

Subdiagrams of Bratteli Diagrams Supporting Finite Invariant Measures

S. Bezuglyi and O. Karpel

*B. Verkin Institute for Low Temperature Physics and Engineering
National Academy of Sciences of Ukraine
47 Lenin Ave., Kharkiv, 61103, Ukraine*

E-mail: bezuglyi@ilt.kharkov.ua,
helen.karpel@gmail.com

J. Kwiatkowski

*Kotarbinski University of Information Technology and Management
Olsztyn, Poland*

E-mail: jkwiat@mat.umk.pl

Received November 13, 2013, revised April 18, 2014

We study finite measures on Bratteli diagrams invariant with respect to the tail equivalence relation. Amongst the proved results on the finiteness of measure extension, we characterize the vertices of a Bratteli diagram that support an ergodic finite invariant measure.

Key words: Bratteli diagrams, ergodic invariant measures, measure support, tail equivalence relation, Cantor set.

Mathematics Subject Classification 2010: 37A05, 37B05 (primary); 28D05, 28C15 (secondary).

1. Introduction and Background

In this note, we continue the study of ergodic measures on the path space X_B of a Bratteli diagram B started in [1–3]. Recall that, given a minimal (or even aperiodic) homeomorphism T of a Cantor set X , one can construct a refining sequence (ξ_n) (beginning with $\xi_0 = X$) of clopen partitions such that every ξ_n is a finite collection of T -towers $(X_v^{(n)} : v \in V_n)$ [7–9]. This fact is in the base of the very fruitful idea: (X, T) can be realized as a homeomorphism φ (Vershik map) acting on the path space of a Bratteli diagram. By definition, a Bratteli diagram B is represented as an infinite graph with the set of vertices V partitioned into the levels V_n , $n \geq 0$, such that the edge set E_n between the levels $n - 1$ and n is determined by the intersection of towers of partitions ξ_{n-1} and ξ_n (the

detailed definition and references are given below). Every T -invariant (hence, φ -invariant) measure μ on X is completely defined by its values $\mu(X_v^{(n)})$ on all towers where $v \in V_n$ and $n \geq 0$. In [1] and [3], the cases of stationary and finite rank Bratteli diagrams (i.e., $|V_n| \leq d$ for all n) were studied. We notice that while studying φ -invariant measures, we can ignore some rather subtle questions about the existence of a Vershik map on the path space (see [4, 5]) and work with the measures invariant with respect to the tail equivalence relation \mathcal{E} (cofinal equivalence relation, in other words).

Our interest and motivation for this work arise from the following result proved in [3]: for any ergodic probability measure μ on a finite rank diagram B there exists a subdiagram \overline{B} of B defined by a sequence $W = (W_n)$, where $W_n \subset V_n$, such that $\mu(X_w^{(n)})$ is bounded from zero for all $w \in W_n$ and n . It was also shown that μ can be obtained as an extension of an ergodic measure on the subdiagram \overline{B} , in other words, \overline{B} supports μ (the detailed definitions can be found below).

What is an analogue of the above result for general Bratteli diagrams? Suppose we take a subdiagram $\overline{B} = \overline{B}(W)$ of a Bratteli diagram B and consider an ergodic probability measure ν on \overline{B} . Then this measure can be naturally extended (by \mathcal{E} -invariance) to a measure $\widehat{\nu}$ defined on the \mathcal{E} -saturation $\widehat{X}_{\overline{B}}$ of the path space $X_{\overline{B}}$. If the cardinality of W_n is growing, then we cannot expect that the measures of the towers corresponding to the vertices from W_n are bounded from below. But we do expect that the rate of changes of $\widehat{\nu}(X_v^{(n)})$ is essentially different for $v \in W_n$ and $v \notin W_n$. We prove that if the measure $\widehat{\nu}$ is finite and the ratio $\frac{|W_n|}{|V \setminus W_n|}$ is bounded, then the minimal value of $\{\widehat{\nu}(X_v^{(n)}) : v \notin W_n\}$ is much smaller than the maximal value $\{\widehat{\nu}(X_v^{(n)}) : v \in W_n\}$. We also get the results for the ratio of the tower heights corresponding to W_n and $V \setminus W_n$.

Another assertion that is proved in the paper is a modification of [3, Theorem 6.1]. We also give a criterion and a sufficient condition for the finiteness of the extended measure $\widehat{\nu}$, using the condition on entries of the incidence matrices. A number of examples related to this issue is also considered in the paper.

Most of definitions and notation used in this paper are taken from [3]. Since the concept of Bratteli diagrams has been studied in a great number of recent research papers devoted to various aspects of Cantor dynamics, we give here only some necessary definitions and notation referring to the pioneering articles [7, 8], (see also [3, 6]) where the reader can find more detailed definitions and the widely used techniques, for instance, the telescoping procedure.

A *Bratteli diagram* is an infinite graph $B = (V, E)$ such that the vertex set $V = \bigcup_{i \geq 0} V_i$ and the edge set $E = \bigcup_{i \geq 1} E_i$ are partitioned into disjoint subsets V_i and E_i where

- (i) $V_0 = \{v_0\}$ is a single point;

(ii) V_i and E_i are finite sets;

(iii) there exists a range map r and a source map s , both from E to V , such that $r(E_i) = V_i$, $s(E_i) = V_{i-1}$, and $s^{-1}(v) \neq \emptyset$, $r^{-1}(v') \neq \emptyset$ for all $v \in V$ and $v' \in V \setminus V_0$.

