HELICITY OF THE TOROIDAL VORTEX WITH SWIRL

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On the basis of solution of the Bragg-Hawthorne equations it is shown that relationship of the helicity of toroidal swirling vortex with circulations along the small and large linked circles depends on distribution of azimuthal velocity in the core of vortex ring and differs from the well-known Moffat relationship – the doubled product of circulations multiplied by the number of links.

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INTRODUCTION

In nature, the toroidal vortices often have a 'swirl' [1] – the orbital motion along the torus directrix. Such objects are the attached ring vortices of tropical cyclones, hurricanes and tornadoes [2], as well as solar toroidal vortices [3] responsible for the 11-year cycle of activity, and many others (see, e.g. [4]). In the presence of swirl there appears the helicity topological integral [5]. Laboratory experiments have confirmed that this may increase the vortex stability [6].

It is known that for two linked vortex contours the helicity should be equal to the product of the circulations multiplied by the doubled number of links [1, 5, 7, 8]. We will present the example with hydrodynamic solution showing that for the toroidal vortex with swirl this ratio has a little different form, reflecting spatial distribution of the vorticity.

1. BRAGG-HAWTHORNE EQUATION AND ITS SOLUTION FOR TOROIDAL VORTEX

Let us consider the axisymmetric stationary flow of an ideal incompressible fluid in the absence of body forces.

We use the Stokes stream function ψ, defined in cylindrical coordinates ($r,\ \varphi,\ z$) in accordance with $V_r = -\partial\psi / r\partial z$, $V_z = \partial\psi / r\partial r$. Here the continuity equation $div\mathbf{V} = 0$ is identically satisfied. Orbital velocity component can now be written in the form $V_\varphi = f(\psi)/r$, where $f(\psi)$ is the known function. We obtain from the expression for vorticity that

$$rot_\varphi \mathbf{V} = -\tilde{\Delta}\psi / r \text{, where } \tilde{\Delta} \equiv r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right) + \frac{\partial^2\psi}{\partial z^2}.$$

The φ-th component of the vorticity can be found from the Euler equation and after simple transformations:

$$rot_\varphi \mathbf{V} = -\frac{f\cdot f'}{r} + \frac{r}{\psi'_r}\frac{\partial\Pi}{\partial r}, \quad \text{where} \quad f' \equiv df / d\psi,$$

$\Pi = p/\rho + V^2/2$ is the Bernoulli integral. The Bragg-Hawthorne equation for ψ are obtained due to equating these two expressions for the azimuthal component of the vorticity [1, 9] (or, equivalently, the Grad-Shafranov equation in MHD case [10]) with given Π and f:

$$\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} = r^2\frac{d\Pi}{d\psi} - f\frac{df}{d\psi}. \qquad (1)$$

Solutions of the equation (1) with $f \neq 0$ describe the stationary axisymmetric flows with swirl. One set of such solutions can be obtained for the case

$$d\Pi / d\psi = Const = \alpha\phi_0\ ,\ \ fdf / d\psi = Const = -\beta R^2\phi_0$$

(α and β are dimensionless constants, and ϕ_0 is dimensional normalization factor). Then the solution of equation (1) for the stream function has the form [10] [1]:

$$\psi = \phi_0\left[\frac{1}{2}\left(\beta R^2 + r^2\right)z^2 + \frac{\alpha-1}{8}\left(r^2 - R^2\right)^2\right]. \qquad (2)$$

Using (2) we obtain V_r и V_z and assign the swirl velocity V_φ (and pressure p). The expression (2) for the case close to the circle $r = R;\ z = 0$ (i.e. with $|r - R| << R$ and small z) transforms into the following expression:

$$\psi = 2^{-1}\phi_0 R^2\left[\left(\beta+1\right)z^2 + (\alpha-1)\left(r-R\right)^2\right]. \qquad (3)$$

Surfaces $\psi(r,z) = Const$ in this solution for $\beta + 1 > 0$ and $\alpha > 1$ are the nested tori with common rotary axis (directrix) $r = R,\ z = 0$ and their meridional sections are ellipses. We also obtain from these equations

$$\Pi = \alpha\phi_0\psi + \Pi_0\ ,\ \ f^2 = f_0^2 - 2\beta R^2\phi_0\psi\ , \qquad (4)$$

where Π_0 and f_0 are integration constants parametrizing the solution.

