SINGULAR SOLUTIONS AND DYNAMIC CHAOS

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It is shown that singular solutions of ordinary differential equations may cause new dynamic chaos conditions and new dynamic chaos modes. In particular, these solutions may lead to the dynamic chaos modes in a completely integrable system. An example of a physical system with dynamics significantly affected by the presence of singular solutions is analyzed. This example is the movement of particles in the central fields.

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INTRODUCTION

Singular solutions of differential equations are well known in mathematics. It is sufficient to point to the fact that the term "singular solutions" has been introduced by Joseph Louis Lagrange (1736-1813). In this paper, by the term "singular solutions" we assume the solutions of the system of ordinary differential equations (ODE) which have points of phase trajectories where the conditions of uniqueness are violated (non-Lipshitz condition). However, the analysis of mathematical models that describe physical processes, it seems, always imposes either explicit or implicit condition of uniqueness (author have not found cases which contradicts this). This condition excludes the singular solutions. As the example, we can provide the definition of the basic properties of the phase space, in which regular and stochastic dynamics of physical systems is investigated [1]: "Trajectories in phase space do not intersect at any particular moment of time…". This restriction immediately eliminates all singular solutions. Such attitude to the singular solutions appears, apparently, due to the fact that they do not correspond to our concept of specific solutions of physical problems. We also note that, according to the literature, quite detailed analyses of singular solutions were given by V.A. Steklov [2]. However, this book is very difficult to find. In [3] was a point out that the singular solutions can generate chaotic dynamics in systems with known analytical solutions for all the phase trajectories (for fully integrated systems). Thus, the presence of the analytical solutions for the phase trajectories does not define the regular dynamics of a system under investigation. It may be assumed that the models considered in [3] together with the numerous models with singular solutions in mathematics are interesting only from the mathematical point of view. Below, we show that such solutions may be also important for solving physical problems. In the second section of this publication some basic mathematical models with the dynamics determined by singular solutions are analyzed. It is shown that their dynamics is chaotic. It is important that such systems have only one degree of freedom and are fully integrable. Thus, the one of the main paradigms of the dynamic chaos theory, namely that the chaos can appear only in a system with more than one degree of freedom, breaks down. The third section contains the physical example, which shows the influence of singular solutions on the dynamics of the system. Such example is a dynamics of the particles (bodies) in the central field. It is shown that the singular solution of this problem significantly affects the dynamics in this field at some values of parameters. The fourth and final chapter summarizes the main results and highlights the features of chaotic modes that emerged as the result of accounting for singular solutions.

1. SPECIAL SOLUTIONS AND CHAOTIC DYNAMICS, GENERATED BY THEM

Here is the simplest example of a mathematical model that has singular solution:

$$\frac{dy}{dx} = \sqrt{y}\,. \tag{1}$$

It is nonlinear first order ordinary differential equation. Its general solution has the following simple form:

$$F(x,y,C) = y - \frac{1}{4}(x+C)^2 = 0\,. \tag{2}$$

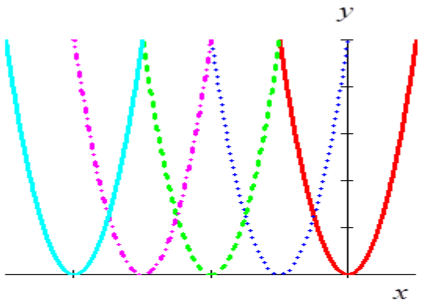

Fig. 1. Quadratic parabola of the solution of (1)