Given a Bratteli diagram B , the n -th incidence matrix $F_n = (f_{v,w}^{(n)})$, $n \geq 0$, is a $|V_{n+1}| \times |V_n|$ matrix such that $f_{v,w}^{(n)} = |\{e \in E_{n+1} : r(e) = v, s(e) = w\}|$ for $v \in V_{n+1}$ and $w \in V_n$. Here the symbol $|\cdot|$ denotes the cardinality of a set.

For a Bratteli diagram $B = (V, E)$, the set of all infinite paths in B is denoted by X_B . The topology defined by finite paths (cylinder sets) turns X_B into a 0-dimensional metric compact space. We will consider only those Bratteli diagrams for which X_B is a Cantor set. The tail equivalence relation \mathcal{E} on X_B says that two paths $x = (x_n)$ and $y = (y_n)$ are tail equivalent if and only if $x_n = y_n$ for n sufficiently large. Let $\overline{W} = \{W_n\}_{n>0}$ be a sequence of (proper, non-empty) subsets W_n of V_n . Set $W'_n = V_n \setminus W_n$. The (vertex) subdiagram $\overline{B} = (\overline{W}, \overline{E})$ is defined by the vertices $\overline{W} = \bigcup_{i \geq 0} W_n$ and the edges \overline{E} that have their source and range in \overline{W} . In other words, the incidence matrix \overline{F}_n of \overline{B} is defined by those edges from B that have their source and range in vertices from W_n and W_{n+1} , respectively.

We use the following notation for an \mathcal{E} -invariant measure μ on X_B and $n \geq 1$ and $v \in V_n$:

- $X_v^{(n)} \subset X_B$ denotes the set of all paths that go through the vertex v ;
- $h_v^{(n)}$ denotes the cardinality of the set of all finite paths (cylinder sets) between v_0 and v ;
- $p_v^{(n)}$ denotes the μ -measure of the cylinder set $e(v_0, v)$ corresponding to a finite path between v_0 and v (since μ is \mathcal{E} -invariant, the value $p_v^{(n)}$ does not depend on $e(v_0, v)$).

If \overline{B} is a subdiagram defined by a sequence $\overline{W} = (W_n)$, then we use the notation $\overline{X}_w^{(n)}$ and $\overline{h}_w^{(n)}$ to denote the corresponding objects of the subdiagram \overline{B} .

Take a subdiagram \overline{B} and consider the set $X_{\overline{B}}$ of all infinite paths whose edges belong to \overline{B} . Let $\widehat{X}_{\overline{B}} := \mathcal{E}(X_{\overline{B}})$ be the subset of the paths in X_B that are tail equivalent to the paths from $X_{\overline{B}}$. In other words, the \mathcal{E} -invariant subset $\widehat{X}_{\overline{B}}$ of X_B is the saturation of $X_{\overline{B}}$ with respect to the equivalence relation \mathcal{E} (or $X_{\overline{B}}$ is a countable complete section of \mathcal{E} on $\widehat{X}_{\overline{B}}$). Let μ be a probability measure on $X_{\overline{B}}$ invariant with respect to the tail equivalence relation defined on \overline{B} . Then μ can be canonically extended to the measure $\widehat{\mu}$ on the space $\widehat{X}_{\overline{B}}$ by the invariance with respect to \mathcal{E} [3]. If we want to extend $\widehat{\mu}$ to the whole space X_B , we can set $\widehat{\mu}(X_B \setminus \widehat{X}_{\overline{B}}) = 0$.

Specifically, take a finite path $\bar{e} \in \bar{E}(v_0, w)$ from the top vertex v_0 to a vertex $w \in W_n$ that belongs to the subdiagram \bar{B} . Let $[\bar{e}]$ denote the cylinder subset of $X_{\bar{B}}$ determined by \bar{e} . For any finite path $s \in E(v_0, w)$ from the diagram B with the same range w , we set $\hat{\mu}([s]) = \mu([\bar{e}])$. In such a way, the measure $\hat{\mu}$ is defined on the σ -algebra of Borel subsets of $X_{\bar{B}}$ generated by all clopen sets of the form $[z]$ where a finite path z has the range in a vertex from \bar{B} . Clearly, the restriction of $\hat{\mu}$ on $X_{\bar{B}}$ coincides with μ . We note that the value $\hat{\mu}(X_{\bar{B}})$ can be either finite or infinite depending on the structure of \bar{B} and B (see below Theorems 2.1 and 2.2). Furthermore, the *support* of $\hat{\mu}$ is, by the definition, the set $X_{\bar{B}}$. Set

$$\hat{X}_{\bar{B}}^{(n)} = \{x = (x_i) \in X_{\bar{B}} : r(x_i) \in W_i, \forall i \geq n\}. \tag{1.1}$$

Then $\hat{X}_{\bar{B}}^{(n)} \subset \hat{X}_{\bar{B}}^{(n+1)}$ and

$$\hat{\mu}(X_{\bar{B}}) = \lim_{n \rightarrow \infty} \hat{\mu}(\hat{X}_{\bar{B}}^{(n)}) = \lim_{n \rightarrow \infty} \sum_{w \in W_n} h_w^{(n)} p_w^{(n)}. \tag{1.2}$$

