RELATIVISTIC MONO-ENERGETIC TRANSPORT COEFFICIENTS IN HOT PLASMAS

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Relativistic mono-energetic drift kinetic equation for hot toroidal plasmas is analyzed. For compatibility with non-relativistic description, non-canonical thermodynamic forces with the additional temperature-dependent term in the first thermodynamic force were introduced. The transport coefficients were defined as a convolution of mono-energetic transport coefficients with Maxwell-Jüttner distribution function and corresponding weight function.

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INTRODUCTION

Non-relativistic neoclassical theory for toroidal devices is commonly considered to be a matured field of research given the extensive body of scientific literature dealing with the topic [1, 2]. One of the most convenient computational methods developed inside the theory for determining neoclassical contributions to the transport of plasma observables such as density, temperature and current is the mono-energetic approach, which leads to the so-called mono-energetic transport coefficients (see, for example, [3] and the references therein).

In contrast to the widespread opinion [1, 2] that relativistic effects in fusion plasmas can be important only for high-energetic groups of electrons with energies close to $m_{e0}c^2$ or exceeding it (say, runaway electrons), relativistic effects was found to produce the non-negligible contribution in collisional transport properties of plasmas for the temperatures about few tens keV due to the features of relativistic thermodynamic equilibrium of electrons [4 - 7]. In order to take these effects into account, one needs to reformulate the mono-energetic drift kinetic equation and the definition of transport coefficients.

In this paper, we derive the neoclassical mono-energetic transport coefficients for relativistic electrons in hot plasmas. Rigorous relativistic description generally requires a covariant formulation [8]. However, while Lorentz invariance is not important for neoclassical transport because the latter is limited to relatively small characteristic velocities, non-covariant formulation is better suited for the task since our goal is to keep the form of equations as close as possible to the formulation generally accepted in non-relativistic neoclassical theory. This approach allows us to calculate the relativistic mono-energetic transport coefficients using already existed mono-energetic non-relativistic transport codes like DKES [9]. Apart from this, it may be considered as the tool for estimation of the applicability range of non-relativistic neoclassical approach.

In section 2, relativistic mono-energetic drift kinetic equation is formulated and a set of non-canonical thermodynamic forces is introduced. It is shown that for compatibility with non-relativistic description it is necessary to include the explicit temperature dependence in the first thermodynamic force. In section 3, the transport coefficients are derived as a convolution of mono-energetic coefficients with the relativistic Maxwell-Jüttner distribution function and corresponding relativistic weight function.

1. MONO-ENERGETIC LINEAR DRIFT KINETIC EQUATION FOR RELATIVISTIC ELECTRONS

Similar to the non-relativistic consideration [1 - 3], in order to obtain the neoclassical mono-energetic transport coefficients for relativistic electrons in hot plasmas (the ions are taken as non-relativistic) on the given magnetic surface with label ρ, we start from the linearized relativistic drift kinetic equation (rDKE) for deviation from the equilibrium, $f_{e1} = f_e - f_{e0}$, induced by gradients of thermodynamic quantities. Using the variables (u,ξ), where $u = v\gamma$ is the momentum per unit mass with $\gamma = (1+u^2/c^2)^{1/2}$, $\xi = (\mathbf{u}\cdot\mathbf{B})/(uB)$ is the pitch and $\mathbf{B}$ is the vector of the magnetic field, the mono-energetic rDKE can be written as

$$V(f_{e1}) - \nu_D(u)L(f_{e1}) = -\left(\mathbf{V}_{dr}\cdot\nabla\rho\right)\frac{\partial f_{e0}}{\partial\rho} - \frac{u\xi}{\gamma}\frac{e}{T_e}\frac{\mathbf{E}\cdot\mathbf{B}}{B}f_{e0}, \quad (1)$$

where $V = \mathbf{V}_{dr}\cdot\nabla_s + \dot{\xi}\,\partial/\partial\xi$ is the mono-energetic Vlasov operator,

$$V = \left(\frac{u\xi}{\gamma}\mathbf{h} + \frac{cE_\rho}{B}\nabla\rho\times\mathbf{h}\right)\cdot\nabla_s - \frac{u(1-\xi^2)}{2\gamma B^2}(\mathbf{B}\cdot\nabla B)\frac{\partial}{\partial\xi}, \quad (2)$$

