LOWER HYBRID RESONANCE: FIELD STRUCTURE AND NUMERICAL MODELING

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The behavior the electromagnetic fields in the vicinity of the lower hybrid resonance point is studied in case of 1D plasma non-uniformity. The first of two found solutions of Maxwell's equations is singular and describes the wave travelling to the lower hybrid resonance layer. This wave if fully absorbed without reflections. Another solution which is regular describes the standing wave. To extend the range of validity of the solutions found, they are matched to the WKB solutions. Three possibilities for numerical solving the wave propagation problem in presence of the lower hybrid resonance zone are discussed in the paper.

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INTRODUCTION

The upper and lower hybrid resonances (UHR and LHR) appear in cold magnetized plasma when the perpendicular diagonal component $\varepsilon_\perp = \mathbf{e}_\perp \cdot \hat{\varepsilon} \cdot \mathbf{e}_\perp$ of the dielectric tensor $\hat{\varepsilon}$ nullifies (here $\mathbf{e}_\perp$ is a unitary vector perpendicular to the steady magnetic field). In case of LHR the WKB solutions predict a regular behavior of the fast magnetosonic wave (FMSW). The wave number of the slow wave (SW) diverges on approach to the LHR layer.

The LHR phenomenon is a base for the lower hybrid heating and current drive. The mode conversion scenario of the minority heating also includes the LHR mechanism for the wave absorption. In a standard minority heating scenario the LHR appears at the plasma periphery, and its role in wave propagation and power balance is not yet studied sufficiently.

In hot plasma in LHR zone, the slow wave converts into ion Bernstein wave. In cases of radio-frequency discharge start-up or a wall conditioning discharge the ions are cold and the wavelength of ion Bernstein wave becomes extremely short. Under such conditions, it is expedient to treat LHR without account of wave conversion.

Presence of the singularity hampers a numerical modeling of wave propagation in plasma when a LHR exists in the calculation domain. A simplest way to proceed is usage of the penalty method in which the singularity in the LHR point is avoided by adding locally an artificial imaginary part to $\varepsilon_\perp$. A more rigorous option is usage of the analytical solutions in the LHR area. The analytical continuation of the Maxwell's equations to the complex plane is the most rigorous approach.

SW FIELD STRUCTURE AT LHR VICINITY

The problem is considered in slab geometry with non-uniformity of plasma along the x coordinate. The magnetic field is directed along z. Using smallness of two parameters $\alpha = \sqrt{k_z^2, k_y^2, k_0^2|\varepsilon_\perp|, k_0^2|g|} / |k_x| << 1$ and $\beta = \sqrt{|\varepsilon_\perp| / |\varepsilon_\parallel|} << 1$, one can obtain an equation for the slow wave [1] from Maxwell's equations:

$$\frac{d}{dx}\left[\frac{1}{\varepsilon_\parallel}\frac{d}{dx}\left(\varepsilon_\perp E_x\right)\right] + \frac{k_{sw}^2}{\varepsilon_\parallel}\left(\varepsilon_\perp E_x\right) = 0 \qquad (1)$$

with $g = -i\mathbf{e}_x \cdot \hat{\varepsilon} \cdot \mathbf{e}_y$, $\varepsilon_\parallel = \mathbf{e}_z \cdot \hat{\varepsilon} \cdot \mathbf{e}_z$, $k_{sw}^2 = -\frac{\varepsilon_\parallel}{\varepsilon_\perp}(k_\parallel^2 - k_0^2 \varepsilon_\perp)$, and in these terms, the dispersion equation for the slow wave is $k_\perp^2 = k_{sw}^2$. Other components of the electric field vector are $E_y = ik_y \int E_x dx$ and $E_z = \int \frac{i\left(k_z^2 - k_0^2 \varepsilon_\perp\right)}{k_z} E_x dx$. At the lower hybrid resonance $\varepsilon_\perp$=0 and k_{sw}=∞. For the propagating wave the WKB solution of this equation is

$$E_x = \frac{C_\pm \sqrt{|\varepsilon_\parallel|}}{\varepsilon_\perp \sqrt{k_{sw}}} \exp(\pm i \int k_{sw} dx) . \qquad (2)$$

