Fizika Nizkikh Temperatur, 2008, v. 34, Nos. 4/5, p. 440–446

Bose–Einstein condensation in a decorated lattice: an application to the problem of supersolid He

D.V. Fil

Institute for Single Crystals, National Academy of Sciences of Ukraine, 60 Lenin Ave., Kharkov 61001, Ukraine
E-mail: fil@isc.kharkov.ua

S.I. Shevchenko

B. Verkin Institute for Low Temperature Physics and Engineering of the National Academy of Sciences of Ukraine
47 Lenin Ave., Kharkov 61103, Ukraine
E-mail: shevchenko@ilt.kharkov.ua

Received November 1, 2007

The Bose–Einstein condensation of vacancies in a three-dimensional decorated lattice is considered. The model describes possible scenario of superfluidity of solid helium, caused by the presence of zero-point vacancies in a dislocation network. It is shown that the temperature of Bose–Einstein condensation decreases under increase of the length of the network segments, and the law of decrease depends essentially on the properties of the vertexes of the network. If the vertexes correspond to barriers with a small transparency, the critical temperature is inversely as the square of the length of the segment. On the contrary, if the vertexes correspond to traps for the vacancies (it is energetically preferable for the vacancies to be localized at the vertexes), an exponential lowering of the temperature of transition takes place. The highest temperature of Bose–Einstein condensation is reached in the intermediate case of vertexes with large transparency, but in the absence of tendency of localization at them. In the latter case the critical temperature is inversely as the length of the segment.

PACS: **67.80.–s** Quantum solids;
67.10.Ba Boson degeneracy.

Keywords: supersolid, zero-point vacancies, dislocations.

1. Introduction

Experimental observation of nonclassical rotational inertia in torsion experiments on solid He [1] (confirmed by a number of other groups [2–4]) has revived interest to the idea on supersolid. The idea goes back to pioneer work by Andreev and Lifshitz [5] where it was shown that the presence of vacancies in quantum crystals at zero temperature (zero-point vacancies) can cause superfluid properties of such systems. However, as was found later [6,7], in ^{4}He crystals the concentration of vacancies is negligibly small and distincts from zero only due to thermal activation. The presence of extensive defects in the crystal can change the situation and make the occurrence of zero-point vacancies energetically favorable. Zero-point vacancies may emerge if the kinetic energy gain exceeds the potential energy losses. Defects destroy ideal periodicity of the lattice, the minimums of potential energy, that correspond to lattice site, become more shallow and it may reduce the losses. If an extensive defect is homogeneous, zero-point vacancies can move freely along the defect. At sufficient concentration of defects they form a network, that provides possibility of flowing of the vacancies through the whole crystal. Under lowering the temperature such a gas of vacancies should go into a superfluid state. In recent papers [8,9] it was established by Monte Carlo simulation that in ^{4}He crystals the grain boundaries and dislocations do possess superfluid properties. On the other hand, as was shown in Ref. 2, a rather long-term annealing (that removes dislocation from the crystal) leads to a complete disappearance of the effect of a step-like change of the period of a torsion oscillator filled with solid helium. Thus, superfluidity of vacancies in a network of dislocations can be considered as probable mechanism of superfluidity of quantum crystals.

The idea on dislocation superfluidity was put forward in Refs. 10, 11 long before the observation of the effect [1]. As was shown in [10,11], the important parameter that determines the temperature of the phase transition in such a system is the length of the segment of the network. However the approach [10,11] did not consider such characteristic of the network as the transparency of the vertexes (more precisely, there was some implicit assumption on its value).

In this paper we consider a simple model of the network that allows to investigate the dependence of the temperature of the Bose–Einstein condensation on the length of the segment and on tunnel characteristics of the vertexes.