We obtain the set of particular solutions of this equation by changing the value of constant C. Each of these particular solutions is the quadratic parabola (Fig. 1). However, all these particular solutions do not contain the solution $y = 0$. However, as it is easy to see from the equation, this expression also satisfies the equation (1), i.e. it is the solution of the equation. This is the singular solution of equation (1). This feature of singular solutions that they cannot be obtained from the general solution by changing the arbitrary constants was one of the first definitions of these solutions. Later it became clear that these solutions have more important feature. Namely, the uniqueness theorem is violates at the points of these solutions. It is easy to see, from equation (1), that in the points of singular solutions ($y = 0$) Lipshitz conditions are not fulfilled. In addition, the figure shows that all partial solutions pass through the points of singular solutions ($y = 0$). There is a general question: How to find singular solutions of differential equations? For ordinary differential equations of the first order to do this is quite easily. There are several ways to find such solutions. We consider only one of them. By its nature, singular solutions are the trajectory or the surface (of different dimensions), which envelope the general solution of differential equations. Therefore, finding such solutions can be associated with finding envelopes of the overall solution. Let us demonstrate this statement on the system of equation (1). To find the envelope, as it is known, we need to differentiate the general solution (2) by an arbitrary constant C and equate the derivative to zero:

$$\frac{\partial F}{\partial C} = \frac{1}{2}\left(x + C\right) = 0 . \qquad (3)$$

From this expression, we must determine the constant C and substitute it in the expression for the general solution (2). As a result, we find a singular solution. This solution in this simple example, of course, coincides with the above solution. Unfortunately, such a simple procedure for finding singular solutions is characteristic only of the first order ODE. To illustrate the difficulty of finding specific solutions for the ODE of higher order, consider one of the possible algorithms for their finding. To do this, write the ODE system in the canonical form:

$$\frac{dx_k}{dt} = f_k\left(t, \vec{x}\right) . \qquad (4)$$

Suppose that we have found general solution to this system of equations. Let this solution has the form:

$$x_k = \varphi_k(t, \vec{C}) . \qquad (5)$$

It is seen that even at this first step in most real situations, we encounter difficulties. Now, by analogy with the general principle of finding envelopes, we need to find a vector of arbitrary constants $\vec{C}$. The easiest way to do it is to use such way. We assume that these constants are functions of time. Then the complete time derivative of the general solution of (5) will have the form:

$$\frac{dx_k}{dt} = \frac{\partial x_k}{\partial t} + \sum_{i=1}^{n} \frac{\partial \varphi_k}{\partial C_i}\frac{dC_i}{dt} .$$

From this expression we can see that if the second term on the right vanishes:

$$\sum_{i=1}^{n} \frac{\partial \varphi_k}{\partial C_i}\frac{dC_i}{dt} = 0, \qquad k = \{1, 2, ...n\}, \qquad (6)$$

then the general solution (5) is left as the solution of the original equation (4) in spite of the dependence of the constants on the time $\vec{C} = \vec{C}(t)$. The system of equations (6) will be the main for the determination of the vector $\vec{C}$. First of all, note that the system (6) has a nontrivial solution for derivatives dC_i / dt only when its determinant will become zero:

$$\det \begin{vmatrix} \dfrac{\partial \varphi_k}{\partial C_1} & ...... & \dfrac{\partial \varphi_k}{\partial C_n} \\ ..... & .... & ..... \\ \dfrac{\partial \varphi_n}{\partial C_1} & ..... & \dfrac{\partial \varphi_n}{\partial C_n} \end{vmatrix} = 0 . \qquad (7)$$

This will be the second relation for finding the vector of constants $\vec{C}$. Suppose that we by using (7), found one of the constants expressed through other constants:

$$C_n = F(t, C_1, C_2, ...C_{n-1}) . \qquad (8)$$

Then their derivation will have such form:

$$\frac{dC_n}{dt} = C_n' = \frac{\partial F}{\partial t} + \sum_{i=1}^{n-1} \frac{\partial F}{\partial C_i} C_i' . \qquad (9)$$

Substitute expressions (8) и (9) into first (n-1) equations (6):

$$\sum_{i=1}^{n-1} \frac{\partial \varphi_k}{\partial C_i}\frac{dC_i}{dt} + \frac{\partial \varphi_k}{\partial C_n} C_n' = \sum_{i=0}^{n-1} \frac{\partial \varphi_k}{\partial C_i}\frac{dC_i}{dt} +$$

$$+ \frac{\partial \varphi_k}{\partial C_n}\left[\frac{\partial F}{\partial t} + \sum_{i=1}^{n-1} \frac{\partial F}{\partial C_i}\frac{dC_i}{dt}\right] = 0,$$

$$k = \{1, 2, ...n-1\} . \qquad (10)$$

The system (10) is linear ODE for finding derivatives dC_i / dt. We find after solving (10):

$$\frac{dC_i}{dt} = F_i(t, A_1, A_2 ... A_{n-1}) \qquad i = \{1, 2, ...n-1\}, \qquad (11)$$

where $\vec{A}$ is vector of new independent constants.