2. Characterization of Subdiagrams Supporting a Measure

Given a Bratteli diagram B , we consider the incidence matrix $F_n = (f_{v,w}^{(n)})$, $v \in V_{n+1}$, $w \in V_n$ and set $A_n = F_n^T$, the transpose of F_n . Together with the sequence of incidence matrices (F_n) , we consider the sequence of stochastic matrices (Q_n) whose entries are

$$q_{v,w}^{(n)} = f_{v,w}^{(n)} \frac{h_w^{(n)}}{h_v^{(n+1)}}, \quad v \in V_{n+1}, w \in V_n.$$

The following result was obtained in [3, Proposition 6.1] for Bratteli diagrams of finite rank. We note here that *this result remains true for arbitrary Bratteli diagrams*, the proof is the same as in [3].

Theorem 2.1. *Let B be a Bratteli diagram with incidence stochastic matrices $\{Q_n = (q_{v,w}^{(n)})\}$ and let \bar{B} be a proper vertex subdiagram of B defined by a sequence of subsets (W_n) where $W_n \subset V_n$.*

(1) *Let μ be a probability invariant measure on the path space $X_{\bar{B}}$ such that the extension $\hat{\mu}$ of μ on $X_{\bar{B}}$ is finite. Then*

$$\sum_{n=1}^{\infty} \sum_{w \in W_{n+1}} \sum_{v \in W_n'} q_{w,v}^{(n)} \mu(\bar{X}_w^{(n+1)}) < \infty. \tag{2.1}$$

(2) If

$$\sum_{n=1}^{\infty} \sum_{w \in W_{n+1}} \sum_{v \in W'_n} q_{w,v}^{(n)} < \infty, \quad (2.2)$$

then any probability invariant measure μ defined on the path space $X_{\overline{B}}$ of the subdiagram \overline{B} extends to a finite measure $\hat{\mu}$ on $\widehat{X}_{\overline{B}}$.

The example below shows that in general case the sufficient condition (2.2) is not necessary and the necessary condition (2.1) is not sufficient.

E x a m p l e. (1) First, we give an example of an infinite measure $\hat{\mu}$ on a Bratteli diagram B such that $\hat{\mu}$ is an extension of a probability measure μ from a subdiagram $\overline{B}(\overline{W})$ and condition (2.1) is satisfied.

Let B be a stationary Bratteli diagram with the incidence matrix

$$F = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix}.$$

Suppose the sequence (W_n) is stationary and formed by the second and third vertices of each level. Then (W'_n) is formed by the first vertex. Since $q_{3,1} = 0$, we have

$$\sum_{n=1}^{\infty} \sum_{v \in W_{n+1}} \sum_{w \in W'_n} q_{v,w}^{(n)} \mu(\overline{X}_v^{(n+1)}) = \sum_{n=1}^{\infty} q_{2,1} \mu(\overline{X}_2^{(n+1)}).$$

Compute

$$q_{2,1} = \frac{h_1^{(n)}}{h_2^{(n+1)}} = \frac{3^{n-1}}{2^n + \sum_{k=0}^{n-1} 2^k 3^{n-1-k}} = \frac{3^{n-1}}{2^n + (3^n - 2^n)} = \frac{1}{3}.$$

It is easy to see that

$$\mu(\overline{X}_2^{(n+1)}) = \frac{2^{n-1}}{3^n}.$$

Then

$$q_{2,1} \mu(\overline{X}_2^{(n+1)}) = \frac{2^{n-1}}{3^{n+1}},$$

and thus condition (2.1) is satisfied. On the other hand, we know that the extension $\hat{\mu}$ is an infinite measure because the Perron–Frobenius eigenvalue of the incidence matrix of \overline{B} is 3, the same as for the odometer corresponding to the first vertex (see [1]).

(2) For any stationary Bratteli diagram, the sufficient condition (2.2) is never satisfied. Thus, to show that (2.2) is not necessary, we can consider any stationary

diagram with the finite full measure $\hat{\mu}$. For instance, one can take a diagram with the incidence matrix

$$F = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix}$$

and μ being the measure on the subdiagram \bar{B} defined as in (1).

In contrast to Theorem 2.1, the following result gives a necessary and sufficient condition for the finiteness of a measure extension.

Theorem 2.2. *Let B, \bar{B}, Q_n, W_n be as in Theorem 2.1 and μ be a probability measure on the path space of the vertex subdiagram \bar{B} . The measure extension $\hat{\mu}(\hat{X}_{\bar{B}})$ is finite if and only if*

$$\sum_{n=1}^{\infty} \sum_{w \in W_{n+1}} \hat{\mu}(X_w^{(n+1)}) \sum_{v \in W'_n} q_{w,v}^{(n)} < \infty \tag{2.3}$$

or

$$\sum_{i=1}^{\infty} \left(\sum_{w \in W_{i+1}} h_w^{(i+1)} p_w^{(i+1)} - \sum_{w \in W_i} h_w^{(i)} p_w^{(i)} \right) < \infty. \tag{2.4}$$