$\mathbf{h} = \mathbf{B}/B$ is the magnetic field unit vector, ∇_s is a gradient within the magnetic surface, $\mathbf{E} = \mathbf{E}_\varphi + E_\rho\nabla\rho$ is the electric field separated to the toroidal (inductive) field $\mathbf{E}_\varphi$ and the radial field, $E_\rho = -\partial\Phi/\partial\rho$ with Φ as plasma potential; $L = (1/2)\partial/\partial\xi((1-\xi^2)\partial/\partial\xi)$ is the Lorentz operator which describes the pitch-angle scattering of electrons and $\nu_D(u)$ is the relativistic electron deflection frequency [10]. The radial component of the relativistic drift velocity can be represented as

$$\dot{\rho} \equiv \mathbf{V}_{dr}\cdot\nabla\rho = \frac{m_{e0}cu^2(1+\xi^2)}{2e\gamma B^3}(\mathbf{B}\times\nabla B)\cdot\nabla\rho. \quad (3)$$

In order to exclude the local dependencies which do not contribute to transport, the local equilibrium f_{e0} can be represented as follows [9]:

$$f_{e0} = \left(1 + \frac{e}{T_e}\int^l\left(\frac{\mathbf{E}\cdot\mathbf{B}}{B^2} - \frac{\langle\mathbf{E}\cdot\mathbf{B}\rangle}{\langle B^2\rangle}\right)Bdl'\right)f_{eMJ}, \quad (4)$$

where the angle brackets ⟨...⟩ denote the flux-surface average and the relativistic thermodynamic equilibrium f_{eMJ} is given by Maxwell-Jüttner distribution function,

$$f_{eMJ} = \frac{n_e}{\pi^{3/2}u_{th}^3}C_{MJ}(\mu_r)e^{-\mu_r(\gamma-1)}, \quad (5)$$

with Boltzmann factor, $n_e = n_{e0}e^{e\Phi/T_e}$, included. Here, $u_{th} = p_{th}/m_{e0}$ is the thermal momentum per unit mass with $p_{th} = (2m_{e0}T_e)^{1/2}$, $\mu_r = m_{e0}c^2/T_e$ and

$$C_{MJ} = \sqrt{\frac{\pi}{2\mu_r}\frac{e^{-\mu_r}}{K_2(\mu_r)}} = 1 - \frac{15}{8\mu_r} + O(1/\mu_r^2), \quad (6)$$

with K_n the modified Bessel function of n-th order.

Then, the right-hand-side of Eq. (1) can be written as

$$RHS = -\frac{u\xi}{\gamma}\frac{e}{T_e}\frac{\langle \mathbf{E}\cdot\mathbf{B}\rangle}{\langle B^2\rangle}Bf_{eMJ}$$
$$-\dot{\rho}\left[\frac{\partial \ln n_e}{\partial\rho} - \left(\frac{3}{2} + R - \kappa\right)\frac{\partial \ln T_e}{\partial\rho} + \frac{eE_\rho}{T_e}\right]f_{eMJ}, \quad (7)$$

where $\kappa = \mu_r(\gamma - 1)$ is the relativistic kinetic energy, normalized by temperature, and

$$R = \mu_r\left(\frac{K_3}{K_2} - 1\right) - \frac{5}{2} = \frac{15}{8\mu_r} + O(1/\mu_r^2) \quad (8)$$

is the relativistic correction term which appears due to the specific feature of Maxwell-Jüttner distribution function [4 - 7].

Now, let us introduce the relativistic thermodynamic forces as follows:

$$A_1(\rho) = \frac{\partial \ln n_e}{\partial\rho} - \left(\frac{3}{2} + R\right)\frac{\partial \ln T_e}{\partial\rho} + \frac{eE_\rho}{T_e},$$
$$A_2(\rho) = \frac{\partial \ln T_e}{\partial\rho}, \quad (9)$$
$$A_3(\rho) = \frac{e}{T_e}\frac{\langle \mathbf{E}\cdot\mathbf{B}\rangle}{\langle B^2\rangle}.$$

They are similar to the non-relativistic "canonical" thermodynamic forces, but there is one very important difference: in contrast to the "canonical" definition, where only the dependencies from the local gradients are present, A_1 contains an additional relativistic factor R and thus represents an explicit function of the electron temperature T_e.

