EFFECT OF TEMPORAL RANDOMIZATION ON THE INTERACTION OF NORMALIZED AND ANOMALOUS TRANSPORT

A.A. Stanislavsky

Institute of Radio Astronomy, Kharkov, Ukraine;
e-mail: alexstan@ira.kharkov.ua

The fractional kinetic equations are a natural consequence of non-Gaussian properties in the behavior of many complex systems. We consider the competition between normalized and anomalous transport in the presence of temporal subordination. The anomalous transport is induced by fractional spatial derivatives as occurs in fractional kinetic theory. It is shown that for large time the power tails of the probability density play a dominant role. This supports and extends the Weitzner-Zaslavsky's result obtained in a simpler case.

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1. INTRODUCTION

Fractional calculus occupies an appreciable place in the description of various kinds of wave propagation in complex media, fractional kinetics of Hamiltonian systems, anomalous diffusion and relaxation, random walks with a long-term memory and flights, pseudochaotic dynamics, etc (see, for example, [1-3] and references therein). The fractional operator is a natural generalization of the ordinary differentiation and integration. Long-term memory effects characterize the fractional operator with respect to time, whereas the nonlocal (long-range) effects characterize it with respect to coordinates. This new type of problems has increased rapidly in areas in which the fractal features of a process or a medium impose the necessity to use nontraditional tools in "regular" smooth physical equations. The language of fractional equations (FE) is in progress now. While the linear FE have attained fairly broad research activity, the study of nonlinear FE is at their very beginning.

The fractional kinetic equations describe non-Gaussian properties in the behavior of stochastic systems. In many cases of physical interest it is reasonable to study simultaneously the Gaussian and anomalous processes [4]. This means that the anomalous processes lead to algebraically decreasing tails of a probability distribution function (PDF), whereas the bulk of the PDF is expected to be mostly Gassian in character. Weitzner and Zaslavsky [5] have investigated the interaction of Gaussian and anomalous dynamics for a simple model in one-dimensional space. They have shown that for large times the fractional derivative term dominates in the solution and leads to power type tails in the probability density. We intend to go on the study and clarify what influence will have subordination on the competition between normalized and anomalous transport.

2. WHAT IS SUBORDINATION?

A subordinated process $Y(U(t))$ is obtained by randomizing the time clock of a random process $Y(t)$ using a new clock $U(t)$, where $U(t)$ is a random process with nonnegative independent increments. The resulting process $Y(U(t))$ is said to be subordinated to $Y(t)$, called the parent process, and is directed by $U(t)$, called the directing process. The directing process is often referred to as the randomized time or else operational time [6]. In general, the subordinated process $Y(U(t))$ can become non-Markovian, though its parent process is Markovian. The PDFs of the parent process $Y(t)$ and directing one $U(t)$ determine the PDF of the subordinated process $Y(U(t))$ via the integral relation:

$$p^{Y(U)}(t,x)=\int_0^\infty p^Y(\tau,x)p^U(t,\tau)d\tau,$$

where $p^Y(\tau,x)$ represents the probability to find the parent process $Y(\tau)$ at x on the operational time τ, and $p^U(t,\tau)$ is the probability to be at the operational time τ on the real time t.

Let the subordinator $U(t)$ be an inverse-time stable process, and the parent process satisfies Lévy (or Gaussian) properties. Then the process $Y(U(t))$ describes the subdiffusion [7].

3. MODEL AND ITS ANALYSIS

Consider the kinetic equation with fractional derivative in time

$$\begin{aligned}&\frac{\partial^\beta P_\beta(x',t')}{\partial t'^\beta}-\frac{P_\beta(x',0)t'^{-\beta}}{\Gamma(1-\beta)}\\&=\frac{\partial^2 P_\beta}{\partial x'^2}+\varepsilon\frac{\partial^\alpha P_\beta}{\partial|x'|^\alpha},\quad 1<\alpha<2,\quad 0<\beta\le 1,\end{aligned}\tag{1}$$

where ε is a constant, $\partial^\alpha/\partial|x'|^\alpha$ the Riesz derivative, $\partial^\beta/\partial t'^\beta$ the differential Riemann-Liouville operator, $P_\beta(x',0)$ the initial condition, $\Gamma(s)$ the gamma function. This equation determines the corresponding PDF. By changing the variables $x=\varepsilon^{\frac{\beta}{2-\alpha}}x'$ and $t=\varepsilon^{\frac{2\beta}{2-\alpha}}t'$ one can get ε out of this equation. Therefore we put it equal to one. The Riesz derivative $\partial^\alpha/\partial|x'|^\alpha$ is easily defined in Fourier transform space as $-|k|^\alpha\tilde{P}_\beta(k,t)$, where $\widetilde{P}_\beta(k,t)$ is the Fourier transform of the function $P_\beta(x',t)$. Then the Fourier transform of the right-hand side of (1) gives $-(k^2+\varepsilon|k|^\alpha)\widetilde{P}_\beta(k,t)$. Thus, for large wavenumber and short wavelength the system exhibits normal, Gaussian transport (if $\beta\neq 1$, then normal diffusion becomes subdiffusion), while for the small wavenumber and large wavelength, the system behaves anomalous in kinetics.

