SUBENSEMBLE CONCEPT IN 2D MAGNETIZED PARTICLE TRANSPORT MODEL

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Transport of magnetized particle undergoing random frozen isotropic electrostatic field is considered. Because of infinitely long field correlation time and particle trapping this problem is of particular interest as a test for closure of statistical equations. The concept of subensembles is incorporated here in the analytical approach we developed earlier. The Lagrangian velocity correlation function is calculated in the drift approximation. To verify validity of the analytical method the results are compared with ones found from a direct numerical simulation. A better quantitative agreement obtained with the use of subensemble concept is shown.

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INTRODUCTION

Understanding of basic mechanisms of particle transport is important for many problems in plasma physics as well as for prediction of impurity migration in various media with random velocity fields such as the atmosphere and the ocean. Description of turbulent transport can be carried out on different levels – from phenomenological models to comprehensive numerical simulation on a basis of kinetic equations. The most direct method to formulate transport equations is statistical averaging of equations of microscopic particle motion in random fields. On this way arise a problem of closure, i.e. expression of higher statistical moments via lower ones, inherited from a statistical nonlinearity of basic equations. There is no regular procedure to make this closure in general case. A closure depends on a particular problem and is a key element in the formulation of statistical approach.

To describe particles in a random field of force or velocity characterized by small Lagrangian correlation time, and respectively small Kubo number, the known Corrsin approximation can be applied. However for trapped resonance particles which for a long time interact with a locally regular field the Corrsin approximation cannot be used. These particles play an important role in transport processes and for their statistical description other closure should be found.

In this paper a spread of magnetized particles in a random electrostatic field is considered. The slower is a field variation over time the more apparent becomes the effect of their trapping. The effect is most pronounced in the limit of frozen turbulence, i.e. when fields are constant in time. Respectively this case is the most difficult for the theoretical description when the result of the calculations is very sensitive to a method of closure. Thus recovering of particle spread in a frozen random field with infinite trapping time is a test for closure procedure. If the procedure is found to be justified for a frozen field, the more reliable it can be applied to a field varying in time.

Particle diffusion caused by external fields was studied analytically and numerically in a number of works. However analytical description of particle trapping effect is not properly built. Relatively new semianalytical approach to statistical description of magnetized particles in external field of electrostatic random waves was given in the works [1] where the authors proposed and developed the decorrelation trajectories method. The main elements of this method are the assumption about subensembles – groups of particles with similar initial conditions that are characterized by specific behavior, and the closure procedure formulated for partial diffusion coefficients in subensembles.

In our paper [2] a different closure procedure based on the equation for mean square displacement of particles was proposed. In the paper [3] the concept of particle subensembles formed in accordance with values of potential and velocity in the initial particle positions was analyzed in detail, and effectiveness of two methods of closure was compared. Our closure gives a qualitative agreement with simulation without use of fitting parameters. Specific behavior was observed in simulation for groups of particles (subensembles) with the same value of potential in the initial particle positions. On contrary for subensembles arranged by the initial particle velocities no specific behavior was observed in simulation.

In this paper we combine elements of both approaches. These are splitting of a particle ensemble on subensembles labeled by a value of potential, and closure on mean square particle displacement. As before no free parameters will be used. Note that a rigorous proof of closure procedure is a complex problem and remains beyond a scope of this work. Nevertheless, some particular steps in a course of formulation of the description as well as the final results were checked by direct numerical simulation. Although the transition to continuum in simulation remains a problem, in other aspects the model for numerical simulation is adjusted with the analytical one.

1. MODEL

We consider motion of strongly magnetized particles in a random electrostatic field. The electric field is statistically uniform and isotropic. To focus on a particle trapping effect we consider frozen fields. Particle motion will be described in a drift approximation neglecting the effects of finite Larmor radius, and this enhances a trapping effect.

A simple model allows establishing links between the assumptions and the results, avoiding interference with less important factors. Isotropy at this stage helps to avoid mutual influence of diffusion in different directions. The effects of the finite Larmor radius are important and will be considered elsewhere.