2. HELICITY

In the following sections we will find the helicity integral [5] for the solution (3)

$$S = \int \mathbf{V}\cdot rot\mathbf{V}dV.$$

We restrict ourselves to the case $\beta + 1 = \alpha - 1$ when sections of the flow by meridional plane are circles:

$$\psi \simeq 2^{-1}\phi_0 R^2(\alpha-1)\left[z^2 + \left(r-R\right)^2\right]. \qquad (5)$$

Any of the tori $\psi(r,z) = Const$ can be considered as a boundary of the area occupied by the swirling vortex flow.

[1] Other examples of the solutions, mainly relate to the investigation of MHD configurations, can be found in the monograph [11] and in the collections of reviews "Queations of plasma theory" edited by M.A. Leontovich (Moscow, Atomizdat, 1963-1982).

We assume that meridional cross-section is the circle with radius a, and the flow outside the torus is potential.

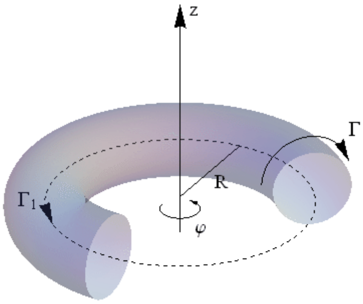

Fig. 1. 3D-scheme of toroidal vortex with swirl

Taking into account the definitions, we obtain expressions for the velocity components. In the case of a thin ring vortex this movement in the meridional plane has the nature of 'solid-body rotation' because the linear velocity of rotation is proportional to the distance to the circular directrix of the vortex ring:

$$V_r \simeq -R\phi_0(\alpha-1)z, \quad V_z \simeq R\phi_0(\alpha-1)(r-R).$$

Here the azimuthal component of the vorticity in the thin ring is practically constant: $\omega_\varphi \simeq -2R\phi_0(\alpha-1)$.

3. THE CASE OF HOMOGENEOUS SWIRL OF VORTEX RING

Firstly, we will consider the case $\beta = 0$, $\alpha = 2$ for which $f(\psi) \equiv f_0 = Const$. This means that the swirl inside the torus is distributed homogeneously: $V_\varphi = f_0 / r \simeq f_0 / R$. Two other velocity components and azimuthal component of the rotor can be written as $V_r \simeq -R\phi_0 z$, $V_z \simeq R\phi_0(r-R)$, $\omega_\varphi \simeq -2R\phi_0$. It is convenient to pass to the polar coordinate system in the meridional plane with origin $r = R$, $z = 0$: $R - r = \eta\cos\theta$, $z = \eta\sin\theta$ (η is the coordinate from rotary axis of the torus along the radius of its cross-section). The vortex sheet $\omega_\theta = -V_\varphi\delta(\eta - a)$ at the boundary $\eta = a$ because the azimuthal velocity is discontinuous here ($\delta(x)$ is Dirac function). So, we assume that it is zero outside the torus. Since we take into account the swirl inside the torus – orbital motion (with angle φ) and vortex motion in meridional section along the small contour (with angle θ), it is convenient to represent the helicity as a sum of two components

$$S = S_\varphi + S_\theta;$$
$$S_\varphi = \int V_\varphi \omega_\varphi dV,$$
$$S_\theta = \int (V_r\omega_r + V_z\omega_z) dV. \tag{6}$$

Thus, resulting components of the helicity S_φ, S_θ have the following form:

$$S_\varphi = \int V_\varphi \omega_\varphi dV \simeq -4\pi^2 R a^2 f_0\phi_0,$$
$$S_\theta = \int V_\theta \omega_\theta dV = -\int_\Sigma V_\theta V_\varphi dS. \tag{7}$$

In the last expression we integrate by the surface of the vortex ring Σ. Since on this surface $V_\theta \simeq Ra\phi_0$ and $V_\varphi \simeq f_0 / R$ then $S_\theta \simeq -4\pi^2 R a^2 f_0\phi_0$. Hence, $S_\theta = S_\varphi$ and, accordingly, for this solution $S = 2S_\varphi$. (We will see below that the expression $S = 2S_\varphi$ is very general). So, the helicity integral for considered case is

$$S = -8\pi^2 R a^2 f_0\phi_0. \tag{8}$$

Express the obtained helicity (7) in therms of velocity circulation Γ on the small contour surrounding the vortex ring once, and Γ_1 on the large contour coinciding with circular directrix of the torus:

$$\Gamma = a\int_0^{2\pi} V_\theta d\theta \simeq 2\pi a^2 R\phi_0,$$
$$\Gamma_1 = R\int_0^{2\pi} V_\varphi d\varphi = 2\pi f_0.$$