P r o o f. Indeed, let $\hat{X}_{\bar{B}}^{(n)}$ be defined as in (1.1). Then $\hat{\mu}(\hat{X}_{\bar{B}}) = \lim_{n \rightarrow \infty} \hat{\mu}(\hat{X}_{\bar{B}}^{(n)})$. Since

$$\hat{X}_{\bar{B}}^{(n)} = \hat{X}_{\bar{B}}^{(1)} \cup (\hat{X}_{\bar{B}}^{(2)} \setminus \hat{X}_{\bar{B}}^{(1)}) \cup \dots \cup (\hat{X}_{\bar{B}}^{(n)} \setminus \hat{X}_{\bar{B}}^{(n-1)}),$$

we obtain

$$\hat{\mu}(\hat{X}_{\bar{B}}^{(n)}) = 1 + \sum_{i=1}^{n-1} \left(\sum_{w \in W_{i+1}} h_w^{(i+1)} p_w^{(i+1)} - \sum_{w \in W_i} h_w^{(i)} p_w^{(i)} \right).$$

This relation proves (2.4). We remark that condition (2.4) is formulated by using the vertices related only to the subdiagram \bar{B} .

On the other hand,

$$\hat{X}_{\bar{B}}^{(n+1)} \setminus \hat{X}_{\bar{B}}^{(n)} = \{x = (x_i) \in \hat{X}_{\bar{B}} : r(x_n) \in W'_n, r(x_i) \in W_i, i \geq n+1\}.$$

and therefore

$$\begin{aligned} \hat{\mu}(\hat{X}_{\bar{B}}^{(n+1)} \setminus \hat{X}_{\bar{B}}^{(n)}) &= \sum_{w \in W_{n+1}} \sum_{v \in W'_n} f_{w,v}^{(n)} h_v^{(n)} p_w^{(n+1)} \\ &= \sum_{w \in W_{n+1}} \sum_{v \in W'_n} q_{w,v}^{(n)} h_w^{(n+1)} p_w^{(n+1)} \\ &= \sum_{w \in W_{n+1}} \hat{\mu}(X_w^{(n+1)}) \sum_{v \in W'_n} q_{w,v}^{(n)}. \end{aligned}$$

Thus,

$$\widehat{\mu}(\widehat{X}_{\overline{B}}) = 1 + \sum_{n=1}^{\infty} \sum_{w \in W_{n+1}} \widehat{\mu}(X_w^{(n+1)}) \sum_{v \in W'_n} q_{w,v}^{(n)}.$$

■

R e m a r k. In Theorem 2.1, we found the necessary condition (2.1) and the sufficient condition (2.2) for the extension $\widehat{\mu}(\widehat{X}_{\overline{B}})$ to be finite. We can show that these conditions, in fact, follow from Theorem 2.2. Indeed, we have

$$q_{w,v}^{(n)} \widehat{\mu}(X_w^{(n+1)}) \geq q_{w,v}^{(n)} \mu(\overline{X}_w^{(n+1)}).$$

Hence, if $\widehat{\mu}(\widehat{X}_{\overline{B}}) < \infty$, then condition (2.1) holds by Theorem 2.2.

To obtain that (2.2) is a sufficient condition for the extension $\widehat{\mu}(\widehat{X}_{\overline{B}})$ to be finite, it suffices to show that there exists $M > 0$ such that

$$f_{w,v}^{(n)} h_v^{(n)} p_w^{(n+1)} \leq M q_{w,v}^{(n)} = M \frac{f_{w,v}^{(n)} h_v^{(n)}}{h_w^{(n+1)}}.$$

We need to show that

$$p_w^{(n+1)} h_w^{(n+1)} = p_w^{(n+1)} \overline{h}_w^{(n+1)} \frac{h_w^{(n+1)}}{\overline{h}_w^{(n+1)}} \leq M$$

for some $M > 0$. Note that in the proof of Proposition 6.1 in [3] (see also the proof of Proposition 2.3 below), it was shown that there exists M such that $\frac{h_w^{(n+1)}}{\overline{h}_w^{(n+1)}} \leq M$ (the proof in [3] was given for the case of Bratteli diagrams of finite rank, but it is easy to see that the same proof works for a general case). Since $p_w^{(n+1)} \overline{h}_w^{(n+1)} < 1$, the same constant M can be used to prove (2.2).

Here is another sufficient condition for $\widehat{\mu}(\widehat{X}_{\overline{B}})$ to be finite.

Proposition 2.3. *Let \overline{B} be a vertex subdiagram of B with a probability measure μ on $X_{\overline{B}}$. If we suppose that*

$$I = \sum_{n=1}^{\infty} \max_{w \in W_{n+1}} \left(\sum_{v \in W'_n} q_{w,v}^{(n)} \right) < \infty,$$

then $\widehat{\mu}(\widehat{X}_{\overline{B}})$ is finite.

