PROBABILISTIC DESCRIPTION OF CASCADE KINETIC PROCESSES

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Kinetic equations of cascade processes are deduced in mostly general form on the basis of methods of branching markovian random process theory. It is done both in the discrete time case and in the continuous time case.

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1. INTRODUCTION

The kinetic evolution processes in non-equilibrium physical many-body systems are described on the basis of probability theory representations by many-particle distribution functions (see, for example, [1]). However, probabilities defined by distribution functions are not observed by direct experiments. Physically measured values are modeled by some mathematical expectations on the probability distribution defined by the set of many-particle distribution functions. These averages are connected with the special class of random values which are called *summing functions* of various order $l = 1,2,...$ [1]. They are represented by sums of typical random values which are contributions of l-particle groups. Only such averages are available to direct observation. The standard problem of non-equilibrium statistical mechanics is the obtaining of evolution equations for these averages. Such equations, as a rule, may be not obtained accurately by the corresponding averaging of the basis of evolution equations for many-particle distribution functions. As a rule, they are obtained in frames of an asymptotical expansion procedure in a small parameter. The most famous example of such a procedure is the Chapmen-Enskog method [2]. It is connected with calculation of non-equilibrium averages when the evolution equation is formulated in the so-called "average field" approximation. Nevertheless, there are some kinetic processes of special type when this approximation gives the exact kinetic equations. The aim of this work is the demonstration of the fact that the so-called *cascade evolution processes* are related to such a random process class [3]. They consist of the particle multiplication by the definite physical mechanism. Each particle of the system is characterized by the collection of physical parameter values. The essential peculiarity of the cascade multiplication process is the evolution independence of each subsystem connected with any separate particle on analogous subsystems. Of course, this independence property is the approximation from the physical point of view. However, at the description of many physical processes, it is justified. The examples of such cascade processes are electron-photon showers [4], the multiplication of nucleons in their collisions with heavy nucleons [5], cascade processes in solids [6], mechanical destruction of solid medium to some separate fragments [7]. In this work, we propose the mathematical model of cascade processes such that they are represented by Markov branching random processes with continuum *particle types*. We show that the closed linear kinetic equations for summing averages are realized for such processes.

2. DISCRETE TIME CASE

Here, we deduce the mentioned evolution equations in the case of discrete time. It is done not only for methodical aim. The using of discrete time is suitable in some phenomenological statistical physical models.

Let particles be elements of physical system under consideration and they are characterized by one positive parameter $r > 0$. For definiteness, we name it as the energy. Then, each i^{th} particle, $i = 1,2,...,n$, is characterized by the value r_i of this parameter. At each time moment $t > 0$, the system is characterized by the random number $\tilde{n}(t)$ of particles and the set $\langle \tilde{r}_1,...,\tilde{r}_{\tilde{n}(t)} \rangle$ of random energies which are the characteristics of all particles. Therefore, we describe the statistical system state at each fixed time moment t by the set of distribution densities $\langle f_n(r_1,...,r_n;t) \geq 0; n \in N \rangle$. They are some symmetrical functions on R_+^n. These densities are defined by such a way that integrals

$$\frac{1}{n!}\int f_n(r_1,...,r_n;t)dr_1...dr_n = p_n(t), \quad n \in N$$

(hereinafter, all integrations on variables r are done on $[0,\infty)$) represent the probability $p_n(t)$ of the fact that there are n particles in the system at the moment t. In this case,

$$\sum_{n=1}^{\infty} p_n(t) = 1 . \qquad (1)$$

It is necessary to formulate the evolution equation of functions $\langle f_n(r_1,...,r_n;t) \geq 0; \quad n \in N \rangle$. We solve this problem supposing that the time t is discrete, i.e. $t = \Delta l$, $l \in N$, $\Delta > 0$. At the next section, we shall formulate the integro-differential equation with continuous time.