Here and further the constants are denoted by C. Note here that $|E_x| \propto |\varepsilon_\perp|^{-3/4}$ and $|E_y|, |E_z| \propto |\varepsilon_\perp|^{-1/4}$. Thus, on approach to LHR point all components of the electric field increase. In the vicinity of LHR the above equation is approximated by

$$\frac{d^2}{dx^2} y + \frac{a}{x} y = 0 , \qquad (3)$$

where $y = \varepsilon_\perp E_x$, $a = -\left.\frac{\varepsilon_\parallel k_z^2}{\partial \varepsilon_\perp / \partial x}\right|_{x=0}$. Here it is assumed that the the LHR point is at x=0. Making substitution $s = \sqrt{x}$ and $v = y / s$, one can come to the Bessel equation for v.

$$\frac{d^2}{ds^2} v + \frac{1}{s}\frac{d}{ds} v + (4a - \frac{1}{s^2}) v = 0. \qquad (4)$$

The approximate solution of this equation is

$$E_x = \frac{1}{\sqrt{u}} Z_1 (2\sqrt{u}), \qquad (5)$$

where $u = ax$, $Z_1 = C_J J_1 + C_Y Y_1$ for u>0. If u<0 then $Z_1 = C_I I_1 + C_K K_1$. To match the solutions, the expansions of the solutions near the matching point, u=0, is made. Then

$$E_x^{(1)}(x \to 0) = \frac{C_Y}{\sqrt{u}} Y_1(2\sqrt{u}) \approx \approx \frac{C_Y 2}{\pi}\left(-\frac{1}{2u} + \frac{1}{2}\ln u + \gamma\right), \quad (6)$$

$$E_x^{(2)}(x \to 0) = \frac{C_J}{\sqrt{u}} J_1(2\sqrt{u}) \approx C_J. \quad (7)$$

Here γ = 0.57722 is Euler-Mascheroni constant. Continuation of expression for $E_x^{(1)}$ to x=-0 results in addition of imaginary unity due to logarithm. Addition of $-iE_x^{(2)}$ to the first solution compensates this. Finally

$$E_x^{(1)} = \begin{cases} \frac{\pi}{2\sqrt{u}}[Y_1(2\sqrt{u}) - iJ_1(2\sqrt{u})], & u > 0 \\ \frac{1}{\sqrt{-u}} K_1(2\sqrt{-u}), & u < 0, \end{cases} \quad (8)$$

$$E_x^{(2)} = \begin{cases} \frac{J_1(2\sqrt{u})}{\sqrt{u}}, & u > 0 \\ \frac{I_1(2\sqrt{-u})}{\sqrt{-u}}, & u < 0. \end{cases} \quad (9)$$

Other components are $E_y = ik_y \int E_x dx$ and $E_z = ik_z \int E_x dx$.

$$\int E_x^{(1)} dx = \frac{\pi}{2a} \begin{cases} -[Y_0(2\sqrt{u}) - iJ_0(2\sqrt{u})], & u > 0 \\ \frac{2}{\pi} K_0(2\sqrt{-u}), & u < 0, \end{cases} \quad (10)$$

$$\int E_x^{(2)} dx = -\frac{1}{a} \begin{cases} J_0(2\sqrt{u}), & u > 0 \\ I_0(2\sqrt{-u}), & u < 0. \end{cases} \quad (11)$$

The first solution has a logarithmic singularity that indicates on residual wave damping. This is confirmed by global behavior of the solution: At u>0 it describes a wave traveling from infinity to the point u=0. There is no reflected wave and, therefore, there is non-zero valued energy flux of negative sign (see also [2, 3]). At u<0 the solution represents a standing wave with zero energy flux. This means that the power is absorbed locally at the LHR point.

It is necessary to note that a more general problem is analyzed in Ref. 4. The authors obtain the solution in integral form which possibly may be reduced to the explicit form given here. Also Ref. 5 should be mentioned in this context.