So, the expression for helicity in this case has usual form:

$$S = -2\Gamma\Gamma_1. \tag{9}$$

4. THE NON-HOMOGENEOUS SWIRL (WITH MAXIMUM SPEED ON THE ROTARY AXIS)

In this section we consider the case of non-homogeneous swirl. Initially, we restrict ourselves by the investigation of the special case where the maximum azimuthal velocity is attained on circular directrix of the vortex ring: $\beta = \alpha - 2 > 0$, and on the surface of the ring the swirl disappears. The expression for the stream function on the boundary of the torus is found from (5)

$$\psi \simeq 2^{-1}(\alpha-1)\phi_0 R^2 a^2,$$

where a is the small radius of the toroidal vortex (the radius of the meridional section). Assuming that on this boundary $V_\varphi = 0$, i.e. $f|_a = 0$, we find the integration constant f_0 in the expression (4): $f_0 = \sqrt{(\alpha-1)(\alpha-2)}\phi_0 R^2 a$. Inside the vortex ring $a^2 \geq \left(z^2 + (r-R)^2\right)$ and the azimuthal velocity is $V_\varphi = \sqrt{(\alpha-1)(\alpha-2)}\,\phi_0 R^2 \sqrt{a^2 - \left(z^2 + (r-R)^2\right)} / r$. The azimuthal component of the vorticity is expressed in the form $\omega_\varphi = -\phi_0(\alpha-1)R^2(r+R)/r^2 \simeq -2\phi_0 R(\alpha-1)$. Thus, for the thin vortex we obtain

$$V_\varphi\omega_\varphi \simeq -2\phi_0^2(\alpha-1)^{3/2}(\alpha-2)R^2\sqrt{a^2 - (z^2 + (r-R)^2)}.$$

Finally we obtain the φ-th component of helicity integrating the expression (6) by the torus volume:

$$S_\varphi = -(8/3)\pi^2 a^3 R^3 \phi_0^2(\alpha-1)\sqrt{(\alpha-1)(\alpha-2)}.$$

To derive the expression S_θ we find the corresponding velocity and vorticity components and integrate by the volume of the ring vortex:

$$S_\theta = -(\alpha-1)^{3/2}(\alpha-2)^{1/2}\phi_0^2 R^2 \int \frac{z^2 + (r-R)^2}{\sqrt{a^2 - (z^2 + (r-R)^2)}} dV.$$

As in previous section we can see that $S_\theta = S_\varphi$ and $S = 2S_\varphi$. The circulation on the small contour (circle with radius a) in this case is

$$\Gamma = a\int_0^{2\pi} V_\theta d\theta \simeq -\pi a^2\omega_\varphi = 2\pi a^2 R\phi_0(\alpha-1).$$

The circulation on the large contour (circular directrix of the torus) is

$$\Gamma_1 = R\int_0^{2\pi} V_\varphi d\varphi = 2\pi R^2 a\phi_0\sqrt{(\alpha-1)(\alpha-2)} .$$

Finally we obtain the helicity for the case of non-homogeneous swirl with its maximum speed on the rotary axis and zero on the boundary expressing it through the circulations:

$$S = -4\Gamma\Gamma_1 / 3 . \qquad (10)$$

We can see that in the case of non-homogeneous swirl, module of the coefficient k for the product of circulations in the expression for the helicity $S = -k\Gamma\Gamma_1$ may be different from two.