In the case of the discrete time t, we consider that the cascade process possesses the memory loss, i.e. the system statistical state which is characterized by the

probability distribution at the time $(l+1)\Delta$ depends only on the probability distribution at the moment $l\Delta$ and it does not remember all previous history. Let us introduce the conditional probability distribution densities $\langle g_{nm}(r_1,...,r_n;t,\Delta\,|\,r'_1,...,r'_m;);m,n\in N\rangle$ of the transition from the state $\langle r'_1,...,r'_m\rangle$ at the time moment $t=l\Delta$ to the state $\langle r_1,...,r_n\rangle$. They are symmetrical functions on argument groups $r'_1,...,r'_m$ and $r_1,...,r_n$ and they satisfy to the normalization condition

$$\sum_{n=1}^{\infty}\frac{1}{n!}\int g_{nm}(r_1,...,r_n;t,\Delta\,|\,r'_1,...,r'_m)dr_1...dr_n=1. \qquad (2)$$

Then, the evolution equation of the process which is the Markov chain in this case, is represented in the form

$$f_n(r_1,...,r_n;t+\Delta)$$
$$=\sum_{m=1}^{\infty}\frac{1}{m!}\int g_{nm}(r_1,...,r_n;t,\Delta\,|\,r'_1,...,r'_m) \qquad (3)$$
$$\times f_m(r'_1,...,r'_m;t)dr'_1...dr'_m.$$

It is necessary to consider all conditional densities g_{mn} as independent ones of the time t when there exists the physical temporal uniformity at the evolution process description. The equation (3) together with the condition (2) guarantees the conservation of the normalization (1) at each time moment.

If the cascade process has such a property that particles does not disappear at each step, then the summation in Eq.(3) spreads only on the values $m=1,...,n$. Let us introduce the conditional distribution densities $g_s(r_1,...,r_s;t,\Delta\,|\,r)$, $s\in N$ which are symmetrical on arguments $r_1,...,r_s$ and satisfied the normalization condition

$$\sum_{s=1}^{\infty}\frac{1}{s!}\int g_s(r_1,...,r_s;t,\Delta\,|\,r)dr_1...dr_s=1. \qquad (4)$$

Each of mentioned densities determines the decay probability of the particle with the energy r into s particles with energies $r_1,...,r_s$ at some fixed time moment t. In addition, keeping in mind that values r_i, $i=1,...,s$ are called energies by conditional way, the energy conservation law $r=r_1+...+r_s$ does not necessary take place at each decay event. Usually, ones consider such cascade processes when, at each evolution step, the decay of each particle into new particles at any fixed time t happens independently of other particles having appeared up to t. In this case, the conditional densities g_{nm} are expressed by means of densities g_s using the so-called *branching conditions*

$$g_{nm}(r_1,...,r_n;t,\Delta\,|\,r'_1,...,r'_m)$$
$$=\sum_{A_1,...,A_m}\prod_{j=1}^{m}g_{s_j}(A_j;t,\Delta\,|\,r'_j) \qquad (5)$$

where the summation is done on all disjunctive subdivisions $\{A_1,...,A_m\}$ of the number set 1,2,...,n, i.e. $A_i\cap A_j=\varnothing$ at $i\neq j$ and $\bigcup_{j=1}^{m}A_j=n$. Besides, the shortened designation $g_s(A;t,\Delta\,|\,r)=$ $=g_s(r_{i_1},...,r_{i_s};t,\Delta\,|\,r)$ has been introduced where $A=\{i_1,...,i_s\}$. On the basis of Eq.(4), it is simple to verify that densities g_{nm} defined by the formula (5) satisfy the normalization condition (2).

