BISTABILITY IN THE DUFFING OSCILLATOR UNDER COMBINED HIGH- AND LOW-FREQUENCY FORCING

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The bistablity onset due to the interaction of high- to low-frequency oscillations in the Duffing oscillator is considered. It is shown that such interaction results in arising of additional bistable states, which has a lower threshold for bistability with respect to the forcing amplitude as compared to the harmonically forced oscillator. This phenomenon is illustrated by considering two examples, including the oscillator with a tandem low- and high-frequency external forcing, and the harmonically forced oscillator with a modulated natural frequency. It is shown that three frequency resonances are responsible for the arising of the additional bistable states. The corresponding threshold values of the forcing amplitudes are found analytically. A comparative analysis of different types of bistable states in the oscillator is presented as well.

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INTRODUCTION

Bistability plays a key role in the dynamics of different electronic, optical, mechanical devices [1 - 3]. For nonlinear systems with external excitation, the bistability is a threshold phenomenon requiring some critical value of the external forcing amplitude for its arising [4, 5]. In the present paper, we will show that the threshold for bistability can be effectively controlled by applying an additional low-frequency external forcing. Besides, the interaction of the high- to low-frequency oscillations results in arising of additional bistable states.

We consider the Duffing oscillator with a combined high- and low-frequency excitation as the mathematical model to illustrate this finding. The high-frequency forcing provides a resonant excitation of the oscillator when the conditions of the principal resonance are fulfilled. Two ways of the application of the low-frequency forcing to the oscillator is considered. The first one is an additive impact of such forcing on the oscillator in the combination with the high-frequency forcing. The second way consists in a low-frequency modulation of the natural frequency of the oscillator.

We consider the conditions for the bistability on set due to the interaction of high- to low-frequency oscillations. Such interaction manifests itself due to three frequency resonances. The application of the secondary averaging techniques has allowed us to obtain analytical conditions for the formation of bistable states and to study their properties in details.

The paper is organized as follows. In Section 2, the case of the tandem low- and high-frequency external forcing of the oscillator is considered. Section 3 deals with the case of the harmonically forced oscillator with a low-frequency modulation of its natural frequency. Section 4 summarizes the obtained results.

1. OSCILLATOR WITH THE ADDITIVE FORCING

We consider the Duffing oscillator, which is subjected to both high- and low-frequency external forcing

$$\frac{d^2x}{dt^2} + \omega_0^2 x = -2\alpha\frac{dx}{dt} - \gamma x^3 - \sigma x^2 + F\cos\omega_e t + N\cos\Omega t. \qquad (1)$$

Here x is the generalized coordinate, ω_0 is the natural frequency of the oscillator, $\alpha > 0$ is the damping coefficient, $\gamma > 0$ is the coefficient of a cubic nonlinearity, F and N are the amplitudes of the external forcing with the corresponding frequencies and Ω. Hereinafter, the principal resonance is considered, which occurs when

$$\omega_e \approx \omega_0. \qquad (2)$$

The properties of the harmonically forced Duffing oscillator (1) when N=0 are well known [4, 5]. Such oscillator demonstrates a bistable behavior, when the amplitude F exceeds some critical value F_{crp}, which reads

$$F_{crp} \equiv \frac{16\alpha\omega_e}{3}\left(\frac{\omega_e\alpha}{\sqrt{3}\gamma}\right)^{1/2}. \qquad (3)$$

In the case when the amplitude $N \neq 0$, the following three frequency resonances can occur in (1):

$$n\cdot\Omega \approx \omega_0 - \omega_e, \qquad (4)$$

with n= 1, 2.

Similar cases of the three frequency interactions have been studied so far from the point of view of the transitions to chaos and excitation of parametric oscillations [6 - 9]. Such resonances can also lead to the appearance of bistability [10].

In order to describe the behavior of the oscillator (1) in the vicinity of the resonances (4), the secondary averaging technique can be used [11]. The application of this technique leads to the following averaged equations for the resonance with $n=1$:

$$\frac{dB}{dt} = -\alpha B - \frac{\sigma NF}{4\Omega\omega_0^4}\sin\psi,$$
$$B\frac{d\psi}{dt} = B(\Delta' - \beta B^2) - \frac{\sigma NF}{4\Omega\omega_0^4}\cos\psi, \qquad (5)$$

where $\Delta' = \omega_0 - \omega_e - \Omega - \frac{2\beta F^2}{4\omega_e^2\Omega^2} - \frac{2\beta N^2}{\omega_0^4}$, $\beta = \frac{3\gamma}{8\omega_e}$.