VIDENING OF ZONE OF SW SOLUTION VALIDITY

The solutions found (8)-(11) are valid in a narrow zone |x|<<L. Here L is the characteristic scale of variation of the dielectric tensor components. Beyond this zone the WKB approximation comes to play. Within it, the electromagnetic field structure is given by formula (2) and the following formulas

$$E_y = \pm \frac{C_\pm k_y \sqrt{|\varepsilon_\parallel|}}{\varepsilon_\perp k_{sw}^2} \exp(\pm i \int k_{sw} dx), \quad (12)$$

$$E_z = \mp \operatorname{sgn} \varepsilon_\parallel \frac{C_\pm \sqrt{|k_{sw}|}}{k_z \sqrt{|\varepsilon_\parallel|}} \exp(\pm i \int k_{sw} dx). \quad (13)$$

To match the solution (8-11) with the WKB solution (2), (12), (13) which, in contrast to the analytical solution, accounts for not only single term in Laurent expansion of k_{sw}^2 over x, the solution (8)-(11) is modified so that its asymptotical behavior coincides with the WKB solution, but the accuracy approximation in the vicinity of the LHR point remains in the same frame as before. In general, the modification consists in replacing of the argument x by x+o(x) and any constant C by C+O(x). This procedure can be justified by a representation of transition from equation (1) to equation (3) as application of a similar procedure instead of omission of some small terms. Following this, the Bessel function arguments in (8-11) is substituted with

$$\Phi = \begin{cases} \int_0^x \sqrt{k_{sw}^2} dx, & x > 0 \\ -\int_0^x \sqrt{-k_{sw}^2} dx, & x < 0. \end{cases} \quad (14)$$

The modified solution can be represented in the form:

$$E_x^{(1,2)} = \frac{\Phi^{1/2} \sqrt{|\varepsilon_\parallel|}}{\varepsilon_\perp \sqrt{k_{sw}}} X_x^{(1,2)}, \quad (15)$$

where $X_x^{(1)} = \begin{cases} \frac{\pi}{2}[Y_1(\Phi) - iJ_1(\Phi)], & x > 0 \\ -K_1(\Phi), & x < 0, \end{cases}$

$$X_x^{(2)} = \begin{cases} J_1(\Phi), & x > 0 \\ I_1(\Phi), & x < 0, \end{cases}$$

$$E_y^{(1,2)} = ik_y \frac{\Phi^{1/2} \sqrt{|\varepsilon_\parallel|}}{\varepsilon_\perp k_{sw}^{3/2}} X_{yz}^{(1,2)}, \quad (16)$$

$$E_z^{(1,2)} = \frac{i(k_z^2 - k_0^2 \varepsilon_\perp)}{k_z} \frac{\Phi^{1/2} \sqrt{|\varepsilon_\parallel|}}{\varepsilon_\perp k_{sw}^{3/2}} X_{yz}^{(1,2)}, \quad (17)$$

where

$$X_{yz}^{(1)} = \begin{cases} -\frac{\pi}{2}[Y_0(\Phi) - iJ_0(\Phi)], & x > 0 \\ -K_0(\Phi), & x < 0, \end{cases}$$

$$X_{yz}^{(2)} = -\begin{cases} J_0(\Phi), & x > 0 \\ I_0(\Phi), & x < 0. \end{cases}$$

The obtained solutions at |x|<<1 behave similarly to above obtained analytical solution (note, that they have different amplitude). At WKB zone, they repeat WKB solutions. For high enough |k_z|, the vicinity of LHR and

WKB zones overlap, and the above solutions are valid everywhere.