The equation (3) with the branching condition (5) completely determine the cascade evolution process by means of the set of transition conditional densities $g_s(r_1,...,r_s;t,\Delta\,|\,r)$, $s\in N$. Our next problem is to obtain kinetic equations (with discrete time) for above mentioned average physical values using Eq. (3). For this, let us introduce into consideration the generation functional $H[u;t]$ of the probability distribution $\langle f_n(r_1,...,r_n;t);n\in N\rangle$. It describes the statistical state of particle system at each time moment by the equivalent way. We define this functional on sufficiently rapidly decreasing functions $u(r)$, $r>0$ by the formula

$$H[u;t]=\sum_{n=1}^{\infty}\frac{1}{n!}\int\left(\prod_{j=1}^{n}u(r_j)\right)f_n(r_1,...,r_n;t)dr_1...dr_n. \qquad (6)$$

Function $G[u;r]$ of the conditional probability distribution is defined by the set of conditional distribution densities $g_s(r_1,...,r_s,t,\Delta\,|\,r)$, $s\in N$,

$$G[u;r]=\sum_{s=1}^{\infty}\frac{1}{s!}\int\left(\prod_{j=1}^{s}u(r_j)\right) \qquad (7)$$
$$\times g_s(r_1,...,r_s;t,\Delta\,|\,r)dr_1...dr_s.$$

Therefore, it depends additionally on the parameter $r>0$. Using definitions (6), (7), we find

$$H[u;t+\Delta]=H[G[u;r];t]. \qquad (8)$$

on the basis of Eq.(3) and the formula (5). Let us define the particle density with energy r by the mathematical expectation

$$\rho(r;t)=\sum_{n=1}^{\infty}\frac{1}{(n-1)!}\int f_n(r_1,...,r_{n-1},r;t)dr_1...dr_n.$$

It is evident that

$$\rho(r;t)=\left(\frac{\delta H[u;t]}{\delta u(r)}\right)_{u\equiv 1}.$$

On the basis of Eq.(8), we find the self-consistent evolution equation

$$\rho(r,t+\Delta)=\int K(r,r';t,\Delta)\rho(r';t)dr' \qquad (9)$$

for this density. Here,

$$K(r,r';t,\Delta)$$
$$=\sum_{n=1}^{\infty}\frac{1}{(n-1)!}\int g_n(r_1,...,r_{n-1},r;t,\Delta\,|\,r')dr_1...dr_{n-1} \qquad (10)$$

is the average number of particles having the energy r and formed from particles having the energy r' at the time moment t during one evolution step. It satisfies the following equation

$$K(r,r';t,\Delta)=\left(\frac{\delta G[u;r']}{\delta u(r)}\right)_{u\equiv 1}.$$

In the analogous way, one may introduce the configuration functions $\rho_l(r_1,...,r_l;t)$ of higher order $l=1,2,3,...$, $\rho_1(r,t)=\rho(r;t)$,

$$\rho_l(r_1,...,r_l;t)=\sum_{n=l}^{\infty}\frac{1}{(n-l)!}\int f_n(r_1,...,r_l,r'_1,...,r'_{n-l};t)dr'_1...dr'_{n-l}.$$

These functions are determined by the generation functional,

$$\rho_l(r_1,...,r_l;t)=\left(\frac{\delta^l H[u;t]}{\delta u(r_1)...\delta u(r_l)}\right)_{u\equiv 1}.$$

However, unlike the equilibrium statistical mechanics system (see, for example, [1]), the evolution equations for these functions are self-consistent, since the changing of the function $\rho_m(r_1,...,r_l;t)$, $m=1,2,...,l$ during one evolution step is determined completely by the density values at the given time moment t. Really, introducing the average

$$K_l(r_1,...,r_l;t,\Delta\,|\,r')=\sum_{n=l}^{\infty}\frac{1}{(n-l)!}\times\int g_n(r_1,...,r_l,r'_1,...,r'_{n-l};t,\Delta\,|\,r')dr'_1...dr'_{n-l}=\left(\frac{\delta^l G[u;r']}{\delta u(r_1)...\delta u(r_l)}\right)_{u\equiv 1},$$

we find from Eq. (8), using the multiple differentiation formula of the composite function, that

$$\rho_l(r_1,...,r_l;t+\Delta)=K_l(\rho_1,...,\rho_l;t),$$

where $K_l(\cdot)$ is the linear operator applying to the densities $\rho_m(r_1,...,r_l;t)$, $m=1,2,...,l$. In particular, at $l=2$, we have

$$\rho_2(r_1,r_2;t+\Delta)=\int K(r_1,r';t,\Delta)K(r_2,r'';t,\Delta)\rho_2(r',r'';t)dr'dr''+\int K_2(r_1,r_2;t,\Delta\,|\,r')\rho_1(r';t)dr'.$$