Drift motion of magnetized particles in constant magnetic and electric fields is governed by the equations

$$v_x = -\frac{\partial}{\partial y}\varphi(\mathbf{r}), \quad v_y = \frac{\partial}{\partial x}\varphi(\mathbf{r}). \qquad (1)$$

With accuracy to a coefficient a potential is a stream function $\varphi(\mathbf{r})$.

Using the Kraichnan approach [4] the equation for distribution function of particles *F(**r**,t)* averaged over realizations of random field $\varphi(\mathbf{r})$ can be obtained in a form

$$F(\mathbf{r},t) = \int d\mathbf{r}' W(\mathbf{r},t;\mathbf{r}',0)\, F(\mathbf{r}',0), \qquad (2)$$

where $W(\mathbf{r}, t, \mathbf{r}', t')$ is the averaged transition probability between two points of coordinate space

$$\frac{\partial}{\partial t} W(\mathbf{r},t;\mathbf{r}',t')$$
$$= \frac{\partial}{\partial r_i}\int_{t'}^{t} d\tau \int d\boldsymbol{\rho}\, W(\mathbf{r},t;\boldsymbol{\rho},\tau)\langle V_i(\mathbf{r})V_j(\boldsymbol{\rho})\rangle \qquad (3)$$
$$\times \frac{\partial}{\partial r'_j} W(\boldsymbol{\rho},\tau;\mathbf{r}',t'),$$

$$W(\mathbf{r},t;\mathbf{r}',t) = \delta(\mathbf{r}-\mathbf{r}'). \qquad (4)$$

It is problematic to find a solution of the nonlinear integro-differential equation (3) for $W(\mathbf{r}, t, \mathbf{r}', t')$ even numerically; therefore it requires a reduction to a simpler form. Along with this a choice of Eq. (3) as a basic one – instead of immediate use of a reduced equation (will be given later) – makes evident what assumptions should be checked out in a course of reduction. Verification of the assumptions can be carried out by means of numerical simulation.

One of the important steps concerns the transition from the nonlocal description to partially local one. For a problem with trapped particles such transition is not obvious, and without proper justification it looks doubtful.

Under conditions of spatial and temporal locality there are integral relations between the Lagrangian correlation function, the diffusion coefficient, and the mean square displacement. We found the Lagrangian velocity correlation function and the mean square displacement from direct simulation. It was shown that by double numerical integration of the Lagrangian correlation function the mean square displacement is recovered with sufficient accuracy. This means that the Taylor relation between the diffusion coefficient *D(t)* and the Lagrangian velocity correlation function $V_L(t)$ can be applied in the case of trapped particles under consideration

$$D(t) = \int_0^t V_L(\tau)\, d\tau. \qquad (5)$$

The Lagrangian velocity correlation function $V_L(t)$ can be given through the Eulerian correlation function $V_E(r^2)$, which for a frozen uniform isotropic field depends only on a distance between two points, and the transition probability

$$V_L(t) = \int d\mathbf{r}\, W(\mathbf{r},t;\mathbf{0},0)\, V_E(r^2),$$
$$V_E(r^2) = \langle V_x(\mathbf{r})V_x(\mathbf{0})\rangle + \langle V_y(\mathbf{r})V_y(\mathbf{0})\rangle. \qquad (6)$$

To calculate the Lagrangian correlation function $V_L(t)$ according to Eq. (6) we have to know the solution of Eqs. (3). As far as it is unknown we need some assumption to establish relation between the Eulerian and Lagrangian correlation functions.

To formulate this assumption we were guided by the following considerations. In the lowest zero approximation a transition from the Eulerian correlation function to the Lagrangian one may be performed by coordinate transformation to the reference frame of free particle. In the problem under consideration in absence of electric field there is no drift motion, so the transition probability is zero. Nevertheless, we will use this analogy taking into account that a particle displacement in the lowest order of approximation is caused by a random field.

