $C(K)$. In the same way as above, we obtain that $J_{1k} \to 0$ $(k \to \infty)$. Therefore, $J_n \to 0$ $(n \to \infty)$. Assertion 7) and, consequently, the whole theorem are proved.

As it is seen later, the analogue of Theorem 10 for δ-subharmonic functions can be proved by changing the reasoning insignificantly.

Theorem 11. *Suppose that υ_n is a sequence of δ-subharmonic functions in a domain $G \subset \mathbb{R}^m$ that converges as a sequence of generalized functions to a generalized function* w*, and the sequence of the Riesz measures μ_n of the functions υ_n is weakly bounded. Then*

1) *the generalized function* w *is a regular generalized function represented by a δ-subharmonic function* w *in G;*

2) *if μ is the Riesz measure of* w*, then $\mu = \mathcal{T} \lim \mu_n \quad (n \to \infty)$;*

3) *if β is a Borel measure with compact support in G such that the function $b(y) = \int h_m(x-y) d\beta(x)$ is continuous and the function $\int h_m(x-y) d|\beta|(x)$ is locally bounded, then*

$$\lim_{n\to\infty} \int \upsilon_n(x) d\beta(x) = \int \mathrm{w}(x) d\beta(x);$$

4) *if γ is a positive finite Borel measure with compact support in G such that the function*

$$h_m(x-y) : \mathbb{R}^m \to L_p(\gamma)$$

is uniformly continuous, then

$$\int |\upsilon_n(x) - \mathrm{w}(x)|^p d\gamma(x) \to 0 \quad (n \to \infty).$$

P r o o f. Since the sequences $\mu_n,\ (\mu_n)_+,\ (\mu_n)_-,\ |\mu_n|$ are $\mathcal{T}$-compact, to prove assertion 1) we can assume without loss of generality that they are $\mathcal{T}$-convergent. Let

$$\mu = \mathcal{T}\lim \mu_n, \quad \bar{\mu} = \mathcal{T}\lim |\mu_n| \quad (n \to \infty).$$

Next, we proceed along the same line as in the proof of assertion 1) in Theorem 10 with the only change: the condition $\mu(\partial G_1) = 0$ should be replaced by the condition $\overline{\mu}(\partial G_1) = 0$.

The proofs of other assertions follow the proofs of the corresponding assertions from Theorem 10.

Note that in Theorem 11 there are no analogues of assertions 3)–5) of Theorem 10. The proofs of these assertions in Theorem 10 are based on some properties of subharmonic functions which do not hold for δ-subharmonic functions.

The following statements can be derived from Theorem 11 and examples 1–3 in [6].

Theorem 12. *Suppose that υ_n is a sequence of δ-subharmonic functions in a domain $G \subset \mathbb{R}^m,\ m > 2$, which, as a sequence of generalized functions, converges to a generalized function* w. *Suppose that the sequence μ_n of the Riesz measures of the functions υ_n is weakly bounded. Then* w *is a regular generalized function represented by a δ-subharmonic function* w *in G. Furthermore, for any domain G_1 compactly embedded in G and for any $p \in [1, \frac{m}{m-2})$ the following limit relation holds:*

$$\lim_{n\to\infty} \int\limits_{G_1} |\upsilon_n(x) - \mathrm{w}(x)|^p dx = 0.$$

In the particular case, where υ_n are subharmonic functions, this theorem coincides with the results obtained by Hörmander in [1].

Theorem 13. *Suppose that υ_n is a sequence of δ-subharmonic functions in a domain $G \subset \mathbb{R}^m,\ m > 2$, which, as a sequence of generalized functions, converges to a generalized function* w. *Suppose that the sequence μ_n of the Riesz measures of the functions υ_n is weakly bounded. Then* w *is a regular generalized function represented by a δ-subharmonic function* w *in G. Furthermore, if the sphere $S(x_0, r) = \{x : \|x - x_0\| = r\}$ is contained in G and $p \in [1, \frac{m-1}{m-2})$, then the following limit relation holds:*

$$\lim_{n\to\infty} \int\limits_{S(x_0,r)} |\upsilon_n(x) - \mathrm{w}(x)|^p ds = 0.$$

Theorem 14. *Suppose that υ_n is a sequence of δ-subharmonic functions in a domain $G \subset \mathbb{R}^m,\ m > 2$, which, as a sequence of generalized functions, converges* *to a generalized function* w*. Suppose that the sequence* μ_n *of the Riesz measures of the functions* υ_n *is weakly bounded. Then* w *is a regular generalized function represented by a* δ*-subharmonic function* w *in* G*. Furthermore, if a compactum* $\sigma \subset G$ *is contained in the* $(m-1)$*-dimensional plane, and* $p \in [1, \frac{m-1}{m-2})$*, then the following limit relation holds:*

$$\lim_{n\to\infty} \int\limits_{\sigma} |\upsilon_n(x) - \mathrm{w}(x)|^p ds = 0.$$

The integrals in Theorems 13 and 14 are the surface integrals of the first kind.

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