3. CASCADE PROCESSES WITH THE CONTINUOUS TIME

As above, we consider cascade processes with random states described by sets $\{r_1,...,r_n\}$ of any random length $n=1,2,...$ at each time moment where $r_i>0$, $i=1,...,n$. Probability distribution of such random states is described by the set of symmetrical densities $\langle f_n(r_1,...,r_n;t)\ge 0; n\in N\rangle$ at each time moment t. They satisfy the normalization condition (1).

Let us introduce the conditional probability distribution densities $\langle g_{nm}(r_1,...,r_n;t\,|\,r'_1,...,r'_m;t');\ m,n\in N\rangle$ of the transition from the state $\langle r'_1,...,r'_m\rangle$ at any time moment t' to the state $\langle r_1,...,r_n\rangle$ at the time moment $t>t'$. They satisfy the normalization condition

$$\sum_{n=1}^{\infty}\frac{1}{n!}\int g_{nm}(r_1,...,r_n;t\,|\,r'_1,...,r'_m;t')dr_1...dr_n=1. \qquad (11)$$

It is evident that the next relation takes place

$$f_n(r_1,...,r_n;t)=\sum_{m=1}^{\infty}\frac{1}{m!}\int g_{nm}(r_1,...,r_n;t\,|\,r'_1,...,r'_m;0)\times f_m(r'_1,...,r'_m;0)dr'_1...dr'_m. \qquad (12)$$

for any $t>0$ taking into account the complete probability formula. In the case when the cascade process is markovian, i.e. the memory about the past is absent, the conditional transition densities g_{nm} satisfy the Chapmen-Kolmogorov equation which has the form

$$g_{nm}(r_1,...,r_n;t\,|\,r'_1,...,r'_m;t')=\sum_{l=1}^{\infty}\frac{1}{l!}\int g_{nl}(r_1,...,r_n;t\,|\,r''_1,...,r''_l;t'')\times g_{lm}(r''_1,...,r''_l;t''\,|\,r'_1,...,r'_m;t')dr''_1...dr''_l \qquad (13)$$

in our case. Then, it follows from Eqs.(12) that these equations connect the densities f_n, $n\in N$ at any different time moments t and t',

$$f_n(r_1,...,r_n;t)=\sum_{m=1}^{\infty}\frac{1}{m!}\int g_{nm}(r_1,...,r_n;t\,|\,r'_1,...,r'_m;t')\times f_m(r'_1,...,r'_m;t')dr'_1...dr'_m. \qquad (14)$$

In particular, if there is the temporal uniformity, conditional densities g_{nm} depend only on the difference $t-t'$ but not on each temporal argument separately.

For a markovian process, the evolution equation for densities g_{nm} and f_n, $m,n\in N$ follows from Eq. (13). Namely, after the substitution $t\Rightarrow t+\Delta$, $t''\Rightarrow t$, and using the smallness of Δ the following expansion can be written down

$$g_{nm}(r_1,...,r_n;t+\Delta\,|\,r'_1,...,r'_m;t)=\delta_{nm}\sum_{P_n}\delta(r_1-r'_1)...\delta(r_n-r'_n)+\Delta\lambda_{nm}(r_1,...,r_n;t\,|\,r'_1,...,r'_m)+o(\Delta). \qquad (15)$$

Here, the first sum is done over all argument $r_1,...,r_n$ permutations and

$$\lambda_{nm}(r_1,...,r_n;t\,|\,r'_1,...,r'_m)=\left(\dot g_{nm}(r_1,...,r_n;t\,|\,r'_1,...,r'_m;t')\right)_{t'=t}.$$

From the functions g_{nm} normalization conservation, it follows that

$$\sum_{n=1}^{\infty}\frac{1}{n!}\int\lambda_{nm}(r_1,...,r_n;t\mid r'_1,...,r'_m)dr_1...dr_n=0.$$