P r o o f. It suffices to show that $I < \infty$ implies $S < \infty$ where

$$S = \sum_{n=1}^{\infty} \sum_{w \in W_{n+1}} \widehat{\mu}(X_w^{(n+1)}) \sum_{v \in W'_n} q_{w,v}^{(n)}$$

is defined in (2.3). Notice that

$$S = \sum_{n=1}^{\infty} \sum_{w \in W_{n+1}} \sum_{v \in W'_n} f_{w,v}^{(n)} h_v^{(n)} p_w^{(n+1)}.$$

We have

$$\begin{aligned} \sum_{w \in W_{n+1}} \sum_{v \in W'_n} f_{w,v}^{(n)} h_v^{(n)} p_w^{(n+1)} &= \sum_{w \in W_{n+1}} \sum_{v \in W'_n} f_{w,v}^{(n)} h_v^{(n)} \frac{p_w^{(n+1)} \bar{h}_w^{(n+1)}}{\bar{h}_w^{(n+1)}} \\ &= \sum_{w \in W_{n+1}} \mu(\bar{X}_w^{(n+1)}) \sum_{v \in W'_n} f_{w,v}^{(n)} \frac{h_v^{(n)}}{\bar{h}_w^{(n+1)}} \\ &\leq \max_{w \in W_{n+1}} \left(\sum_{v \in W'_n} f_{w,v}^{(n)} \frac{h_v^{(n)}}{\bar{h}_w^{(n+1)}} \right) \sum_{w \in W_{n+1}} \mu(\bar{X}_w^{(n+1)}). \end{aligned}$$

Since μ is a probability measure on $X_{\bar{B}}$, we obtain that

$$\sum_{w \in W_{n+1}} \mu(\bar{X}_w^{(n+1)}) = 1.$$

We show that there exists $M > 0$ such that $\frac{h_w^{(n)}}{\bar{h}_w^{(n)}} < M$ for all $w \in W_n$ and all sufficiently large n . Indeed, set

$$M_n = \max_{w \in W_n} \frac{h_w^{(n)}}{\bar{h}_w^{(n)}}.$$

Fix any $v \in W_{n+1}$. Then

$$\begin{aligned} \frac{h_w^{(n+1)}}{\bar{h}_w^{(n+1)}} &= \frac{1}{\bar{h}_w^{(n+1)}} \left(\sum_{v \in W_n} f_{w,v}^{(n)} h_v^{(n)} + \sum_{v \in W'_n} f_{w,v}^{(n)} h_v^{(n)} \right) \\ &\leq \frac{M_n}{\bar{h}_w^{(n+1)}} \sum_{v \in W_n} f_{w,v}^{(n)} \bar{h}_v^{(n)} + \frac{1}{\bar{h}_w^{(n+1)}} \sum_{v \in W'_n} f_{w,v}^{(n)} h_v^{(n)} \\ &= M_n + \frac{h_w^{(n+1)}}{\bar{h}_w^{(n+1)}} \sum_{v \in W'_n} f_{w,v}^{(n)} \frac{h_v^{(n)}}{h_w^{(n+1)}} \\ &= M_n + \frac{h_w^{(n+1)}}{\bar{h}_w^{(n+1)}} \sum_{v \in W'_n} q_{w,v}^{(n)} \\ &\leq M_n + \frac{h_w^{(n+1)}}{\bar{h}_w^{(n+1)}} \varepsilon_n, \end{aligned}$$

where

$$\varepsilon_n = \max_{w \in W_{n+1}} \left(\sum_{v \in W'_n} q_{w,v}^{(n)} \right).$$

Since $I < \infty$, the value of ε_n tends to zero as n tends to ∞ . From the above inequalities we obtain

$$\frac{h_w^{(n+1)}}{h_w^{(n)}}(1 - \varepsilon_n) \leq M_n \text{ and } M_{n+1} \leq \frac{M_n}{1 - \varepsilon_n}.$$

Finally,

$$M_n \leq \frac{M_1}{\prod_{k=1}^{\infty} (1 - \varepsilon_k)} = M.$$

Since $I < \infty$, we get that M is well-defined.

Thus,

$$\begin{aligned} \sum_{w \in W_{n+1}} \sum_{v \in W'_n} f_{w,v}^{(n)} h_v^{(n)} p_w^{(n+1)} &< M \max_{w \in W_{n+1}} \left(\sum_{v \in W'_n} f_{w,v}^{(n)} \frac{h_v^{(n)}}{h_w^{(n+1)}} \right) \\ &= M \max_{w \in W_{n+1}} \left(\sum_{v \in W'_n} q_{w,v}^{(n)} \right). \end{aligned}$$

■

E x a m p l e. In this example, we consider a class of Bratteli diagrams B whose incidence matrices F_n of size $|V_{n+1}| \times |V_n|$ can be written as follows:

$$F_n = \begin{pmatrix} a_n & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix}, \quad n \geq 1,$$

where $a_n \geq 2$ for $n \geq 1$. Consider the subdiagram \bar{B} defined by the leftmost vertex at each level such that W_n consists of a single vertex and $\bar{F}_n = (a_n)$. Let μ be the unique probability invariant measure on $X_{\bar{B}}$. Then

$$p_w^{(n)} = \frac{1}{a_0 \cdots a_{n-1}}$$

for $w \in W_n$.

Recall that, by Theorem 2.2, $\hat{\mu}(X_{\bar{B}})$ is finite if and only if the series

$$S = \sum_{n=1}^{\infty} \sum_{w \in W_{n+1}} \sum_{v \in W'_n} f_{w,v}^{(n)} h_v^{(n)} p_w^{(n+1)}$$

converges. Thus,

$$\left(\widehat{\mu}(\widehat{X}_{\overline{B}}) < \infty\right) \iff \left(S = \sum_{n=1}^{\infty} \frac{1}{a_0 \cdots a_n} \sum_{v \in W'_n} h_v^{(n)} < \infty\right).$$

For any $v \in W'_n$, the height $h_v^{(n)}$ is independent of v ; denote it by $h_0^{(n)}$. Hence,

$$S = \sum_{n=1}^{\infty} \frac{|V_n| - 1}{a_0 \cdots a_n} h_0^{(n)}.$$

We formulate below some conditions for the convergence and divergence of the series S .