The averaged equations for the case n=2 are:

$$\frac{dB}{dt} = -\alpha B - \frac{\beta F N^2}{4\Omega\omega_0^5}\sin\psi, \qquad (6)$$

$$B\frac{d\psi}{dt} = B(\Delta'' - \beta B^2) - \frac{\beta F N^2}{4\Omega\omega_0^5}\cos\psi,$$

where $\Delta'' = \omega_0 - \omega_e - 2\Omega - \frac{\beta F^2}{2\omega_e^2\Delta^2} - \frac{2\beta N^2}{\omega_0^4}$.

The amplitude B and the phase ψ are related with the initial coordinate x by the following relations:

$$x = U\cos\omega_e t + V\sin\omega_e t + \frac{N}{\omega_0^2}\cos\Omega t, \qquad (7)$$

$$\frac{dx}{dt} = -U\omega_e\sin\omega_e t + V\omega_e\cos\omega_e t - \frac{N\Omega}{\omega_0^2}\sin\Omega t,$$

with

$$U = B\cos(n\Omega t + \psi) + R/\Delta, \qquad (8)$$
$$V = B\sin(n\Omega t + \psi).$$

Analysis of the stationary states of (5) and (6) shows that these systems have bistable states. For the system (5), the condition for the bistability is

$$NF > (NF)_{crs1} \equiv \frac{8\Omega\omega_0^4\alpha}{3\sigma}\left(\frac{2\alpha\sqrt{3}}{\beta}\right)^{1/2}, \qquad (9)$$

and for the system (6) this condition is as follows

$$N^2F > (N^2F)_{crs2} \equiv \frac{8\Omega\omega_0^5\alpha}{3\beta}\left(\frac{2\alpha\sqrt{3}}{\beta}\right)^{1/2}. \qquad (10)$$

Thus, these bistable states appear only as a result of the interaction of the high- and low-frequency oscillations. This conclusion follows from the fact that the threshold for the bistable states to arise is determined by the product of the external forcing amplitudes. Note, that for the resonance with n=1 the bistability arises with the simultaneous presence of the quadratic and cubic nonlinearities in the oscillator (1). In opposite to this, for the resonance with n=2, the quadratic nonlinearity is not required for the bistability onset.

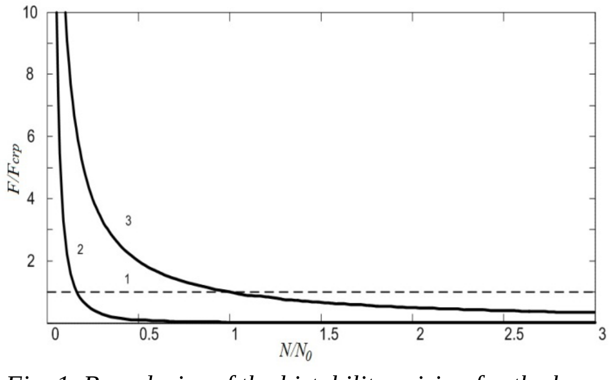

Fig. 1. Boundaries of the bistability arising for the harmonically (dashed line) and the bi-harmonically (solid curves) forced oscillator for ω_0=1, α=0.001; Ω=0.05, γ=0.001; σ=0.001

The conditions for the bistability appearance are illustrated in Fig. 1 on the parameter plane with the normalized amplitudes F and N as the coordinates. The value of F_{crp} given by (3) and $N_0 = 2\Omega\omega_0^3/\sigma$ have been used for the normalization. Here the straight line F/F_{crp}=1 divides the parameter plane on two regions. Bistable states of the harmonically forced oscillator (N=0) appear, if the parameters are chosen from the region above this straight line. Bistable states due to the three frequency resonance (4) with n=1 and n=2 arise in the areas above the curve 2 and curve 3, correspondently. It is important to note, that according to Fig. 1 the threshold for the bistability onset with respect to the amplitude of the high-frequency forcing in the three-frequency resonances can be much lower as compared to that for the two-frequency resonance (2). Thus, the additional low-frequency forcing enables for an efficient control of the threshold for bistability.