2. The model

In what follows we will model a superfluid dislocation as one-dimensional lattice chain with the period a and the length $l = qa$. It is implied that the chain has nonzero concentration of zero-point vacancies n_1 (one-dimensional concentration). The chains (segments) are joined into a regular three-dimensional network. Two edge sites of each segment are the vertexes of the network (we will call such sites the central ones). To be more specific, we consider that the network obeys cubic symmetry. In fact, we model the network as a decorated* cubic lattice with the period l and the number of sites in the elementary cell equal $3q-2$ (Fig. 1). We are interested in the case of large q. The vacancies that moves in a such lattice are described in the tight-binding approximation. The model contains two parameters: t, the amplitude of tunnelling between nearest neighbor internal sites in segments, and t_1, the amplitude of tunnelling between a central site and a nearest neighbor internal site. The one-site energies are assumed to be the same for all sites. In the broader sense, the model describes a network formed by one-dimensional wires, along which bosons can move. The central sites play the role of scatterers that connect the wires.

The number of zero-point vacancies N is supposed to be much smaller than the numbers of sites in the decorated lattice, i.e., the filling factor for the vacancies satisfies the inequality $\nu = n_1 a << 1$. For the further analysis it is convenient to introduce three-dimensional concentration of vacancies $n \approx 3q\nu / l^3$. We note that even at small filling factor $\nu << 1$ the number of vacancies per elementary cell can be much larger than unity ($nl^3 >> 1$) if the condition $\nu >> 1/q$ is satisfied. Below we will consider the filling factors belonging to the diapason $1/q << \nu << 1$.

At small filling factors one can neglect the interaction between the vacancies. The temperature of Bose–Einstein condensation T_0 for a noninteracting gas of bosons in a compound lattice is determined by the equation

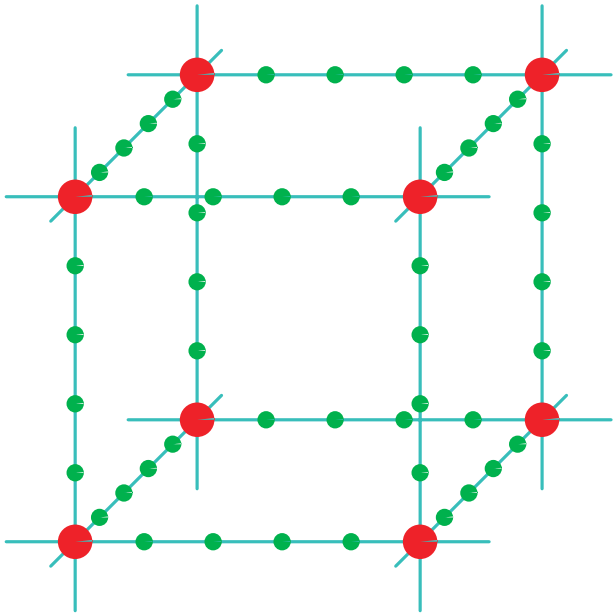

Fig. 1. Elementary cell of the decorated lattice ($q = 5$). Large circles represent central sites, small ones, the internal sites of the segments.

$$N = \sum_{\lambda,\mathbf{k}} \left[\exp\left(\frac{\varepsilon_\lambda(\mathbf{k}) - \mu_0}{T_0} \right) - 1 \right]^{-1}, \qquad (1)$$

where $\varepsilon_\lambda(\mathbf{k})$ is the spectrum of bosons in a lattice, λ is the band index, $\mathbf{k}$ is the wave vector, μ_0 is the chemical potential at $T \le T_0$ (that coincides with the energy of the bottom of the lowest band). In Eq. (1) the index λ runs from 1 to $3q-2$ (the number of sites in the unit cell), and summation in taken over $\mathbf{k}$ belonging to the first Brillouin zone ($-\pi / l < k_i < \pi / l$).