The formulas (8) and (11) allow potentially find all the components of the vector $\vec{C}$. Substituting them in the expression for the general solution of (5), we will find set of singular solutions.

As we can see, the implementation of the considered algorithm for finding singular solutions is generally a separate challenge. Only in rare cases it can be implemented analytically.

Let us present a simple example of a nonlinear ODE, on which can be illustrated the above algorithm to obtain singular solutions, as well as all the difficulties that can arise while. It is a model example [4]. So, let there be given:

$$xy' + y'^2 + z' - y = 0, \qquad (12)$$
$$z'y + y'z' - z = 0.$$

In accordance with the above algorithm, we need to have a general solution of the system (12). In this case, these solutions are easy to find:

$$y = c_1 x + c_1^2 + c_2, \qquad (13)$$
$$z = c_2 x + c_1 c_2 .$$

Let us find determinant (7) for these solutions:

$$\begin{vmatrix} \left(x + 2c_1\right) & 1 \\ c_2 & \left(x + c_1\right) \end{vmatrix} = \left(x + 2c_1\right)\left(x + c_1\right) - c_2 = 0 . \qquad (14)$$

From (14) we will find c_2:

$$c_2 = \left(x + 2c_1\right)\left(x + c_1\right). \qquad (15)$$

Let us the expression for c_2 substitute in system of equations (6). For our example this system has the form:

$$\left(x + 2c_1\right)c_1' + c_2' = 0,$$

$$c_2 c_1' + (x + c_1)c_2' = 0 . \qquad (16)$$

As a result of the substitution we get $c_1' = -0.5$ – from the first equation of the system (16). Using this relation and the expression (15), we find one of the sets of singular solutions:

$$y = -0.25 \cdot x^2 + Ax + 3A^2,$$
$$z = 0.5 \cdot x^2 A + 2 \cdot x \cdot A^2 + 2A^3 . \qquad (17)$$

Expressions (17) are a one-parameter set of singular solutions in which is a new parameter – arbitrary constant A. Another set of special solutions can be obtained using the second equation (16). A complete solution of the model system (12) we shall not write out. Now we can see, that use of the above algorithm for finding singular solutions for real systems is a complex problem. We will not use it here. The main result that can be get out of this algorithm is that an increase in the number of degrees of freedom of the system being studied follow to the growing set of singular solutions. Therefore, we can expect that for complex systems with many degrees of freedom, they will play a more significant role than for simple systems. In this paper we will be interested, first of all, those particularly of singular solutions that are in uncertainty of those phase trajectories that come to the points of these solutions. In most cases, it is easy to solve problems. It is enough to analyze the right-hand sides of equations (4) for the implementation Lipshitz conditions for them. Even easier: should look at the derivatives of right-hand sides from the dependent variables in, for example, systems (4). If these derivatives in some points of the phase space will be infinitely large, then these areas will be those areas in contact with which there appears uncertainty. When taking into account the singular solutions it can be easily implemented modes of dynamic chaos. For this it is necessary that the phase trajectories of the dynamical system periodically get to the points of singular solutions. In extreme cases, as we shall see in the following section, these trajectories should be approached enough close to points of singular solutions. As a typical example, let us consider the dynamics of the system, which is described by the following equations:

$$\dot{x}_0 = x_1; \qquad \dot{x}_1 = \left(\frac{x_1^2}{2x_0}\right) - 0.5 \cdot x_0. \qquad (18)$$