Finally, we obtain the mono-energetic rDKE, which can be solved by the DKES code [9]:

$$\left(\xi\mathbf{h} + \frac{c}{B}\frac{\gamma E_\rho}{u}\nabla\rho\times\mathbf{h}\right)\cdot\nabla_s f_e - \frac{\gamma\nu_D}{u}L(f_e) =$$
$$-\frac{\gamma}{u}\dot{\rho}\left[A_1(\rho) + \kappa A_2(\rho)\right]f_{eMJ} - \xi B A_3(\rho) f_{eMJ}. \quad (10)$$

Due to the lack of derivatives of f_{e1} with respect to u and ρ in Eq. (10), their values can be treated as parameters that leads to a considerable simplification of the drift kinetic equation from five phase-space variables to three (two angles at the magnetic surface and pitch). Similar to the non-relativistic formulation [3], solution of Eq. (10) is defined only by parameters $\gamma E_\rho/u$ and $\gamma\nu_D(u)/u$, which are, actually, nothing else as E_ρ/v and ν_D/v, respectively. This approach is sufficient to cover the main features of the neoclassical radial transport when applied for calculations of the transport coefficients and fluxes.

2. RELATIVISTIC MONO-ENERGETIC TRANSPORT COEFFICIENTS

Similar to [3], let us look for the solution of Eq. (10) as

$$f_{e1} = \frac{R_0 u_d}{u}\left[A_1 + \kappa A_2\right]f_{eMJ}\hat{f}_e + R_0 A_3 f_{eMJ}\hat{g}_e, \quad (11)$$

where R_0 is the reference value of the torus major radius, $u_d = m_{e0}c^2/(2e\gamma R_0 B_0)$ is characteristic of the radial drift velocity and B_0 is the reference value of magnetic field strength. Then the original drift kinetic equation splits into a system of two independent dimensionless differential equations:

$$\frac{\gamma R_0}{u}V(\hat{f}_e) - \frac{\gamma R_0}{u}\nu_D(u)L(\hat{f}_e) = -\frac{\gamma}{u_d}\dot{\rho},$$
$$\frac{\gamma R_0}{u}V(\hat{g}_e) - \frac{\gamma R_0}{u}\nu_D(u)L(\hat{g}_e) = -\xi b, \quad (12)$$

with $b = B/B_0$. Here, first equation describes the radial transport due to radial gradients, contained in A_1 and A_2, and second equation describes the parallel transport due to a parallel electric field, contained in A_3 (the factor γ in the Eq. (12) for $\hat{f}_e$ is kept with the only purpose to keep the form of equations as similar as possible to the corresponding non-relativistic equations).

Within neoclassical formalism, the relationships between the flux-surface-averaged fluxes, I_i, and the thermodynamic forces which drive them, A_i, can then be expressed as

$$I_i = -n_e\sum_{j=1}^{3}L_{ij}A_j, \quad (13)$$

where L_{ij} is the matrix of transport coefficients.

As was shown in [5], the relativistic flux-surface-averaged flow I_1, which is related to the radial component of the particle flux density, $\mathbf{\Gamma}_e$, can be written in the same form as the non-relativistic one,

$$I_1 = \langle \mathbf{\Gamma}_e\cdot\nabla\rho\rangle = \left\langle \int d^3u\,\dot{\rho} f_{e1}\right\rangle. \quad (14)$$

Next, I_2, which is the radial component of the energy flux density, $\mathbf{Q}_e$, is equal

$$I_2 = \left\langle \frac{\mathbf{Q}_e}{T_e}\cdot\nabla\rho\right\rangle = \left\langle \int d^3u\,\kappa\dot{\rho} f_{e1}\right\rangle. \quad (15)$$

And the last, I_3, is the parallel component of the electron current density, $\mathbf{J}_e$, is equal

$$I_3 = \frac{\langle \mathbf{J}_e\cdot\mathbf{B}\rangle}{eB_0} = \left\langle \int d^3u\frac{u}{\gamma}\xi b f_{e1}\right\rangle. \quad (16)$$

Expressing the fluxes I_i through the thermodynamic forces, the mono-energetic solutions of Eq. (12) may be used to determine the transport coefficients by energy convolution with the local Maxwell-Jüttner distribution function,

$$L_{ij} = \frac{2}{\sqrt{\pi}}C_{MJ}(\mu_r)\int d^3\kappa\sqrt{\kappa}e^{-\kappa}\gamma\sqrt{\frac{\gamma+1}{2}}D_{ij}(\kappa)h_i h_j, \quad (17)$$

where $h_1 = h_3 = 1$, $h_2 = \kappa$ and $D_{ij}(\kappa)$ are the mono-energetic transport coefficients, defined below. If one compares the expression for relativistic energy convolution given by Eq. (17) to the corresponding non-relativistic formula [3], one may find that an additional relativistic factor $\gamma\sqrt{(\gamma+1)/2}$ appears under integral, along with expected normalization coefficient $C_{MJ}(\mu_r)$, which arise from Maxwell-Jüttner distribution function and the use of relativistic kinetic energy, $\kappa = \mu_r(\gamma - 1)$, instead of non-relativistic one, $K = m_{e0}v^2/2T_e$.