5. HELICITY OF A RING VORTEX. GENERAL CASE

Let us go back to the general representation of helicity as the sum of longitudinal and transversal components, correspondently: $S = S_\varphi + S_\theta$. In the case of axial symmetry of the flow we represent the vorticity component in the following form: $\boldsymbol{\omega}_\theta = rot\mathbf{V}_\varphi \equiv \left[\nabla, \mathbf{V}_\varphi\right]$, $\boldsymbol{\omega}_\varphi = rot_\varphi \mathbf{V}_\theta$. Consider the difference

$$S_\theta - S_\varphi = \int\left(\mathbf{V}_\theta\cdot\boldsymbol{\omega}_\theta - \mathbf{V}_\varphi\cdot\boldsymbol{\omega}_\varphi\right)dV$$

$$= \int\left(\mathbf{V}_\theta\cdot\left[\nabla, \mathbf{V}_\varphi\right] - \mathbf{V}_\varphi\cdot\left[\nabla, \mathbf{V}_\theta\right]\right)dV$$

and apply the identity

$$div\left[\mathbf{a},\mathbf{b}\right] \equiv \nabla\cdot\left[\mathbf{a},\mathbf{b}\right] = \mathbf{b}\cdot\left[\nabla,\mathbf{a}\right] - \mathbf{a}\cdot\left[\nabla,\mathbf{b}\right]$$

to the vectors $\mathbf{V}_\varphi$ and $\mathbf{V}_\theta$. Then we obtain

$$S_\theta - S_\varphi = \int\nabla\cdot\left[\mathbf{V}_\varphi, \mathbf{V}_\theta\right]dV = \int_\Sigma\left[\mathbf{V}_\varphi, \mathbf{V}_\theta\right]\cdot\mathbf{n}\, dS = 0$$

for the case if azimuthal velocity component is equal to zero on the boundary of the integration range Σ. Thus, in the general case of axisymmetric flow in circular vortex

$$S_\theta = S_\varphi ; \; S = 2S_\varphi . \qquad (11)$$

At the same time the expression of helicity through circulations $S = -k\Gamma\Gamma_1$ for toroidal vortex is not universal: coefficient k for product of circulations may be different from two. Indeed, let us consider a thin vortex ring with swirl of some general form $rV_\varphi = f(\psi) \neq const$. In the considered solution of (5) $\psi = \psi(\eta)$ and $f = g(\eta)$, where

$$g(\eta) \simeq \sqrt{{f_0}^2 - (\alpha-1)(\alpha-2){\phi_0}^2 R^4\eta^2} .$$

Then, taking into account expressions for the helicity, we get $S = 2S_\varphi = -16\pi^2(\alpha-1)R\phi_0\int_0^a \eta\cdot g(\eta)d\eta$. If the swirl is homogeneous ($\alpha = 2$), then $g(\eta) \equiv f_0$ and $k = 2$. In the case of non-homogeneous swirl we have

$$k = \frac{4}{f_0 a^2}\int_0^a \eta\cdot g(\eta)d\eta . \qquad (12)$$

The difference of k from the value $k = 2$ is caused by the fact that in the case of swirling vortex the linked contours belong to the same toroidal vortex and are not independent. There is still a topological integral of helicity, which can be expressed through the product of circulations of two linked contours. But here coefficient k is a functional of distribution of swirl over the cross-section of the torus, and also depends on the choice of contours by which the velocity circulation is calculated.

6. DISCUSSION

We explain the above results on the example of non-homogeneous swirl with maximum velocity on the torus circular axis (4). The value of helicity integral for the swirling vortex ring with specified circulation velocity on the small (Γ) and large (Γ_1) contours depends on distribution of the swirl along small radius of the torus. At the same time, swirl distribution is unambiguously associated with distribution of the vorticity meridional component.

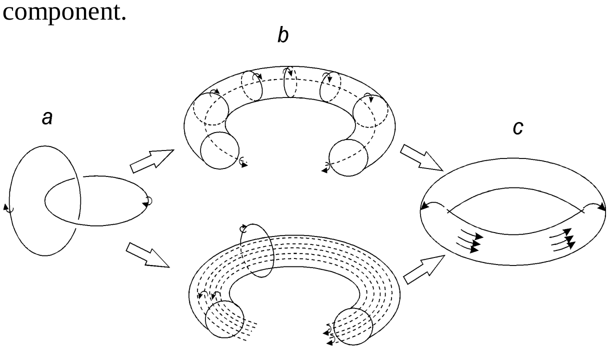

Fig. 2. Two linked vortex threads. Small circular arrows show the direction of circular motion near each of the threads. (In the remaining pictures the meaning of circular arrows is the same.) (a). The upper picture: circles in the meridional plane represent the family of vortex threads (vortex shroud) obtained as a result of 'smearing' of vertical vortex thread over the torus surface. The lower picture: circles in horizontal planes represent the family of vortex threads obtained as a result of 'smearing' of horizontal vortex thread over the torus volume (b). Vortex ring with homogeneous swirl: the circular arrows show the direction of movement in the meridional and horizontal sections (in the latter case, the arrows have the same length, since the swirl is homogeneous) (c)

If to replace the continuous vorticity distribution with discrete one, separating the azimuthal component from the poloidal (meridian) component, without changing the helicity value, we will get two families of linked vortex threads.