The Hamiltonian of the system has the form

$$H = -\sum_{\mathbf{i}} \sum_{\alpha=x,y,z} [t_1 (b^+_{\mathbf{i},\upsilon} b_{\mathbf{i},(\alpha,1)} + b^+_{\mathbf{i},(\alpha,q-1)} b_{\mathbf{i}+\mathbf{n}_\alpha,\upsilon}) + t \sum_{\xi=1}^{q-2} b^+_{\mathbf{i},(\alpha,\xi)} b_{\mathbf{i},(\alpha,\xi+1)} + \text{h.c.}], \qquad (2)$$

where $b^+_{\mathbf{i},\eta}$ ($b_{\mathbf{i},\eta}$) is the operator of creation (annihilation) of a boson in the site η in the i-th cell, $\mathbf{i}$ is the radius-vector of the i-th cell, $\mathbf{n}_x = (l,0,0)$, $\mathbf{n}_y = (0,l,0)$, $\mathbf{n}_z = (0,0,l)$ are the primitive vectors of translation. The following notation for η is used: $\eta = \upsilon$, the central site, $\eta = (\alpha,\xi)$, the ξ-th internal site in a segment aligned in α direction.

Applying the Fourier-transformation

$$b_{\mathbf{i},\eta} = \frac{1}{\sqrt{N_i}} \sum_{\mathbf{k}} b_{\mathbf{k},\eta} e^{i\mathbf{k}\mathbf{i}} \qquad (3)$$

* In difference with the simple cubic lattice the decorated lattice contains additional elements — lattice chains along the bonds.

(where N_i is number of unit cells), we rewrite the Hamiltonian as

$$H = \sum_{\eta_1,\eta_2} M_{\eta_1,\eta_2}(\mathbf{k}) b^+_{\mathbf{k},\eta_1} b_{\mathbf{k},\eta_2}. \qquad (4)$$

The matrix $\mathbf{M}(\mathbf{k})$ has dimension $(3q-2)\times(3q-2)$ and is presented in the following block form:

$$\mathbf{M}(\mathbf{k}) = -t \begin{pmatrix} 0 & \mathbf{T}_x & \mathbf{T}_y & \mathbf{T}_z \\ \mathbf{T}_x^+ & \mathbf{D}_{q-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{T}_y^+ & \mathbf{0} & \mathbf{D}_{q-1} & \mathbf{0} \\ \mathbf{T}_z^+ & \mathbf{0} & \mathbf{0} & \mathbf{D}_{q-1} \end{pmatrix}. \qquad (5)$$

Here $\mathbf{D}_{q-1}$ is the $(q-1)\times(q-1)$ matrix that corresponds to the tunnelling between internal sites:

$$\mathbf{D}_{q-1} = \begin{pmatrix} 0 & 1 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix}, \qquad (6)$$

and $\mathbf{T}_\alpha$ and $\mathbf{T}_\alpha^+$ are of $1\times(q-1)$ and $(q-1)\times 1$ matrixes that describe the tunnelling between the central site and the nearest internal site:

$$\mathbf{T}_\alpha = (\tau\, 0 \ldots 0 \ldots \tau e^{-ik_\alpha l}), \quad \mathbf{T}_\alpha^+ = \begin{pmatrix} \tau \\ 0 \\ \ldots \\ 0 \\ \tau e^{ik_\alpha l} \end{pmatrix}, \qquad (7)$$

$\tau = t_1 / t$ is a key parameter of the model (the ratio of amplitudes of tunnelling between an internal site and a central site and between two internal sites).

The spectrum of bosons satisfies the dispersion equation $\det(\varepsilon\mathbf{I} - \mathbf{M}) = 0$. Using the expressions (5)–(7) for $\mathbf{M}$, we obtain the explicit form of the dispersion equation

$$[\Delta_{q-1}(\tilde{\varepsilon})]^2 [\tilde{\varepsilon}\Delta_{q-1}(\tilde{\varepsilon}) - 6\tau^2 \Delta_{q-2}(\tilde{\varepsilon}) - (-1)^q 2\tau^2 \sum_\alpha \cos(k_\alpha l)] = 0, \qquad (8)$$

where $\tilde{\varepsilon} = \varepsilon / t$ and $\Delta_q(\tilde{\varepsilon}) = \det(\tilde{\varepsilon}\mathbf{I} + \mathbf{D}_q)$.