Finally, the relativistic mono-energetic transport coefficients D_{ij} for electrons are defined here as follows:

$$D_{11} = D_{12} = D_{21} = D_{22} = -\frac{u_d^2 R_0}{2u}\left\langle \int_{-1}^{+1} d\xi \frac{\dot{\rho}}{u_d} \hat{f}_e \right\rangle,$$

$$D_{13} = D_{23} = -\frac{u_d R_0}{2}\left\langle \int_{-1}^{+1} d\xi \frac{\dot{\rho}}{u_d} \hat{g}_e \right\rangle, \qquad (18)$$

$$D_{31} = D_{32} = -\frac{u_d R_0}{2\gamma}\left\langle \int_{-1}^{+1} d\xi \xi b \hat{f}_e \right\rangle,$$

$$D_{33} = -\frac{u_d R_0}{2\gamma}\left\langle \int_{-1}^{+1} d\xi \xi b \hat{g}_e \right\rangle.$$

Of these mono-energetic coefficients, D_{11} is related for description of the radial transport, D_{33} of the parallel transport, D_{13} is characteristic of the Ware pinch and D_{31} of the bootstrap current. Only three of these coefficients are independent, however, as $D_{13} = -D_{31}$ due to Onsager symmetry.

CONCLUSIONS

Following the standard approach to neoclassical theory, the relativistic mono-energetic drift-kinetic equation for hot electrons is considered. Due to a specific features of the Maxwell-Jüttner distribution function, the relativistic correction term appears in the first thermodynamic force. By splitting the mono-energetic rDKE in two independent equations which correspond to the different thermodynamic forces, the set of transport coefficients is obtained. Using this scheme, relativistic transport coefficients can be found by re-interpretation of the solution from the non-relativistic transport codes. The solution of rDKE for given values of $\gamma E_\rho/u$ and $\gamma\nu_D(u)/u$ and velocity $\gamma/u = v$, should be interpreted as the same non-relativistic function of pitch-angle with different velocity v and parameters E_ρ/v and $\nu_D(u)/v$, such that these parameters should coincide numerically. Then the transport coefficients can be calculated through the convolution of mono-energetic transport coefficients with Maxwell-Jüttner distribution function and specific relativistic weight factor.

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РЕЛЯТИВИСТСКИЕ МОНОЭНЕРГЕТИЧЕСКИЕ КОЭФФИЦИЕНТЫ ПЕРЕНОСА В ГОРЯЧЕЙ ПЛАЗМЕ

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Проанализировано релятивистское моноэнергетическое дрейфовое кинетическое уравнение для горячей тороидальной плазмы. Для совместимости с нерелятивистским формализмом были введены неканонические термодинамические силы, содержащие дополнительный температурно-зависимый член в первой термодинамической силе. Коэффициенты переноса получены в виде свёртки моноэнергетических коэффициентов переноса с функцией распределения Максвелла-Юттнера, описывающей термодинамическое равновесие в релятивистском газе, и соответствующей весовой функцией.

РЕЛЯТИВІСТСЬКІ МОНОЕНЕРГЕТИЧНІ КОЕФІЦІЄНТИ ПЕРЕНОСУ В ГАРЯЧІЙ ПЛАЗМІ

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Проаналізовано релятивістське моноенергетичне дрейфове кінетичне рівняння для гарячої тороїдальної плазми. Для сумісності з нерелятивістським формалізмом були введені неканонічні термодинамічні сили, що містять додатковий температурно-залежний член у першій термодинамічній силі. Коефіцієнти переносу отримані у вигляді згортка моноенергетичних коефіцієнтів переносу з функцією розподілу Максвелла-Юттнера, що описує термодинамічну рівновагу в релятивістському газі, та з відповідною ваговою функцією.