The dynamics of such a system has been studied in [3]. Here the most important characteristics of this system are represented. First of all, it is easy to show that the function $\varphi = (x_0 - R)^2 + x_1^2 - R^2 = 0$ is an integral of the system (18). Moreover parameter R (the radius of the circles) can take arbitrary values. The system (18) is a model of an oscillator with nonlinear friction. The phase portrait of the system (18) is shown in Fig. 2. Integral curves in this case are the circles. The center of these circles are located on the axis $x_1 = 0$. The radii of circles are equal to the distance of center to the zero point ($x_0 = 0; x_1 = 0$). This point is common point to all circles. In addition, this point is a singular solution of (18). Looking at the integral equation (18), it is difficult to imagine that the dynamics of the system (18) may be irregular. However, numerical calculations show that it is irregular. Indeed, Fig. 3 shows the dependence variable x_0 on the time. It is seen that the phase trajectory after passing the point of singular solution ($x_0 = x_1 = 0$) can jump from one circle to another circle. And these jumps occur randomly. Practically any change accuracy of calculation change the time dynamics of this system. In addition, spectral analysis of the system (18) shows that her spectrum is broad and the correlation function decays quite quickly (Fig. 4).

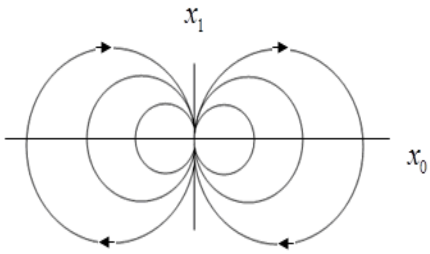

Fig. 2. Phase portrait of the system (18)

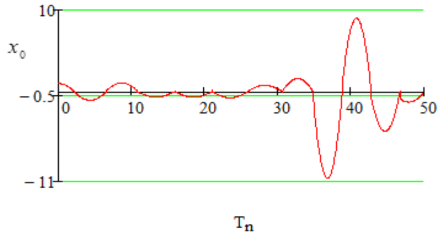

Fig. 3. Time dependence of the variable x_0.
One can see transitions image point from one circle to another

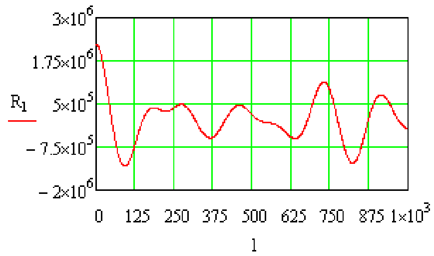

Fig. 4. Autocorrelation function of the variable x_0

The system (18) is not unique. In [3] has been stadied the dynamics of other system:

$$\frac{dx_0}{dt} = x_0 \cdot x_1 + \gamma \cdot x_1 \equiv F_1; \frac{dx_1}{dt} = x_1^2 - x_0^4 - \gamma \cdot x_0 \equiv F_2. \quad (19)$$

It has been shown that in the vicinity of zero this system has an area in which the uniqueness theorem is break down. In addition, the phase trajectories are periodically fall into this region. The dynamics of this system is irregular. In [3] also shows how you can construct set of systems that have the desired properties.

2. KEPLER PROBLEM

The above examples do not contain a clear physical meaning. In this section we show that taking into account singular solutions can be substantial and for well-known physical models. The first physical example of a system with one degree of freedom is the problem of the motion of the particle in a central field. This problem due to the existence of the integral, which expresses the law of conservation of angular momentum, is reduced to the task having only one degree of freedom. Moreover, take into consideration integral of the energy, this problem is fully integrated. Below we show, despite its integrability in its dynamics can be observed modes that resemble modes with dynamic chaos. Such dynamics occur when the phase trajectories pass near the points of singular solutions. The dynamics of particles in the central field can be described by the following system of equations:

$$\dot{r} = v$$

$$\dot{v} = \frac{M^2}{m^2 r^3} - \frac{\alpha}{mr^2} - \frac{\beta \cdot \cos(\omega \cdot t)}{r^\gamma}, \qquad (20)$$
$$\dot{\varphi} = M / mr^2,$$

here $M \equiv mr^2\dot{\varphi} = const$ angular moment of the impulse.