Let us take into account that the investigated cascade process is jump-like, i.e. the described stochastic evolution is represented by the sequence of jumps from one state to another occurring at random time moments $\langle\tilde{t}_k;k\in N\rangle$. Then, at small Δ, with assumption that there exists the nonzero average time between two sequential jumps, the probability $g_{nm}dr_1...dr_n$ is the sum of probabilities of two events: the probability

$$\Delta\nu_{nm}(r_1,...,r_n;t\mid r'_1,...,r'_m)dr_1...dr_n=o(\Delta),$$
$$\nu_{nm}(r_1,...,r_n;t\mid r'_1,...,r'_m)\geq 0$$

of the fact that one transition has taken place $\langle r_{1'},...,r_{m'}\rangle\Rightarrow\langle r_1,...,r_n\rangle$ and the probability

$$\delta_{nm}\sum_{P_n}\delta(r_1-r'_1)...\delta(r_n-r'_n)$$
$$\times\left(1-\Delta\nu_m(r_1,...,r_m;t)\right)dr_1...dr_m+o(\Delta)$$

of the fact that there is no transition. From this, it follows that

$$\lambda_{nm}(r_1,...,r_n;t\mid r'_1,...,r'_m)$$
$$=\nu_{nm}(r_1,...,r_n;t\mid r'_1,...,r'_m)\qquad(16)$$
$$-\delta_{nm}\nu_m(r_1,...,r_m;t)\prod_{j=1}^{m}\delta(r_j-r'_j),$$

$$\nu_m(r_1,...,r_m;t)$$
$$=\sum_{n=1}^{\infty}\frac{1}{n!}\int\nu_{nm}(r'_1,...,r'_n;t\mid r_1,...,r_m)dr'_1...dr'_m,\qquad(17)$$

that is the so-called *detail balance principle*. Tending $\Delta\to\infty$ after the substitution of the expansion (15) to Eq.(12), we find

$$\dot{f}_n(r_1,...,r_n;t)$$
$$=\sum_{m=1}^{\infty}\frac{1}{m!}\int\nu_{nm}(r_1,...,r_n;t\mid r'_1,...,r'_m)$$
$$\times f_m(r'_1,...,r'_m;t)dr'_1...dr'_m$$
$$-\nu_n(r_1,...,r_n;t)f_n(r_1,...,r_n;t).\qquad(18)$$

Let us express *the transition frequencies* $\nu_{nm}(r_1,...,r_n;t\mid r'_1,...,r'_m)$ through more elementary quantities, i.e. frequencies of particle decays in the case when the process is cascade-like. As above, let us take first into account that particles do not disappear and, therefore, the summation in Eq. (18) spreads only on values $m=1,...,n$. We introduce the conditional distribution densities $g_s(r_1,...,r_s;t+\Delta\mid r,t')$, $s\in N$ of the decay of the particle with energy r to s particles with energies $r_1,...,r_s$ at the time moment $t+\Delta$. They are similar to densities $g_s(r_1,...,r_s;t,\Delta\mid r)$ and, therefore, they satisfy the normalization condition (4). Further, we construct the conditional densities g_{nm} on the basis of introduced densities using the *branching condition*

$$g_{nm}(r_1,...,r_n;t+\Delta\mid r'_1,...,r'_m,t')$$
$$=\sum_{A_1,...,A_m}\prod_{j=1}^{m}g_{s_j}(A_j;t+\Delta\mid r'_j,t'),\qquad(19)$$

where designations are similar to those used in Eq. (5). Substituting the representation analogous to one performed by Eq. (16) but at a small Δ,

$$g_s(r_1,...,r_s;t+\Delta\mid r',t)=\delta_{1s}\delta(r_1-r')$$
$$\times(1-\Delta\mu(r';t))+\Delta\mu_s(r_1,...,r_s;t\mid r')+o(\Delta),$$

into Eq.(19), we obtain that Eq.(15) takes place with the condition (16) where

$$\nu_{nm}(r_1,...,r_n;t\mid r'_1,...,r'_m)$$
$$=\sum_{\varnothing\neq A\subset I_n}\sum_{l=1}^{m}\sum_{P_{m-1}}\left(\prod_{j\in\bar{A}}\delta(r_j-r'_{Pj})\right)\mu_{n-m+1}(A;t\mid r'_l).\qquad(20)$$