3. Band structure of the spectrum

As follows from the dispersion equation (8), some bands are reduced to degenerate levels. The energies of these levels are given by the equation

$$\Delta_{q-1}(\tilde{\varepsilon}) = 0. \qquad (9)$$

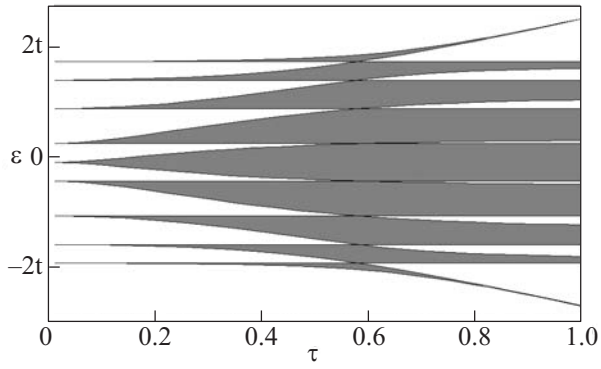

Fig. 2. The band structure for bosons in the decorated lattice (for $q = 9$). The allowed bands are shown grey.

Equation (9) coincides with the dispersion equation for an isolated chain with $q-1$ sites. The energies of the levels are equal to

$$\varepsilon_s = -2t \cos\frac{\pi s}{q} \qquad (10)$$

($s = 1, 2, \ldots, q-1$). The degree of degeneracy of each level is $2N_i$. The wave functions of such states have zero weight in the central sites that is in agreement with the absence of dispersion.

The spectrum of the bands with a finite dispersion satisfies the equation

$$\tilde{\varepsilon}\Delta_{q-1}(\tilde{\varepsilon}) - 6\tau^2\Delta_{q-2}(\tilde{\varepsilon}) - (-1)^q 2\tau^2 \sum_\alpha \cos(k_\alpha l) = 0. \qquad (11)$$

The left hand side of Eq. (11) is a q-th order polynomial. It has q distinct solutions that corresponds to q bands (allowed bands). The widths of the bands essentially depend on the parameter τ. It is illustrated in Fig. 2.

One can see from this figure, that there is a special value $\tau = \tau_c = 1/\sqrt{3}$ at which all allowed bands have the largest width, and there is no gaps between the bands. At $\tau > \tau_c$ and $\tau < \tau_c$ the allowed bands become narrower and energy gaps (forbidden bands) appear. The effect of narrowing is strongest for the lowest and the highest bands. As we will see later, the temperature of Bose–Einstein condensation is determined mainly by the width of the lowest band. Qualitatively, the dependence of the width of bands on τ is explained as follows. At small τ it is energetically preferable for the vacancies to localize inside segments, and the central sites play the role of the barriers with a small transparency. As is known, in particular, from the Kronig–Penney model (see, for example, Ref. 12), in such a situation there is a sharp narrowing of the lowest band. At $\tau > \tau_c$ it is energetically preferable for the vacancies to localize at central sites, and overlapping between such localized states becomes very small.

To obtain quantitative estimation for the Bose–Einstein condensation temperatures it is necessary to find the spectrum of the bands for arbitrary q. It follows from the definition of $\Delta_q(\widetilde{\varepsilon})$ that this function satisfies the recurrent relation

$$\Delta_q(\widetilde{\varepsilon}) = \widetilde{\varepsilon}\Delta_{q-1}(\widetilde{\varepsilon}) - \Delta_{q-2}(\widetilde{\varepsilon}) \tag{12}$$

($\Delta_1(\widetilde{\varepsilon}) = \widetilde{\varepsilon}$, $\Delta_2(\widetilde{\varepsilon}) = \widetilde{\varepsilon}^2 - 1$). Using the relation (12) and applying the method of mathematical induction one can prove that

$$\Delta_q(2\cos\gamma) = \frac{\sin[(q+1)\gamma]}{\sin\gamma} \tag{13}$$

and

$$\Delta_q(\pm 2\cosh\gamma) = (\pm 1)^q \frac{\sinh[(q+1)\gamma]}{\sinh\gamma}. \tag{14}$$

Equations (13) and (14) allow to reduce the Eqs. (9) and (11) to compact trigonometrical equations. In particular, the substitution $\widetilde{\varepsilon} = 2\cos\gamma$ reduces Eq. (9) to the equation $\sin q\gamma = 0$ (with the additional condition $\sin\gamma \neq 0$) which solutions correspond to the energies (10).