For definiteness in (20) we introduced Coulomb (for the motion of charged particles) or gravity (Kepler problem) potential. In addition, we took into account a small external perturbation $\beta << 1$. In particular, if $\gamma = 4$, $\omega = 0$ it can mean that the shape of the attracting center differs from a strictly spherical shape. If $\gamma = 4$, $\omega \neq 0$ it may be an ion that is in an external periodic electric field.

It should be noted, that the third equation in (20) does not affect the dynamics of the first two equations. The dynamics of the angular variable is uniquely determined by the radial dynamics. The system (20) has one degree of freedom in this case.

If we take into account the energy integral, the problem of the dynamics in the central field is fully integrated. Such solutions can be found in many of the courses of mechanics (see, e.g., [5]).

Our task is to show that, despite the complete integrability, the dynamics of such system can be very complicated and, in some sense, chaotic. This case is similar to that which was discussed in the previous section. The difference lies mainly in the fact that it is a well-studied physical problem. The second difference is that the phase trajectories of the system are not strictly go through a singular solution.

These trajectories are going close to the area of violations of the uniqueness theorem, not getting into it directly. The value of proximity which necessary for realization of chaotic regimes is determined by parameters of the system, the accuracy of numerical calculations and the value of small external perturbations. Let us illustrate these features. First, consider the dynamics of the system (20) in the absence of disturbance $\beta = 0$. Select the next parameter: $r(0) = 10$; $\dot{r}(0) = 0.1$; $\varphi(0) = 0$; $a = 0.3$; $b = 2$; There $a \equiv M / m$; $b \equiv \alpha / m$.

Typical results of numerical calculations of a Keplerian dynamics of the system (20) are shown in Fig. 5.

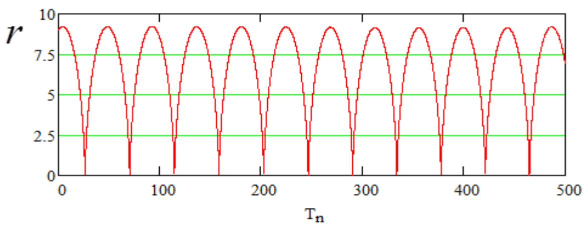

Fig. 5. The typical time dependence of the distance of the particle from the center of attraction

To further illustrate the regular dynamics of a particle in Fig. 6 shows the result of the influence of asymmetric form the center of attraction. It is known that the deviation of the attractive potential of a spherically symmetric shape the perihelion rotates at an angle different from 2π. In addition, the same feature perihelion motion observed if we take into account the effects of relativism. There arises, so-called rosette trajectory of Somerfield (Fig. 6). From these figures it is clear that the dynamics is regular. And it can be shown that the smaller the value of the angular momentum (option), the closer the trajectory of light particles will approach the attracting center. The coordinates of the center of attraction is a singular solution.

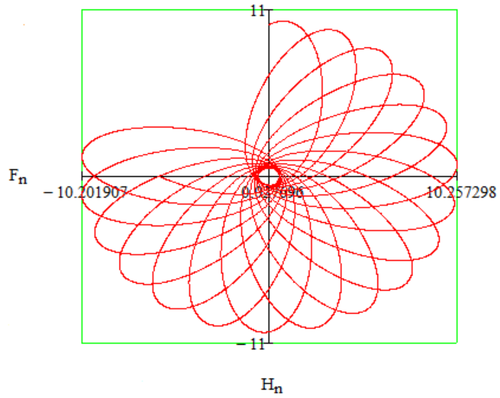

Fig. 6. Rosette trajectory, similar to the Sommerfed trajectory. The system parameters (20):
$r(0) = 10$; $\dot{r}(0) = 0.1$; $\varphi(0) = 0$;
$\beta = 0.1$; $\omega = 0$, $\gamma = 4$; $a = 1.5$; $b = 2$