FMSW FIELD STRUCTURE

The equation for FMSW can be obtained from the Maxwell's equations neglecting E_z. It can be written in the following self-conjugate form:

$$\frac{d}{dx}\left(A\frac{d}{dx}E_y\right)+BE_y=0, \qquad (18)$$

where $A=\frac{k_z^2-k_0^2\varepsilon_\perp}{k_y^2+k_z^2-k_0^2\varepsilon_\perp}$ and

$$B=k_0^2\varepsilon_\perp-\frac{k_0^4g^2}{k_y^2+k_z^2-k_0^2\varepsilon_\perp}-k_y\frac{d}{dx}\left(\frac{k_0^2g}{k_y^2+k_z^2-k_0^2\varepsilon_\perp}\right).$$

As mentioned before, the LHR point is a regular point for FMSW. If the LHR zone is narrow, the polynomial representation of the FMSW field is quite accurate. To find it, the Taylor expansion for the coefficients *A* and *B* should be made. To lowest order, the couple of the basic solutions is:

$$E_y^{(1)}=1-\frac{B|_{x=0}}{2A|_{x=0}}x^2,\quad E_y^{(2)}=x-\frac{A'|_{x=0}}{2A|_{x=0}}x^2. \qquad (19)$$

The *x* component of the electric field could be then found from the *x* component of Maxwell's equations.

NUMERICAL TREATMENT OF LHR

Numerical modeling that uses discretization with finite difference or finite element methods is hampered by singularities at LHR location. The matrix of the linear equations system which is produced by the discretization may be ill-conditioned. Even well-conditioned matrix may produce wrong results since the singularity cannot be reproduced by polynomial or other smooth functions.

The singularity disappears in natural way if the dissipative effects, such as binary collisions, are accounted for in the dielectric tensor. However, if the collision frequency is much less than the radio-frequency, too fine mesh is necessary to reproduce the electric fields.

One of the methods to fix the singularity is the penalty method which consists in adding an artificial imaginary part to $\varepsilon_\perp$ that covers the LHR vicinity only (see Ref. 1). The method is quite practical, but its apparent disadvantage is that it could result in some inaccuracy in the solution.

Another approach could be realized using the analytical solutions in the LHR zone. The solutions presented in preceding sections of the paper could be used and matched with the numerical solutions at the left and right margin of the LHR zone. This approach is more complicated than the penalty method both in programming and in amount of the calculations, but the accuracy expected should be higher. However, the accuracy of the calculations has an upper limit determined by the accuracy of the approximate analytical solutions.

A prospective method for treating LHR is the method of the analytical continuation. Following this algorithm, the coordinate *x* is extended to the complex plane. All quantities and functions in Maxwell's equations are assumed to be analytical. The path of integration is chosen so that it circumvents the LHR point. In this way the singularity is avoided. The method has an advantage of numerical convergence that is provided by application of discretization at the whole domain. This method is tested in the section below.

APPLICATION OF ANALYTICAL CONTINUATION TO MAXWELL'S EQUATIONS

Analytical continuation is implemented at 1D cylindrical code [6] to the radial coordinate. The coordinate is modified to $r^*=r+iu$. In the numerical experiments $u=C_{ac}L[1-(r-r_0)^2/L^2]^2$ at the segment $L\text{-}r_0<r<L+r_0$ and zero outside it. The plot of the modified radial coordinate is given in Fig. 1.