Using Eqs. (3), (5), and (6) we obtain for the second moment of the transition probability $\langle r^2 \rangle$

$$\langle r^2 \rangle = 2\int_0^t D(t)\, dt. \qquad (7)$$

The closure proposed for the problem is given by the relation

$$V_L(t) = V_E(\langle r^2(t)\rangle). \qquad (8)$$

Eqs. (5), (7), and (8) makes a closed set of equations and can be solved numerically. A satisfactory quantitative agreement of the results obtained from this approach and direct simulation of particle motion is shown in papers [2, 3].

2. SUBENSEMBLES

The method can be improved by more detailed description of particle motion. For this we take into account that their motion governed by Eq. (1) occurs along the equipotential lines $\varphi(\mathbf{r}) = \text{const}$. Simulation shows a specific behavior of different groups of particles which belong to equipotential surfaces of different levels [3]. The difference is noticeable between individual particle trajectories; they are shorter at peaks or hollows of potential and longer in valleys. As well a statistical analysis shows that the mean square displacement of particles in subensembles grows, and after some time comes to saturation. Distinction between subensembles is manifested by difference in saturation times of mean square displacement and also its levels. Kurtosises of particle displacement in subensembles are different as well. Along with this, no specific behavior for groups of particles with different initial velocities was found in numerical simulation.

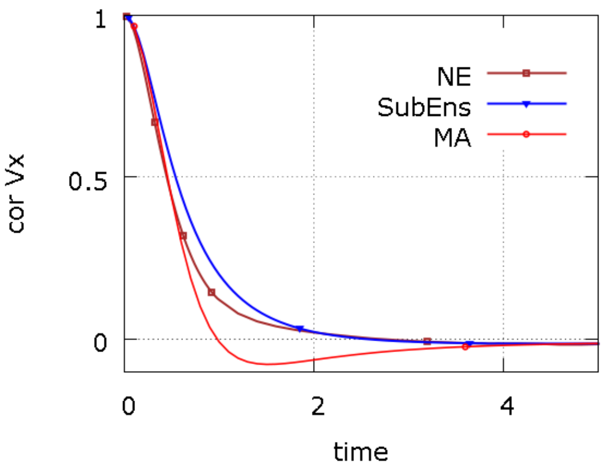

Fig. 1. The Lagrangian velocity correlation function found from numerical simulation (NE), in the basic model without splitting on subensembles (MA), and in the model with subensembles (SubEns)

Amplitude of the partial Euler correlation function in each subensemble is determined by the level of the equipotential surface. The contribution of subensembles to statistical characteristics of the whole ensemble of particles is proportional to their weights. According to the central limit theorem distribution of particle over subensembles is Gaussian. Direct numerical test shows that for finite number of particles in numerical simulation the Gaussian distribution over initial values of potential is a good approximation.

It should be noted that the form of Lagrangian correlation functions obtained from direct simulation show diversity for those subensembles which are characterized by a large absolute value of a potential, close to the maximum. These subensembles consist from strongly trapped particles which are moving on contour lines of small size. Thus because of a small dispersion and relatively small number of strongly trapped particles distinction of their contribution can be neglected.

3. RESULTS

The random field $\varphi(\mathbf{r})$ in numerical simulation was taken as a superposition of harmonics with N_k wave numbers oriented over N_ϕ directions, and with random phases a_i and a_j

$$\varphi(\mathbf{r}) = \varphi(r,\theta) = \sum_{i=1}^{N_k}\sum_{j=1}^{N_\phi} \varphi_i \cos\left(k_i r \cos\left(\phi_j - \theta\right) + \alpha_i + \alpha_j\right). \quad (9)$$

The wave intensity is distributed over wave numbers according to

$$\varphi_i^2 = \frac{2}{\sqrt{\pi}} \frac{\varphi_0^2}{N_k N_\phi} \frac{k_{\max}}{\Delta k} \exp\left(-\frac{k_i^2}{\Delta k^2}\right).$$

Then corresponding Eulerian correlation function is of the form

$$\langle \varphi(\mathbf{r})\varphi(\mathbf{0})\rangle = \varphi_0^2 \exp(-p) I_0(p), \quad p = \frac{1}{8}\Delta k^2 r^2, \quad (10)$$

where I_0 is the modified Bessel function. As it was mentioned the distribution of amplitude φ_0 over subensembles is Gaussian.