Using the results of Refs. [5,7], it is easy to represent the solution of Eq. (1) in the form

$$P_\beta(x,t)=\int_0^\infty F_\beta(z)P_1(x,t^\beta z)dz,$$

where the function

$$F_\beta(z)=\frac{1}{2\pi i}\int_{Br}e^{u-zu^\beta}u^{\beta-1}du$$

is expressed in terms of the Bromwich integral. The PDF $P_1(x,t)=P_{\beta=1}(x,t)$ is the same one denoted by $P(x,t)$ in Section 2 in the paper of Weitzner and Zaslavsky. For the initial condition $P_\beta(x,0)=\delta(x)$ the PDF $P_\beta(x,t)$ is written as a Fourier integral

$$P_\beta(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dk\,e^{-ikx}E_\beta[-t^\beta(k^2+k^\alpha)],$$

where $E_\beta(y)=\sum_{n=0}^\infty y^n/\Gamma(\beta n+1)$ is the one-parameter Mittag-Leffler function. From above it follows that

$$\int_0^\infty F_\beta(z)z^\gamma dz=\frac{\Gamma(\gamma+1)}{\Gamma(\beta\gamma+1)}.$$

Next, we examine the simple relevant moment

$$M_\beta=\int_{-\infty}^{\infty}|x|dxP_\beta(x,t).$$

The asymptotic expansion M_β for large t takes the form

$$M_\beta\sim\frac{2}{\pi}\left[\frac{\pi t^{\beta/\alpha}}{\beta\Gamma(\beta/\alpha)\sin\pi/\alpha}+\frac{\pi t^{\beta(1-1/\alpha)}}{\alpha\beta\Gamma(\beta(1-1/\alpha))\sin\pi/\alpha}+O(t^{2\beta-3\beta/\alpha})\right]. \quad (2)$$

The remaining terms give the corrections from anomalous transport.

Denote Q_β the solution of the equation only with anomalous transport

$$\frac{\partial^\beta Q_\beta(x,t)}{\partial t^\beta}-\frac{Q_\beta(x,0)t^{-\beta}}{\Gamma(1-\beta)}=\frac{\partial^\alpha Q_\beta}{\partial|x|^\alpha},\quad 1<\alpha<2,\quad 0<\beta\leq 1.$$

Following the same procedure as for M_β, we obtain

$$M_{Q_\beta}\sim\frac{2t^{\beta/\alpha}}{\beta\Gamma(\beta/\alpha)\sin\pi/\alpha}. \quad (3)$$

We are now prepared to compare the contribution of normalized and anomalous kinetics in dependence of the value of the parameters α and β, and we start with the simplest comparison, namely between M_β and M_{Q_β}.

4. RESULTS AND DISCUSSION

To sum up, in leading order and for $1<\alpha<2$ and $0<\beta<1$, it is seen that $M_\beta\sim M_{Q_\beta}$. However, the difference between M_β and M_{Q_β} is not small unless $t^{\beta/\alpha}>>t^{\beta(1-1/\alpha)}$. For α not far from one, this relation holds, but as α approaches 2, one requires increasingly larger values of t in order that $t^{\beta/\alpha}$ dominates $t^{\beta(1-1/\alpha)}$. The asymptotic expansion for M_β fails at $\alpha=2$, and must then be very poor for small values of $2-\alpha$. If the value t is sufficiently large, then the anomalous transport under subordination is the limiting form for the case with both Gaussian and anomalous transport subordinated, however there may be significant corrections. It should be pointed out that the long-term memory effects (in the form of the fractional temporal derivative) do not change a character of the interaction between the normalized and anomalous transport distinctly.

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ВЛИЯНИЕ ВРЕМЕННОЙ СУБОРДИНАЦИИ НА ВЗАИМОДЕЙСТВИЕ НОРМАЛЬНОЙ И АНОМАЛЬНОЙ КИНЕТИК

А.А. Станиславский

Дробные кинетические уравнения появляются вследствие негауссовых свойств поведения сложных систем. Мы рассматриваем соперничество между нормальным (гауссовым) и аномальным транспортом при наличии субординации. Аномальный транспорт приводит к появлению дробных производных по пространственным переменным в кинетическом описании систем. Показано, что на больших временах степенные хвосты функции распределения вероятности играют доминирующую роль. Это подтверждает результат Вейцнера-Заславского, полученный в более простом случае, и расширяет границы его применимости.

ВПЛИВ ТИМЧАСОВОЇ СУБОРДИНАЦІЇ НА ВЗАЄМОДІЮ НОРМАЛЬНОЇ Й АНОМАЛЬНОЇ КІНЕТИК

О.О. Станіславський

Дробові кінетичні рівняння з'являються внаслідок негаусових властивостей поводження складних систем. Ми розглядаємо суперництво між нормальним (гаусовим) і аномальним транспортом при наявності субординації. Аномальний транспорт приводить до появи дробових похідних по просторовим перемінним у кінетичному описі систем. Показано, що на великих часах степені хвости функції розподілу імовірності відіграють домінуючу роль. Це підтверджує результат Вейцнера-Заславського, отриманий в більш простому випадку, і розширює межі його застосування.