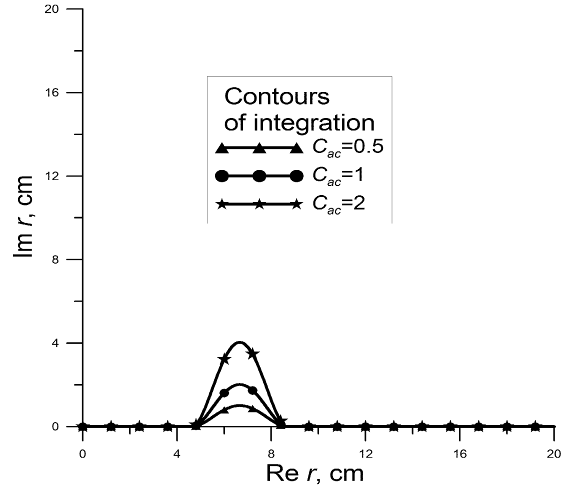

Fig. 1. Modified radial coordinate

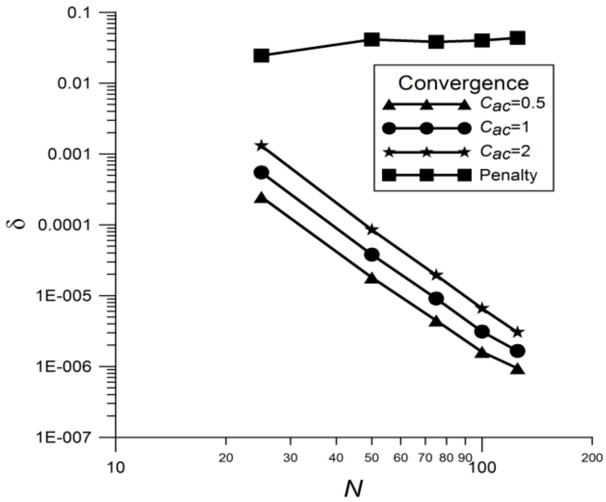

Fig. 2. The relative error in electric fields at the area where the integration contour is at real axis compared with the calculation without analytical continuation. The curve 'penalty' corresponds to applying the penalty method in the area

In the particular case under consideration the excursion of r^* to the complex plane is local and made around the regular point (no LHR point in the domain). The solution at the area where the excursion is made should depend on the integration path. But outside this area the solution should be independent on which excursion was made, and this is the point to check.

Fig. 2 shows the relative difference in electric fields at the area where the integration contour is at real axis compared with the calculation without analytical continuation. The difference quickly goes to zero when making the mesh denser. This calculation demonstrates applicability of the analytical continuation to the wave propagation problems.

CONCLUSIONS

The behavior the electromagnetic fields in the vicinity of the LHR point is studied in case of 1D plasma non-uniformity. Analytical solutions for slow wave are found in the LHR vicinity. The first of two solutions is singular and describes the wave travelling to the LHR layer. This wave if fully absorbed without reflections. Another solution which is regular describes the standing wave. To extend the range of validity of the solutions found, they are matched to the WKB solutions. The fast magnetosonic wave has no singularities in the LHR zone and could described with polynomials.

Presence of the singularity hampers a numerical modeling of wave propagation in plasma when a LHR exists in the calculation domain. A simplest way to proceed is usage of the penalty method in which the singularity in the LHR point is avoided by adding locally an artificial imaginary part to $\varepsilon_\perp$. A more rigorous option is usage of the analytical solutions in the LHR area. For the slow and fast waves, the above obtained solutions may be used. The analytical continuation of the Maxwell's equations to the complex plane is the most rigorous approach, and its applicability is checked using a numerical example.

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НИЖНИЙ ГИБРИДНЫЙ РЕЗОНАНС: СТРУКТУРА ПОЛЕЙ И ЧИСЛЕННОЕ МОДЕЛИРОВАНИЕ

В.Е. Моисеенко, T. Wauters, A. Lyssoivan

Поведение электромагнитных полей в непосредственной близости от нижнего гибридного резонанса изучено в случае одномерной неоднородности плазмы. Первое из двух найденных решений уравнений Максвелла сингулярно и описывает волну, падающую на слой нижнего гибридного резонанса. Эта волна полностью поглощается без отражений. Другое решение, которое регулярно, описывает стоячую волну. Чтобы расширить область применимости найденных решений, они приведены к виду ВКБ-решений. Обсуждаются три возможности для численного решения задачи распространения волн в присутствии зоны нижнего гибридного резонанса.

НИЖНІЙ ГІБРИДНИЙ РЕЗОНАНС: СТРУКТУРА ПОЛІВ І ЧИСЕЛЬНЕ МОДЕЛЮВАННЯ

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Поведінка електромагнітних полів в безпосередній близькості від нижнього гібридного резонансу вивчено в разі одновимірної неоднорідності плазми. Перше з двох знайдених рішень рівнянь Максвелла сингулярно і описує хвилю, що падає на шар нижнього гібридного резонансу. Ця хвиля повністю поглинається без відбиття. Інше рішення, яке регулярно, описує стоячу хвилю. Щоб розширити область застосовності знайдених рішень, вони приведені до вигляду ВКБ-рішень. Обговорюються три можливості для чисельного рішення задачі поширення хвиль у присутності зони нижнього гібридного резонансу.