An example of the resonance curves for the three-frequency resonance with n=1 is shown in Fig. 2 for different values of the low-frequency amplitude N and for other parameters corresponding to Fig. 1. Here the high-frequency amplitude F is chosen to be equal $F_{crp}/2$. It means that the bistability cannot arise in the case of the harmonically forced oscillator. The bistability observed on the resonance curves in this figure is due to the interaction of the high- to low-frequency oscillations.

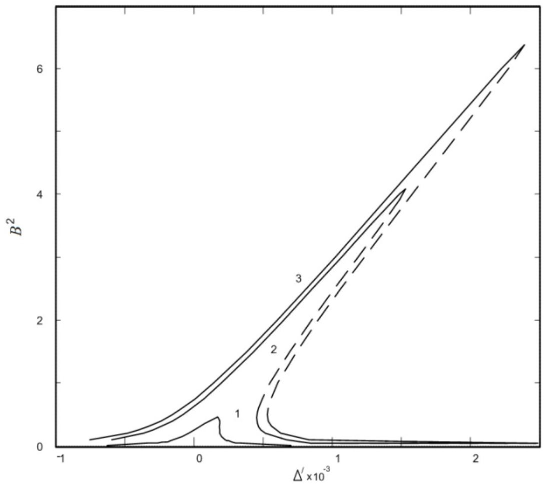

Fig. 2. Resonance curves for the three-frequency resonance with n=1 for different values of the low-frequency amplitude: $N=N_{crp}/2$ (1); $N=2N_{crp}$ (2); $N=2.5N_{crp}$ (3). Solid curves show stable states, and dashed curves show unstable ones

2. OSCILLATOR WITH MODULATED NATURAL FORCING

In this section we consider the harmonically driven Duffing oscillator with the natural frequency, which is slowly varying in time:

$$\frac{d^2x}{dt^2} + \omega^2(t)x = -2\alpha\frac{dx}{dt} - \gamma x^3 - F\cos\omega_e t \cdot \qquad (11)$$

Here $\omega(t) = \omega_0(1 + m\cos\Omega t)$ is the natural frequency time variation function, where m<<1 is the modulation coefficient, $\omega_0$is the mean value of the frequency and Ω is the frequency of the modulation which is small as compared to ω_0. As in the previous section we study the resonant excitation of the oscillator by the external harmonic force at the frequency ω_e, when the condition (2) holds. In opposite to (1), we do not include in (11) the quadratic nonlinear term, since it does affect here the oscillator dynamics.

In addition to the two-frequency resonance (2), one of the three-frequency resonances (4) with n=1 also takes place in (11). It is possible to show that for this resonance, the oscillator dynamics is described by the following averaged equations with respect to the amplitude B and the phase ψ of the excited oscillations:

$$\frac{dB}{dt} = -\alpha B - \frac{mF}{8\omega_e\Omega}\sin\psi,$$

$$B\frac{d\psi}{dt} = B(\Delta' - \beta B^2) - \frac{mF}{8\omega_e\Omega}\cos\psi, \tag{12}$$

where $\Delta' = \omega_0 - \omega_e - \Omega - 2\beta R^2/\Omega^2$, $\beta = 3\gamma/8\omega_e$.

From the above system, we obtain that the bistability arises if the following threshold condition is fulfilled:

$$\frac{mF}{4\Omega} > F_{crp}, \tag{13}$$

where F_{crp} is defined by the expression (3).

It should be pointed out that this bistable state also appears only as a result of high- to low-frequency interaction. This follows from the fact that the threshold for the bistabilities (13) is defined by the product of the high-frequency forcing amplitude and the modulation coefficient.

Let us compare the thresholds (3) and (13) obtained, respectively, for the two-frequency resonance (2) and for the three-frequency resonance. For this, we rewrite (13) in the following form

$$\frac{F}{F_{crp}} > \frac{4\Omega}{m}. \tag{14}$$

Since Ω<<1, the above condition can be satisfied if the modulation coefficient m is small, that means that for the bistability to arise due to the resonance (4), the amplitude of the external forcing amplitude can be smaller than F_{crp}. This condition of the bistability onset is illustrated in Fig. 3 on the parameter plane with the coordinates F/F_{crp} and m. The straight line $F/F_{crp}=1$ (line 1 in Fig. 3) divides the parameter plane into two regions. When the oscillator parameters are taken from the region above this line, the bistability can arise with the presence of only high-frequency external harmonic forcing without the oscillator natural frequency modulation. In the region below this line such bistability cannot arise.