At this point, the uniqueness theorem is violated. Fig. 6 illustrates the fact that the integral curves in the neighborhood of this point are "thicken". Therefore, even a small perturbation of the trajectories in this area may transfer moving body from one trajectory to another trajectory. In this case previously closely spaced particles in other regions of the phase space can be located far away. The forces that relocate the moving body from one trajectory to another can be very small and can generally be random. But even in the case when they are regular but periodic, and the period of these forces does not coincide with a period of body motion around the center of attraction, the dynamics of the body may be random. Indeed, let us introduce a small regular periodic perturbation ($\beta = 0.0001$, $\gamma = 0$). All other parameters of the system (20) remain unchanged. The dynamics of the moving body in this case is shown in Fig. 7. From these figures it is seen that the moving body jumps randomly from one trajectory to another trajectory. The spectrums of this movement are wide, and the correlation function decays rapidly. This result we have obtained with high accuracy of calculations. If the numerical calculations carried out at a lesser degree of precision then a similar picture (the emergence of the irregular dynamics) appears in the absence of external perturbation.

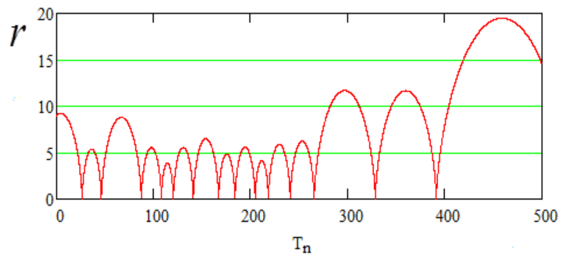

Fig. 7. The typical time dependence of the distance of the particle from the center of attraction

CONCLUSIONS

Here are the most important results obtained in this work:

1. The most significant result is the result that singular solutions must take into account in the analysis of many physical processes.

2. As shown in Section 2, set of singular solutions increases with the number of degrees of freedom of the system under study. This means that the role of singular solutions is particularly high in the dynamics of complex systems.

3. Draw attention that in general, the search for special solutions is challenging. However, for practical purposes, for the estimates, it is sufficient to determine the region of phase space, in which is a violation of the uniqueness theorem. In this case, if the phase trajectory of the studied system in its dynamics often visits this area, the dynamics of these systems will be chaotic.

4. Note that in such tasks as body motion in a central field, the appearance of singular solutions is a natural and objective characterize the dynamics of the system under study.

5. It should be noted that even in the case when the phase trajectories do not fall into area with singular solutions, but are enough close to it, consideration of these singular solutions can also be very significant. As we have saw in the example the motion of bodies in a central field, in this case, even a very small, but the unaccounted external forces can qualitatively change the dynamics of the system.

6. Thus, the account of singular solutions significantly expands the range of physical systems in which can be realized regimes with chaos.

7. It should also pay attention to the fact that the physical nature of the onset of chaos, which is generated by singular solutions, is different from the nature of occurrence of the usual dynamic chaos. The most significant difference is the appearance of chaotic dynamics in fully integrated systems. Unpredictable divergence phase trajectories occur only in the vicinity of singular solutions.

8. In the present work, we focuses our attention on the role of singular solutions in the emergence of regimes with dynamic chaos. Singular solutions can play a significant role in many other cases. One thing is clear that they must be taken into account in the study of the dynamics of physical systems.

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ОСОБЫЕ РЕШЕНИЯ И ДИНАМИЧЕСКИЙ ХАОС

В.А. Буц

Показано, что учет особых решений систем обыкновенных дифференциальных уравнений может приводить к новым условиям появления режимов с динамическим хаосом. В частности, показана возможность возникновения режимов с динамическим хаосом в полностью интегрируемых системах. Приведен пример физической системы, динамика которой может существенно зависеть от наличия особых решений. Такой системой является система, которая описывает движение тел в центральном поле.

ОСОБЛИВІ РОЗВ’ЯЗКИ ТА ДИНАМІЧНИЙ ХАОС

В.О. Буц

Показано, що врахування особливих розв’язків систем звичайних диференціальних рівнянь може призводити до нових умов появи режимів з динамічним хаосом. Зокрема, показана можливість виникнення режимів з динамічним хаосом у системах, що повністю інтегруються. Наведено приклад фізичної системи, динаміка якої може істотно залежати від наявності особливих рішень. Такою системою є система, яка описує рух тіл у центральному полі.