Integral of the helicity for the system of two linked vortex threads is determined by the product of circulations along a contour enclosing these threads: $S = \pm 2\Gamma\Gamma_1$.

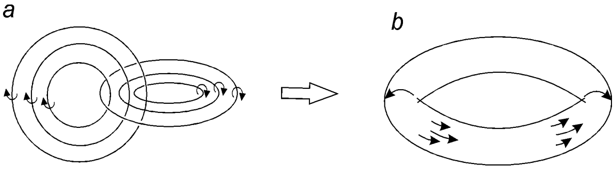

Fig. 3. Two families of linked vortex threads with incomplete links of vertical and horizontal threads (a). The ring vortex with inhomogeneous swirl. The circular arrows show the direction of movement in meridional and horizontal sections (in the latter case, the arrows are of different lengths, since the swirl is non-homogeneous) (b)

As to continuous distribution, let us ‘smear’ one of the vortex threads over the surface of circular torus, preserving the meridionality of vorticity $\omega_\varphi = 0, \omega_\theta \neq 0$ (see Fig. 2,b, upper picture), and the other thread – over the inside of the same torus (see Fig. 2,b, lower picture), so that the vorticity was oriented along the azimuth $\omega_\varphi \neq 0, \omega_\theta = 0$ (preserving the value of circulations Γ, Γ_1). At that the value of helicity integral will not change.

If to ‘smear’ the thread with meridional vorticity over the torus volume (not the surface), the result will be different: not all pairs of partial threads, belonging to different families, will be linked. This is the reason why the coefficient k may be less than 2.

CONCLUSIONS

For a vortex with swirl (orbital motion) the helicity is nonzero, but relation to the product of the linked contours circulations differs from the known formula $S = \pm 2\Gamma\Gamma_1 \cdot I$, where factor I is the Gaussian integral of links with integer values [8, 12, 13]. In our case, for thin vortex rings with circular cross-section the coefficient k may vary in the limits $4/3 \leq k < \infty$, if the values Γ and Γ_1 of velocity circulation are determined on the above-mentioned circuits (small and large generatrixes of the torus). Graphic explanation of this difference is presented in Figs. 2 and 3. The case when $k = 2$ corresponds to homogeneous swirl, and the cases when $4/3 \leq k < 2$ and $2 < k < \infty$ correspond to non-homogeneous swirl with maximum and minimum of azimuthal velocity on circular directrix of the torus. The minimum possible value $k = 4/3$ corresponds to the considered parabolic case of azimuthal velocity distribution in the absence of swirl on the vortex boundary.

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СПИРАЛЬНОСТЬ ТОРОИДАЛЬНОГО ВИХРЯ С НЕОДНОРОДНОЙ ЗАКРУТКОЙ

Е.Ю. Банникова, В.М. Конторович, С.А. Пославский

На основе решения уравнений Брэгга-Хоторна показано, что связь спиральности тонкого тороидального вихря при наличии закрутки с циркуляциями вдоль малой и большой зацеплённых окружностей зависит от распределения азимутальной скорости в ядре кольцевого вихря и отличается от известного соотношения Моффата – удвоенного произведения таких циркуляций, умноженного на число зацеплений.

СПІРАЛЬНІСТЬ ТОРОЇДАЛЬНОГО ВИХОРУ З НЕОДНОРІДНОЮ ЗАКРУТКОЮ

О.Ю. Баннікова, В.М. Конторович, С.О. Пославський

На основі розв’язку рівнянь Брегга-Хоторна показано, що зв'язок спіральності тонкого тороїдального вихору при наявності закрутки з циркуляціями вздовж малого та великого зачеплених кіл залежить від розподілу азимутальної швидкості в ядрі кільцевого вихору і відрізняється від відомого співвідношення Моффата – подвоєного добутку таких циркуляцій, помноженого на кількість зачеплень.