The borderline of the bistability appearance defined by the ratio (14) is shown by curve 2 in Fig. 3.

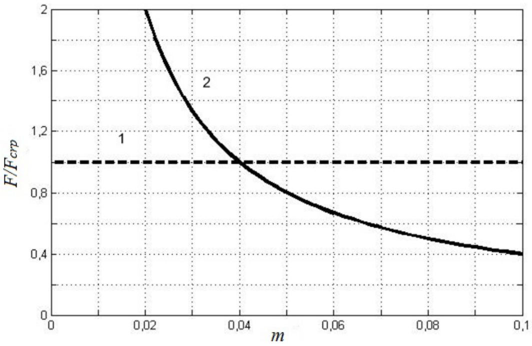

Fig. 3. Bifurcation diagram on the parameter plane the normalized forcing amplitude and the coefficient of modulation for the resonance (2) and for the resonance (4) with n=1 for the normalized modulation frequency Ω=0.01

The parameter plane shows that the threshold value of F for this bistability can be lowered several times with respect to F_{crp} even with a small modulation coefficient. For example, with $m=0.1$ the threshold is lowered by the factor 2.5.

An example of the frequency response curve with the modulated natural frequency of the oscillator is given in Fig. 4. The curve is plotted for $F=0.8F_{crp}$. The peak 1 corresponds to the two-frequency resonance (2). There is no bistability on this resonance due to the selected value of F. Bistability arises here (peak 2 on the resonance curve) due to the three-frequency resonance (4), since it requires a lower value of the forcing amplitude for this bistability onset. By increasing the amplitude over F_{crp}, it is possible to have a bistability at the peak (1) of the resonance curve as well. It means that a pair of bistable states can exists in such oscillator. Each of the states can be controlled independently. It is possible to show mathematically that the distance between the resonances on the frequency scale of the external forcing is approximately equal to the frequency of modulation.

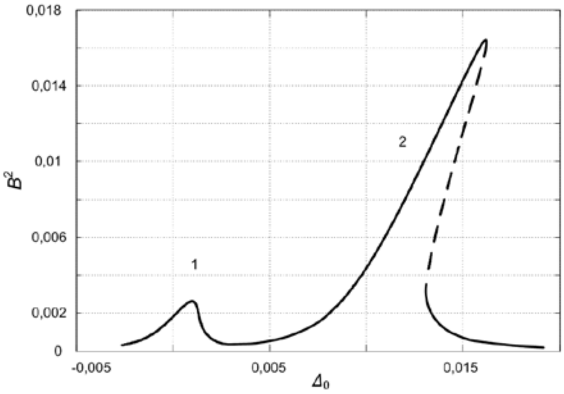

Fig. 4. Frequency response curve of the oscillator with modulated natural frequency. Solid line corresponds to stable states, and dashed line to saddle ones. F/F_{crp}= 0.8, m=0.1, α = 0.001, γ=1, Ω =0.01

CONCLUSIONS

Thus, we have demonstrated that the interaction of high- to low-frequency oscillations in the Duffing oscillator leads to arising of additional bistable states. The threshold for such additional states can be lower as compared to the bistable state that is attributed to the harmonically forced Duffing oscillator. We have illustrated the above findings by two examples, including the oscillator with a tandem low- and high-frequency external forcing, and the harmonically forced oscillator with modulated natural frequency. In the both cases, three frequency resonances are responsible for the arising of the additional bistable states. The application of the secondary averaging technique has allowed us to obtain analytical conditions for the bistability onset due to such resonances. The obtained results are important for applications. They indicate, for example, that the number of bistable states can be increased just by applying a low frequency forcing to the oscillator. The low-frequency forcing also enables for efficient control of the threshold for the bistability.