Here, $\mu_s(r_1,...,r_s;t\mid r')\geq 0$ is the frequency of the decays of the particle with energy r' to s particles with energies $r_1,...,r_s$,

$$\sum_{s=1}^{\infty}\frac{1}{s!}\int\mu_s(r'_1,...,r'_s;t\mid r)dr'_1...dr'_s=\mu(r;t).\qquad(21)$$

Using definitions (17) and (20), we find

$$\nu_m(r_1,...,r_m;t)=\sum_{l=1}^{m}\mu(r_l;t).\qquad(22)$$

Substituting Eq. (20) and Eq. (22) to Eq. (18), we come to the equation system numerated by index $n\in N$. It determines completely the cascade process

$$\dot{f}_n(r_1,...,r_n;t)$$
$$=\sum_{\varnothing\neq A\subset I_n}\int\mu_{n-m+1}(A;t\mid r')f_m(\bar{A},r';t)dr'\qquad(23)$$
$$-\sum_{l=1}^{n}\mu(r_l;t)f_n(r_1,...,r_n;t).$$

Let us introduce into consideration the generation functional $H[u;t]$ of the set $\langle f_n(r_1,...,r_n;t);n\in N\rangle$ corresponding to Eq. (6). Having defined the functional

$$M_t[u;r]=\sum_{s=1}^{\infty}\frac{1}{s!}\int\left(\prod_{j=1}^{s}u(r_j)\right)\qquad(24)$$
$$\times\mu_s(r_1,...,r_s;t\mid r)dr_1...dr_s,$$

on the basis of the equation system (23), we obtain the evolution equation of the functional $H[u;t]$,

$$\dot{H}[u;t]=\int\frac{\delta H[u;t]}{\delta u(r')}\left(M_t[u;r']-u(r')\mu(r';t)\right)dr'.\qquad(25)$$

From this equation, by the differentiation over $u(r)$ at $u\equiv 1$, we obtain the equation of the particle number density,

$$\dot{\rho}(r;t)=\int\mu(r,r';t)\rho(r';t)dr'-\mu(r;t)\rho(r;t),\qquad(26)$$

where $\mu(r,r';t) = \left(\delta M_1[u;r']/\delta u(r)\right)_{u\equiv 1}$ and the equality $M_1[1;r'] = \mu(r';t)$ is used. As in the case of discrete time, by the induction on the order l of correlation function

$$\rho_l(r_1,...,r_l;t) = \left(\delta^l H[u;t]/\delta u(r_1)...\delta u(r_l)\right)_{u\equiv 1},$$

one may obtain that its evolution equation has the form

$$\dot{\rho}_l(r_1,...,r_l;t) = M_l(\rho_1,...,\rho_l),$$

where M_l is the linear integral operator applying to the set of correlation functions $\rho_1,...,\rho_l$.

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ВЕРОЯТНОСТНОЕ ОПИСАНИЕ КАСКАДНЫХ КИНЕТИЧЕСКИХ ПРОЦЕССОВ

Ю.П. Вирченко, Р.Е. Бродский

Даётся общий вывод кинетических уравнений для каскадных процессов, основанный на представлениях теории марковских ветвящихся случайных процессов, как в случае дискретного, так и в случае непрерывного времени.

ІМОВІРНОСНЕ ОПИСАННЯ КАСКАДНИХ КІНЕТИЧНИХ ПРОЦЕСІВ

Ю.П. Вірченко, Р.Є. Бродський

Дається загальне виведення кінетичних рівнянь для каскадних процесів, основане на представленнях теорії марківських гілчастих випадкових процесів, як у випадку дискретного, так і у випадку неперервного часу.