For finding the spectrum we use the substitution $\widetilde{\varepsilon} = 2\cos(\pi + \gamma)$ that reduces Eq. (11) to the following equation for γ:

$$\cot\gamma \sin(q\gamma)(1-3\tau^2) + 3\tau^2\left[\cos(q\gamma) - \frac{1}{3}\sum_\alpha \cos(k_\alpha l)\right] = 0. \tag{15}$$

At $\tau = \tau_c = 1/\sqrt{3}$, Eq. (15) can be easily solved that yields the spectrum

$$\varepsilon_j(\mathbf{k}) = -2t\cos\left(\frac{2\pi}{q}\left[\frac{j}{2}\right] - (-1)^j \frac{1}{q}\arccos\frac{\sum_\alpha \cos(k_\alpha l)}{3}\right) \tag{16}$$

($j = 1, 2, \ldots q$), where square brackets indicate the integer part. One can see from (16) that at any q the bandgaps are equal to zero. We display also approximate expression for the half-width of lower bands

$$W_j \approx \frac{t\pi^2}{q^2}\left(j - \frac{1}{2}\right) \tag{17}$$

($j << q$). As follows from (17), the half-widths are in inverse proportion with the square of length of the segment (at fixed a).

At $\tau < \tau_c$ one can find an approximate solutions of Eq. (15) in a diapason of small γ (that correspond to the lower bands). We specify the case of τ not too close to τ_c, when the condition $(1-3\tau^2)q >> 1$ is satisfied. The solution of Eq. (15) can sought as $\gamma = \gamma_j + \widetilde{\gamma}_j$, where $\gamma_j = \pi j / q$ ($j << q$) and $\widetilde{\gamma}_j << \gamma_j$. The approximate expression for $\widetilde{\gamma}_j$ as obtained from (15) reads as

$$\widetilde{\gamma}_j \approx -\frac{3\tau^2}{1-3\tau^2}\frac{\pi j}{q^2}\left[1 - \frac{(-1)^j}{3}\sum_\alpha \cos(k_\alpha l)\right]. \tag{18}$$

It gives the following expression for the spectrum of lower bands

$$\varepsilon_j(\mathbf{k}) \approx -2t\left\{\cos\frac{\pi j}{q} + \frac{\pi^2 j^2}{q^3}\frac{3\tau^2}{1-3\tau^2}\left[1 - \frac{(-1)^j}{3}\sum_\alpha \cos(k_\alpha l)\right]\right\}. \tag{19}$$

In this case the half-width of the lowest band is equal to

$$W_1 \approx \frac{2t\pi^2}{q^3}\frac{3\tau^2}{1-3\tau^2}, \tag{20}$$

i.e., it is inversely proportional the third power of the length of the segment. We note that in the Kronig–Penney model at small transparency of the barriers the width of the bottom band is also inversely proportional to the cube of distance between the barriers (see, for instance, [12]).

If τ exceeds τ_c (and $q >> 1$), a sharp narrowing of the lowest band takes place, and this band drops below the level $-2t$. Using the substitution $\widetilde{\varepsilon} = -2\cosh\gamma$, we obtain from (11) the following equation

$$\coth\gamma \sinh(q\gamma)(1-3\tau^2) + 3\tau^2\left[\cosh(q\gamma) - \frac{1}{3}\sum_\alpha \cos(k_\alpha l)\right] = 0. \tag{21}$$

At $q(3\tau^2 - 1) >> 1$ it is convenient to seek for a solution of Eq. (21) in the form $\gamma = \gamma_0 + \widetilde{\gamma}$, where γ_0 is given by the equation

$$\coth\gamma_0 = \frac{3\tau^2}{3\tau^2 - 1} \tag{22}$$

(i.e. $\gamma_0 = \ln(6\tau^2 - 1)/2$), and an exponentially small correction $\widetilde{\gamma}$ can be obtained directly from (21):

$$\widetilde{\gamma} \approx \frac{2}{3}\sinh\gamma_0 \cosh\gamma_0 \, e^{-q\gamma_0}\sum_\alpha \cos(k_\alpha l). \tag{23}$$