REFERENCES

1. Hyatt M. Gibbs. *Optical Bistability: Controlling Light with Light*. Academic Press, 1985.

2. L.A. Lugiato. Theory of optical bistability // *Progress in Optics* / Edited by E. Wolf. 1984, № 21, p. 69-216.
3. G.P. Agarwal, H.J. Carmichael. Optical bistability through nonlinear dispersion and absorption // *Phys. Rev. A*. 1979, v. 19, № 5, p. 2074-2086.
4. C.Hayashi. *Nonlinear Oscillations in Physical systems*. McGraw-Hill Book Company, 1964.
5. H.J. Pain. *The Physics of Vibrations and Waves*. John Wiley & Sons, Ltd., 2005.
6. D.V. Shygimaga, D.M. Vavriv, V.V. Vinogradov. Chaos due to the interaction of high- and low-frequency modes // *IEEE Trans. Circuits & Syst.* December 1998, v. 45, p. 1255-1259.
7. V.A. Buts, D.M. Vavriv, O.G. Nechayev, D.V. Tarasov. A simple method for generating electromagnetic oscillations // *IEEE Trans. on Circuits and Systems II*. 2015, v. 62, № 1, p. 36-40.
8. A.B. Belogortsev, D.M. Vavriv, O.A. Tretyakov. Destruction of quasiperiodic oscillations in weakly nonlinear systems // *Appl. Mech. Rev*. 1994, v. 46, № 7, p. 372-384.
9. D.M. Vavriv, V.B. Ryabov, S.A. Sharapov, K. Ito. Chaoticstates of weakly and strongly nonlinear oscillators with quasiperiodic excitation // *Phys. Rev. E*. 1996, v. 53, p. 103-111.
10. D.M. Vavriv, A.Y. Nimets. Torus hysteresis // *Radio Physics and Radio Astronomy*. 2014, v. 19, №3, p. 267-275.
11. Yu.A. Mitropolskiy. *Asimptotic Methods in the Theory of Nonlinear Oscillation*. Gordon & Breach, New York, 1961.

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БИСТАБИЛЬНОСТЬ В ОСЦИЛЛЯТОРЕ ДУФФИНГА ПРИ ОДНОВРЕМЕННОМ ВЫСОКОЧАСТОТНОМ И НИЗКОЧАСТОТНОМ ВОЗДЕЙСТВИИ

А.Ю. Немец, Д.М. Ваврив

Рассматривается случай возникновения бистабильности в осцилляторе Дуффинга в результате взаимодействия высокочастотных и низкочастотных колебаний. Показано, что такое взаимодействие приводит к возникновению дополнительных бистабильных состояний, обладающих меньшим пороговым значением амплитуды внешней силы по сравнению с гармонически возбуждаемым осциллятором. Это явление проиллюстрировано на примере двух случаев: возбуждения осциллятора одновременным внешним воздействием высокочастотных и низкочастотных колебаний, и гармонически возбуждаемого осциллятора с модулированной собственной частотой. Показано, что дополнительные бистабильные состояния возникают в результате трехчастотного взаимодействия. Аналитически определены соответствующие пороговые значения амплитуды внешней силы, воздействующей на осциллятор. Приведен сравнительный анализ различных типов бистабильных состояний в осцилляторе.

БІСТАБІЛЬНІСТЬ В ОСЦИЛЯТОРІ ДУФФІНГА ПРИ ОДНОЧАСНОМУ ВИСОКОЧАСТОТНОМУ ТА НИЗЬКОЧАСТОТНОМУ ВПЛИВІ

А.Ю. Німець, Д.М. Ваврів

Розглянуто випадок виникнення бістабільності в осциляторі Дуффінга в результаті взаємодії високочастотних та низькочастотних коливань. Показано, що така взаємодія призводить до появи додаткових бістабільних станів, що мають меншу величину порогового значення амплітуди зовнішньої сили в порівнянні з гармонічно збуджуваним осцилятором. Це явище проілюстровано на прикладі двох випадків: збудження осцилятора одночасним зовнішнім впливом високочастотних та низькочастотних коливань, та гармонічно збуджуваного осцилятора з модульованою власною частотою. Показано, що додаткові бістабільні стани виникають в результаті трьохчастотної взаємодії. Аналітично розраховані відповідні порогові значення амплітуди зовнішньої сили, що впливає на осцилятор. Приведено порівняльний аналіз різноманітних типів бістабільних станів в осциляторі.