(i) We observe that $h_0^{(n)}$ does not depend on a_{n-1} and a_n . Hence, for any values of the parameters $\{a_i\}_{i=1}^{n-2}$ and $\{|V_i|\}_{i=1}^n$ we can choose $a_{n-1}a_n$ large enough to guarantee that the series S converges.

(ii) Suppose for simplicity that $a_0 = 1$. Then we have

$$h^{(1)} = (1, \dots, 1)^T$$

and

$$h_0^{(n+1)} = h_0^{(n)}(|V_n| - 1) + h_1^{(n)}.$$

Since $a_n > 1$, we obtain

$$h_0^{(n+1)} \geq h_0^{(n)}|V_n|.$$

Thus, for every n , the inequality

$$h_0^{(n)} \geq |V_1| \cdots |V_{n-1}|$$

holds. Therefore,

$$S \geq \sum_{n=1}^{\infty} \frac{|V_1| \cdots |V_{n-1}|(|V_n| - 1)}{a_0 \cdots a_n}. \tag{2.5}$$

If

$$\sum_{n=1}^{\infty} \frac{|V_1| \cdots |V_{n-1}|(|V_n| - 1)}{a_0 \cdots a_n} = \infty,$$

then $\widehat{\mu}(\widehat{X}_{\overline{B}}) = \infty$. In other words, we can see that if the number of vertices in (W'_n) is sufficiently large, then the extension of the measure μ is infinite.

(iii) Denote

$$b_n = \frac{|V_n| - 1}{a_0 \cdots a_n} h_0^{(n)},$$

then

$$\frac{b_{n+1}}{b_n} = \frac{(|V_{n+1}| - 1)(|V_n| - 1 + h_1^{(n)})}{a_{n+1}(|V_n| - 1)}.$$

If $(|V_n| - 1)a_n^{-1} \geq 1$ for all sufficiently large n , then the series S diverges and the measure extension is infinite.

(iv) It is obvious that $h_0^{(n)} \geq a_0 \dots a_{n-2}$. Hence,

$$S \geq \sum_{n=1}^{\infty} \frac{|V_n| - 1}{a_{n-1}a_n}.$$

Thus,

$$\left(\sum_{n=1}^{\infty} (|V_n| - 1)(a_{n-1}a_n)^{-1} = \infty \right) \implies \left(\hat{\mu}(\hat{X}_B) = \infty \right).$$

Denote

$$S_1 = \sum_{n=1}^{\infty} \frac{|V_1| \dots |V_{n-1}|(|V_n| - 1)}{a_0 \dots a_n}$$

and

$$S_2 = \sum_{n=1}^{\infty} \frac{|V_n| - 1}{a_{n-1}a_n}.$$

We have seen that $S_1 = \infty$ implies that $\hat{\mu}$ is infinite and $S_2 = \infty$ implies the same. Let us compare the series S_1 and S_2 to find out whether their convergence implies the finiteness of the measure $\hat{\mu}$. Suppose $|V_n| = n^2$ and $a_n = n^2$. Then

$$S_2 = \sum_{n=1}^{\infty} \left(\frac{1}{n(n-1)} + \frac{1}{n^2(n-1)} \right)$$

converges, but S_1 still diverges (the general term is $\frac{n^2-1}{n^2}$) and the measure $\hat{\mu}$ is infinite. Thus, S_2 cannot provide us with a necessary and sufficient condition for the finiteness of $\hat{\mu}$.

(v) On the other hand, it follows from

$$h_1^{(n+1)} = h_0^{(n)}(|V_n| - 1) + a_n h_1^{(n)} \leq h_1^{(n)}(a_n + |V_n| - 1)$$

that

$$h_0^{(n)} \leq h_1^{(n)} \leq (a_n + |V_n| - 1) \dots (a_{n-1} + |V_{n-1}| - 1)$$

for every n . Then we have

$$\begin{aligned} S &\leq \sum_{n=1}^{\infty} \frac{(a_1 + |V_1| - 1) \dots (a_{n-1} + |V_{n-1}| - 1)(|V_n| - 1)}{a_0 \dots a_n} \\ &= \sum_{n=1}^{\infty} \frac{|V_n| - 1}{a_n} \left(1 + \frac{|V_1| - 1}{a_1} \right) \dots \left(1 + \frac{|V_{n-1}| - 1}{a_{n-1}} \right). \end{aligned}$$

Hence $\widehat{\mu}(\widehat{X}_{\overline{B}}) < \infty$ if the sequence $(|V_n| - 1)a_n^{-1}$ tends to zero fast enough such that the above series converges.

Therefore, if $(|V_n| - 1)a_n^{-1} \geq 1$ for all sufficiently large n , then the measure extension is infinite; if the measure extension is finite, then the sequence $(|V_n| - 1)a_n^{-1}$ tends to zero fast enough.

To simplify the formulation of the next statement, we assume that $f_{w,v} > 0$ for every $w \in W_{n+1}$, $v \in W'_n$ and $n > 0$, i.e., for every $w \in W_{n+1}$ there is an edge to some vertex from W'_n . This assumption is not restrictive since one can use the telescoping procedure to ensure the positivity of all entries of F .