The spectrum of the lowest band reads as

$$\varepsilon_1(\mathbf{k}) \approx -E_0\left[1 + \frac{2\sinh^2\gamma_0}{3} e^{-q\gamma_0}\sum_\alpha \cos(k_\alpha l)\right], \tag{24}$$

where

$$E_0 = 2t\cosh\gamma_0 = 2t\frac{3\tau^2}{\sqrt{6\tau^2 - 1}}.$$

Thus, the lowest band becomes exponentially narrow with the half-width

$$W_1 \approx 4t \cosh \gamma_0 \sinh^2 \gamma_0 e^{-q\gamma_0}.$$

All the others bands lay above the level $-2t$. The spectra for the lower bands (starting from $j=2$) are determined by the Eq. (19) if one replaces j for $j+1$ in its left hand part. The eigenfunctions for the lowest band have the maximum weight in the central site and it falls quickly with the distance from the central site.

4. Temperature of Bose–Einstein condensation

Due to the presence of degenerate levels in the spectrum the Eq. (1) for the temperature of Bose–Einstein condensation (BEC) is modified to

$$N = \sum_{j=1}^{q} \sum_{\mathbf{k}} \frac{1}{\exp\left(\dfrac{\varepsilon_j(\mathbf{k})-\mu_0}{T_0}\right)-1} + 2N_i \sum_{s=1}^{q-1} \frac{1}{\exp\left(\dfrac{\varepsilon_s-\mu_0}{T_0}\right)-1}. \tag{25}$$

We are interested in systems with a rather large concentration of vacancies ($n >> l^{-3}$). In this case the BEC temperature is much larger than the width of the lowest band. It allows to use the approximate expression for the Bose distribution function for such a band

$$\left[\exp\left(\frac{\varepsilon_1(\mathbf{k})-\mu_0}{T_0}\right)-1\right]^{-1} \approx \frac{T_0}{\varepsilon_1(\mathbf{k})-\mu_0}.$$

The rest bands and degenerate levels give non-negligible contribution into (25), only if their energies, counted from a bottom of the lowest band, are of order or less than the BEC temperature. The main contribution into (25) yields the lowest band. Therefore, for evaluation of what bands and levels should be taken into account, one can use the estimate $T_0 \sim W_1 n l^3$.

Let us first consider $\tau < \tau_c$ (and $q >> 1/(1-\tau^2/\tau_c^2)$. In this range of the parameters the half-width of the lowest band $W_1 \propto q^{-3}$, while the energy gap between the first and second band $\Delta_{g,1} \propto q^{-2}$. At small filling factors ν the three-dimensional concentration satisfies the condition $nl^3 = 3q\nu << q$. Hence, $T_0 << \Delta_{g,1}$ and it is enough to take into account in Eq. (25) only the lowest band and the lowest degenerate level (that lies at the top of the lowest band). As a result, Eq. (25) is reduced to

$$nl^3 \approx \frac{1}{\pi^3} \int_0^{\pi}\int_0^{\pi}\int_0^{\pi} d\tilde{k}_x d\tilde{k}_y d\tilde{k}_z \frac{T_0}{W_1\left[1-\dfrac{1}{3}(\cos\tilde{k}_x+\cos\tilde{k}_y+\cos\tilde{k}_z)\right]} + \frac{T_0}{W_1} \approx 2.5\frac{T_0}{W_1}. \tag{26}$$