Velocity correlation function is obtained by double differentiation of the potential correlation function over coordinates in a two-dimensional space

$$V_E(r^2) = -\Delta\langle \varphi(\mathbf{r})\varphi(\mathbf{0})\rangle . \quad (11)$$

Eqs. (10), (11) gives the Eulerian velocity correlation function for Eqs. (5), (7), (8). This system of equations was solved numerically. In results the Lagrangian velocity correlation function, the running diffusion coefficient and evolution of the particle mean square displacement were obtained. On the other hand the Lagrangian velocity correlation function can be found from a direct simulation of particle motion governed by Eqs. (1), (9).

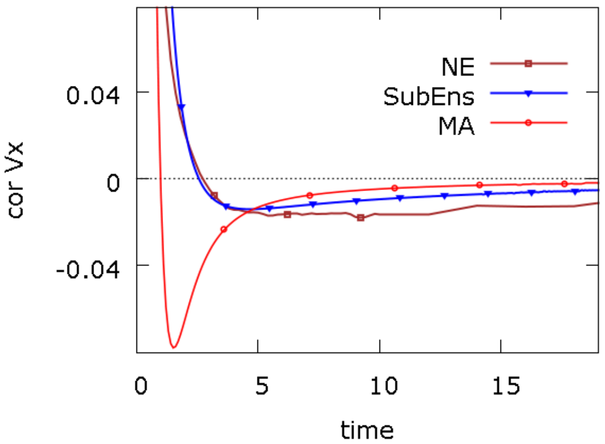

Fig. 2. The same in the other scale

In Fig. 1 the Lagrangian velocity correlation functions found from numerical simulation, in the basic model without splitting on subensembles, and in the model which account for specific particle motion in subensembles are compared.

The effect of particle trapping observed in numerical simulation is reflected by the negative values of the Lagrangian correlation function on a large time interval. As could be seen the correlation decreases in time but remains finite and tends slowly to zero. In particular realization of a frozen field particle motion is completely deterministic, thus the correlation time goes to infinity. These features are reflected by the basic model; however it gives an exaggerated negative value of the velocity correlation function. Calculations based on the subensemble concept improve the quantitative agreement with the results of numerical simulation as it is shown in different scale in Fig. 2.

The proposed model allows generalization to a variable random field with a finite correlation time. It also can be expanded to account for the effects of the finite Larmor radius.

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КОНЦЕПЦИЯ ПОДАНСАМБЛЕЙ В ДВУМЕРНОЙ МОДЕЛИ ПЕРЕНОСА ЗАМАГНИЧЕННЫХ ЧАСТИЦ

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Рассмотрен перенос замагниченных частиц под действием случайного замороженного изотропного электростатического поля. Из-за бесконечно большого времени корреляции поля и захвата частиц эта проблема имеет особый интерес для проверки замыкания статистических уравнений. Аналитический подход, развитый нами ранее, дополнен концепцией подансамблей. Рассчитана лагранжева корреляционная функция скорости в дрейфовом приближении. Для проверки достоверности аналитического метода его предсказания сравниваются с результатами прямого численного моделирования. Показано, что использование концепции подансамблей улучшает количественное согласование результатов.

КОНЦЕПЦІЯ ПІДАНСАМБЛІВ У ДВОВИМІРНІЙ МОДЕЛІ ПЕРЕНЕСЕННЯ ЗАМАГНІЧЕНИХ ЧАСТИНОК

В.І. Засенко, О.М. Черняк, А.Г. Загородній

Розглянуто перенесення замагнічених частинок під дією випадкового замороженого ізотропного електростатичного поля. Через нескінченно довгий час кореляції поля та захоплення частинок ця проблема має особливий інтерес для перевірки замикання статистичних рівнянь. Аналітичний підхід, розвинутий нами раніше, доповнено концепцією піддансамблів. Розраховано лагранжеву кореляційну функцію швидкості в дрейфовому наближенні. Для перевірки достовірності аналітичного методу його передбачення порівнюються з результатами прямого числового моделювання. Показано, що використання концепції піддансамблів поліпшує кількісне узгодження результатів.