Corollary 2.4. *Let $B, \overline{B}, Q_n, W_n$ be as in Theorem 2.1 and μ be a probability measure on the path space of the vertex subdiagram \overline{B} . Let the measure extension $\widehat{\mu}(\widehat{X}_{\overline{B}})$ be finite. Then*

$$\sum_{n=1}^{\infty} \min_{w \in W_{n+1}} \max_{v \in W'_n} q_{w,v}^{(n)} < \infty.$$

In particular,

$$\sum_{n=1}^{\infty} \min_{w \in W_{n+1}} \max_{v \in W'_n} \frac{h_v^{(n)}}{h_w^{(n+1)}} < \infty. \tag{2.6}$$

P r o o f. By Theorem 2.2, we have

$$\begin{aligned} \widehat{\mu}(\widehat{X}_{\overline{B}}) &= 1 + \sum_{n=1}^{\infty} \sum_{w \in W_{n+1}} \widehat{\mu}(X_w^{(n+1)}) \sum_{v \in W'_n} q_{w,v}^{(n)} \\ &\geq 1 + \sum_{n=1}^{\infty} \sum_{w \in W_{n+1}} \widehat{\mu}(X_w^{(n+1)}) \max_{v \in W'_n} q_{w,v}^{(n)} \\ &\geq 1 + \sum_{n=1}^{\infty} \min_{w \in W_{n+1}} \max_{v \in W'_n} q_{w,v}^{(n)} \sum_{w \in W_{n+1}} \widehat{\mu}(X_w^{(n+1)}). \end{aligned}$$

Since

$$\sum_{w \in W_{n+1}} \widehat{\mu}(X_w^{(n+1)}) \rightarrow \widehat{\mu}(\widehat{X}_{\overline{B}}) > 0,$$

there is a constant $C > 0$ such that $\sum_{w \in W_{n+1}} \widehat{\mu}(X_w^{(n+1)}) > C$ for all n . Hence we obtain

$$\sum_{n=1}^{\infty} \min_{w \in W_{n+1}} \max_{v \in W'_n} q_{w,v}^{(n)} < \infty.$$

Since $f_{w,v} > 0$ for every $w \in W_{n+1}$, $v \in W'_n$ and $n > 0$, then there follows relation (2.6). ■

R e m a r k. Let B be a stationary Bratteli diagram. If B is simple, then there is a unique ergodic invariant measure ν on X_B . Suppose that λ is the Perron–Frobenius eigenvalue for the incidence matrix of B . Then all the heights $h_v^{(n)}$ of B grow as λ^n . Thus, for any choice of $W_n \subset V_n$, for any $v \in W'_n$ and $w \in W_{n+1}$, the ratio $\frac{h_v^{(n)}}{h_w^{(n+1)}}$ will tend to $\frac{1}{\lambda}$ as n tends to infinity. Hence, by Corollary 2.4, there is no proper subdiagram \bar{B} such that ν could be the extension of an invariant ergodic measure from $X_{\bar{B}}$.

In the case of a non-simple stationary diagram B , the minimal support of an ergodic invariant measure is some simple stationary subdiagram $\bar{B}(W)$ whose incidence matrix \bar{F} has the Perron–Frobenius eigenvalue $\bar{\lambda}$. Then, for every $w \in W_n$, the height $h_w^{(n)}$ grows again as $\bar{\lambda}^n$, but for every $v \in W'_n$, the height $h_v^{(n)}$ grows as δ^n , where $\delta < \bar{\lambda}$ (see [1]).

We recall that for a finite rank Bratteli diagram the support of any probability measure μ is determined by a vertex subdiagram $\bar{B}(W)$, $W = (W_n)$, whose vertices v satisfy the condition: there exists some $\delta > 0$ such that $\mu(X_v^{(n)}) > \delta$ for all sufficiently large n and all $v \in W_n$ (see [3]). In particular, a Bratteli diagram B is of exact finite rank if the condition $\mu(X_v^{(n)}) > \delta$ holds for all vertices $v \in V_n$. Clearly, the above result cannot be true for general Bratteli diagrams. Nevertheless, we can find another characterization for vertices that belong to the support of a probability measure by studying how the measure of towers $X_v^{(n)}$ changes when v is in the subdiagram and when it is not in the subdiagram.

R e m a r k. Let $\hat{\mu}$ be the extension of the measure μ defined on an exact finite rank subdiagram \bar{B} of a Bratteli diagram B . Suppose that $\hat{\mu}(\hat{X}_{\bar{B}}) < \infty$. Then we have

$$\begin{aligned} \max_{v \in W'_n} \hat{\mu}(X_v^{(n)}) &\leq \sum_{v \in W'_n} \hat{\mu}(X_v^{(n)}) \\ &= \hat{\mu}(\hat{X}_{\bar{B}}) - \sum_{w \in W_n} \hat{\mu}(X_w^{(n)}) \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Since the measure of any tower $\bar{X}_w^{(n)}$ is bounded from zero, it follows that

$$\lim_{n \rightarrow \infty} \frac{\max_{v \in W'_n} \hat{\mu}(X_v^{(n)})}{\min_{w \in W_n} \mu(\bar{X}_w^{(n)})} = 0, \tag{2.7}$$

and therefore

$$\lim_{n \rightarrow \infty} \frac{\max_{v \in W'_n} \hat{\mu}(X_v^{(n)})}{\min_{w \in W_n} \hat{\mu}(X_w^{(n)})} = 0. \tag{2.8}$$

It is very plausible that (2.8) is true for any uniquely ergodic Bratteli subdiagram \overline{B} . However, this is still under the question. On the other hand, we are able to prove the following result.