It is instructive to write down the answer for the BEC temperature in terms of one-dimensional concentration of vacancies n_1, the distance between nearest sites a, and the length of the segment l:

$$T_0 = 1.2\pi^2 \frac{3\tau^2}{1-3\tau^2} 2ta^2 \frac{n_1 a}{l^2}. \tag{27}$$

At $\tau \approx \tau_c$ the energy gaps between the bands approach zero and one should take into account many bands and levels. Since the main contribution into (25) is given by the levels and the bands, which energies counted from the bottom of the lowest band are smaller than T_0, it is enough to take into account only such levels and bands. Besides, it is possible to approximate all bands, except the first one, by the degenerate levels (with degree of degeneracy N_i), located between degenerate levels (10). As a result, Eq. (25) is reduced to

$$nl^3 \approx \frac{T_0}{W_1}\frac{1}{\pi^3} \int_0^{\pi}\int_0^{\pi}\int_0^{\pi} d\tilde{k}_x d\tilde{k}_y d\tilde{k}_z \frac{1}{2\left[\dfrac{1}{\pi}\arccos\left(\dfrac{1}{3}\sum_{\alpha}\cos\tilde{k}_\alpha\right)\right]^2} + 2\sum_{s=1}^{s_{\max}} \frac{T_0}{2W_1 s^2} + \sum_{j=1}^{j_{\max}} \frac{T_0}{2W_1(j+\frac{1}{2})^2}, \tag{28}$$

where $s_{\max}$ and $j_{\max}$ are numbers of levels and bands, which energy is of order of T_0. In view of fast convergence of the sums in Eq. (28) they can be extended to infinity. The integral in Eq. (28) can be evaluated numerically. As a result, we obtain $T_0 \approx 0.2 W_1 n l^3$. Using (17) we find

$$T_0 \approx 3ta^2 \frac{n_1}{l}. \tag{29}$$

At $\tau > \tau_c$ the energy gap that separates the lowest band from the lowest level and the second band is exponentially large in comparison with the width of the bottom band. Hence, it is enough to consider in (25) only the lowest band. It gives $T_0 \approx 0.7 W_1 n l^3$. At large $q = l/a$ this temperature is exponentially small. For example, at $\tau = 1$ the BEC temperature is equal to

$$T_0 \approx 9tn_1 l e^{-l \ln 5/2a}. \tag{30}$$

We should note that since the states in the lowest band correspond to the vacancies localized near the central site, the result (30) is more sensitive to the interaction than the results for other τ. Due to the tendency to localization at central sites, the interaction can be neglected, only if number of vacancies per elementary cell is less or of order of unity. Therefore, the result (30) should be considered as qualitative one.

5. Conclusion

The model of decorated lattice considered in this paper describes three physically distinct situations, depending on value of τ. The case of small τ corresponds the situation, when intersections of dislocations play the role of barriers with a small transparency for the vacancies. In this case the BEC temperature is inversely proportional to the square length of the segment. The case $\tau > \tau_c$ describes the situation, where the vacancies tend to localize at the dislocation intersections. In such a situation the BEC temperature decreases exponentially with the increase of the length of the segment. At last, the case $\tau \approx \tau_c$ corresponds the situation where vacancies can move freely through the intersections, and do not localize at them. The latter situation is the most preferable for the BEC: only linear decrease of the temperature of transition with the increase of the length of the segment takes place. We consider that the functional dependences of the BEC temperature on the length of the segment are more or less universal and do not depend on mechanisms that cause the appearance of the barriers (or the centers of localization) on the dislocation intersections.

It is of interesting to estimate BEC temperature for the most preferable case (29). If one defines the effective mass of vacancies on a simple (not decorated) cubic lattice $M^* = \hbar^2 / 2ta^2$, the result (29) can be presented in the form

$$T_0 \approx \frac{3}{2} \frac{\hbar^2}{M^*} \frac{n_1}{l}. \tag{31}$$

The answer (31) up to the numerical factor of order of unity coincides with the estimate given in Ref. 11. As was shown in [9] by numerical simulation, the superfluid part of dislocation contains a number of lattice sites in its cross-section. Therefore, a more realistic model of superfluid dislocation is not a single lattice chain but a bunch of chains. The Eq. (31) does not contain any parameters of the decorated lattice and it can be applied for general model of superfluid dislocation. In the latter case the quantity n_1 should be understood as linear concentration per dislocation. Implying that the effective mass of vacancies is approximately equal to the mass of ^{4}He atom, we evaluate (31) as $T_0 \approx (n_1 / l) \cdot 18$ K·Å^2. At linear concentration of vacancies $n_1 = 1$ Å^{-1} (a value obtained in [9]) the temperature $T_0 \approx 0.1$ K is reached for the length of the segment $l = 180$ Å, that corresponds to two-dimensional density of dislocations $n_d = 3 \cdot 10^{11}$ cm^{-2}. This is the upper estimate for T_0 and it shows that in real situation the BEC temperature is rather low.

In conclusion, we discuss shortly the question on the relation between the Bose–Einstein condensation and the superfluidity of vacancies in the network of dislocations. At temperatures lower than the BEC temperature the long-range phase correlations are established in the system. In our case the phases on different segments become correlated. It allows to describe the system (below T_0) by a complex order parameter that reflects the possibility of nondissipative flow along the dislocations. (An interaction between the vacancies, necessary for superfluidity, is always presented in real systems. We just imply this interaction to be repulsive and rather small.) Above T_0 the flowing without relaxation is impossible. But the specifics of the system studied consists in that the temperature T_0 and the temperature of degeneracy for the gas of vacancies on a segment T_d are quite different: $T_d = tn_1^2 a^2 >> T_0$. Therefore, the relaxation time for the flow at $T_d >> T > T_0$ can be very large, and above T_0 a peculiar quasi-superfluid phase can be realized (see Ref. 11, and also discussion in Ref. 9). Let us note that in the experiment [13] the attempt of direct observation of Bose–Einstein condensation in solid helium was made (through the measurements of temperature dependence of the kinetic energy per atom). According to the results of Ref. 13, in the temperature range where nonclassical rotational inertia is observed (measurements were done down to 0.07 K), the Bose–Einstein condensate does not arise yet.

The authors would like to thank L.A. Pastur for helpful discussions. This study was supported in part by the CRDF grant No 2853.

1. E. Kim and M.H.W. Chan, *Nature* **427**, 225 (2004); *Science* **305**, 1941 (2004); *Phys. Rev. Lett*. **97**, 115302 (2006).
2. A.S.C. Rittner and J.D.Reppy, *Phys. Rev. Lett*. **97**, 165301 (2006).
3. M. Kondo, S. Takada, Y. Shibayama, and K. Shirahama, *J. Low Temp. Phys.* **148**, 695 (2007).
4. A. Penzev, Y. Yasuta, and M. Kubota, *J. Low Temp. Phys.* **148**, 677 (2007).
5. A.F. Andreev and I.M. Lifshitz, *Zh. Exp. Teor. Fiz.* **56**, 2057 (1969) [*Sov. Phys. JETP* **29**, 1107 (1969)].
6. M.W. Meisel, *Physica* **B178**, 121 (1992).
7. B.A. Fraass, P.R. Granfors, and R.O. Simmons, *Phys. Rev.* **B39**, 124 (1989).
8. L. Pollet, M. Boninsegni, A.B. Kuklov, N.V. Prokof'ev, B.V. Svistunov, and M. Troyer, *Phys. Rev. Lett*. **98**, 135301 (2007).

9. M. Boninsegni, A.B. Kuklov, L. Pollet, N.V. Prokof'ev, B.V. Svistunov, and M. Troyer, *Phys. Rev. Lett*. **99**, 035301 (2007).
10. S.I. Shevchenko, *Fiz. Nizk. Temp.* **13**, 115 (1987) [*Sov. J. Low Temp. Phys.* **13**, 61 (1987)].
11. S.I. Shevchenko, *Fiz. Nizk. Temp.* **14**, 1101 (1988) [*Sov. J. Low Temp. Phys.* **14**, 553 (1988)].
12. J.H. Davies, *The Physics of Low-Dimensional Semiconductors: an Introduction*, Cambridge University Press (1997).
13. M.A. Adams, J. Mayers, O. Kirichek, and R.B.E. Down, *Phys. Rev. Lett*. **98**, 085301 (2007).