Proposition 2.5. *Let B be a Bratteli diagram with the incidence matrices $F_n = \{(f_{v,w}^{(n)})\}$. Let $\overline{B} = \overline{B}(W)$ be a proper vertex subdiagram of B such that $\frac{|W_n|}{|V \setminus W_n|} \leq C$ for every n and some constant $C > 0$. Suppose $\hat{\mu}$ is a finite invariant measure on the path space X_B which is obtained as the extension of a probability measure μ defined on $X_{\overline{B}}$. Then*

$$\lim_{n \rightarrow \infty} \frac{\min_{v \in W'_n} \hat{\mu}(X_v^{(n)})}{\max_{w \in W_n} \mu(\overline{X}_w^{(n)})} = 0. \tag{2.9}$$

P r o o f. Let $W'_n = V_n \setminus W_n$. For every n , we have

$$\begin{aligned} \hat{\mu}(\widehat{X}_{\overline{B}}) &= \sum_{v \in W'_n} \hat{\mu}(X_v^{(n)}) + \sum_{w \in W_n} \hat{\mu}(X_w^{(n)}) \\ &\geq |W'_n| \min_{v \in W'_n} \hat{\mu}(X_v^{(n)}) + \sum_{w \in W_n} \hat{\mu}(X_w^{(n)}). \end{aligned}$$

Thus,

$$\min_{v \in W'_n} \hat{\mu}(X_v^{(n)}) \leq \frac{\hat{\mu}(X_B) - \sum_{w \in W_n} \hat{\mu}(X_w^{(n)})}{|W'_n|}.$$

Since $\mu(X_{\overline{B}}) = 1$, we obtain

$$\max_{w \in W_n} \mu(\overline{X}_w^{(n)}) \geq \frac{1}{|W_n|}.$$

Hence,

$$\frac{\min_{v \in W'_n} \hat{\mu}(X_v^{(n)})}{\max_{w \in W_n} \mu(\overline{X}_w^{(n)})} \leq \frac{|W_n|(\hat{\mu}(X_B) - \sum_{w \in W_n} \hat{\mu}(X_w^{(n)}))}{|W'_n|}.$$

Notice that

$$\hat{\mu}(X_B) - \sum_{w \in W_n} \hat{\mu}(X_w^{(n)}) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

This proves that equality (2.9) holds. ■

R e m a r k. Since $\widehat{\mu}(X_w^{(n)}) \geq \mu(\overline{X}_w^{(n)})$ for every $w \in W_n$ and every n , we obtain the following simple corollary of the proved result

$$\lim_{n \rightarrow \infty} \frac{\min_{v \in W'_n} \widehat{\mu}(X_v^{(n)})}{\max_{w \in W_n} \widehat{\mu}(X_w^{(n)})} = 0.$$

Acknowledgment. This article was finished when the first named author was a visitor of the Department of Mathematics, University of Iowa. He is thankful to the department for hospitality and support.

References

- [1] *S. Bezuglyi, J. Kwiatkowski, K. Medynets, and B. Solomyak*, Invariant Measures on Stationary Bratteli Diagrams. — *Ergodic Theory Dynam. Syst.* **30** (2013), 973–1007.
- [2] *S. Bezuglyi and O. Karpel*, Homeomorphic Measures on Stationary Bratteli Diagrams. — *J. Funct. Anal.* **261** (2011), 3519–3548.
- [3] *S. Bezuglyi, J. Kwiatkowski, K. Medynets, and B. Solomyak*, Finite Rank Bratteli Diagrams: Structure of Invariant Measures. — *Trans. Amer. Math. Soc.* **365** (2013), 2637–267.
- [4] *S. Bezuglyi, J. Kwiatkowski, and R. Yassawi*, Perfect Orderings on Finite Rank Bratteli Diagrams. — *Canad. J. Math.* **66** (2014), 57–101.
- [5] *S. Bezuglyi and R. Yassawi*, Perfect Orderings on General Bratteli Diagrams. Preprint, 2013.
- [6] *F. Durand*, Combinatorics on Bratteli Diagrams and Dynamical Systems. In: *Combinatorics, Automata and Number Theory*, V. Berthé and M. Rigo (eds.), *Encyclopedia of Mathematics and its Applications* **135**, Cambridge University Press (2010), 338–386.
- [7] *T. Giordano, I. Putnam, and C. Skau*, Topological Orbit Equivalence and C^* -Crossed Products. — *J. Reine Angew. Math.* **469** (1995), 51–111.
- [8] *R.H. Herman, I. Putnam, and C. Skau*, Ordered Bratteli Diagrams, Dimension Groups, and Topological Dynamics. — *Int. J. Math.* **3(6)** (1992), 827–864.
- [9] *K. Medynets*, Cantor Aperiodic Systems and Bratteli Diagrams. *C. R., Acad. Sci. Paris, Ser. 1* **342** (2006), 43–46.