INTERMITTENCY IN HAMILTONIAN SYSTEMS

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We consider 2D map with the singularity. Here we observe an intermittency behavior. This system can be interpreted in two ways. In the first way this map can arise like a result of quantization of the continuous Hamiltonian system with one degree of freedom. In the second way we can interpret this map like a Poincaré section of some 2D Hamiltonian system. As is well known the behavior of a Poincaré section defines the system behavior as a whole. We investigate the mechanism of the chaos generation near singularity. We show that singularity can generate a stochastic sea in Hamiltonian systems under any value of a perturbation. Originating modes have intermittent structure.

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1. MAP WITH SINGULARITY

We consider conservative map $\mathbb{R}^2 \to \mathbb{R}^2$:

$$\begin{aligned} x_{n+1} &= x_n + a y_n, \\ y_{n+1} &= y_n - a x_{n+1} + \frac{b}{x_{n+1}}. \end{aligned} \qquad (1)$$

It is easy to find that the map conserves the phase volume. It can be an exact Poincaré map of some Hamiltonian system with higher dimension. Then information obtained under the exploration can be transferred to properties of continuous Hamiltonian systems. In another case we can interpret it like a result of approximate quantization of a continuous Hamiltonian system [1,2]. New properties of this map would correspond to effects of quantization. Such duality of nature of the maps occurrence expands the possibilities of its properties interpretation.

Map has two parameters a и b. But we can assign $b = a$, since substitution

$$\tilde{x} = \sqrt{b}\, x$$
$$\tilde{y} = \sqrt{b}\, y$$

transforms the map (1) to the universal form

$$\begin{aligned} x_{n+1} &= x_n + a y_n \\ y_{n+1} &= y_n - a x_{n+1} + \frac{a}{x_{n+1}}. \end{aligned} \qquad (2)$$

Singularity line is a main object, which defines structure of the map phase space. This line forms an angle $\alpha = arctg(a) \approx a$ with ordinate axis. Let us consider map of the line intersecting the ordinate axis under angle α. We get that its image has asymptotic rays, which have angles α и β with ordinate axis at the infinity

$$\beta = arctg\left(tg 2\alpha + \frac{tg^3\alpha}{1 + tg^2\alpha} \right) \approx 2\alpha \quad , a \ll 1 .$$

Thus, the map rotates at the angle α at large distance from the infinity and the structure of phase space would have $[2\pi/a]$ order symmetry under small values of a.

2. STRUCTURE OF PHASE SPACE

The phase portrait of the map is shown it Fig. 1. Singularity in the phase space allows the trajectories to reach the infinity under finite number of steps. Original coordinates of such trajectories lie on preimages of the singularity line. Singularity and series of its preimages provides the absence of classical behavior of systems in a large region of phase space. Thus it is naturally to divide the phase space into regions (Fig. 3).

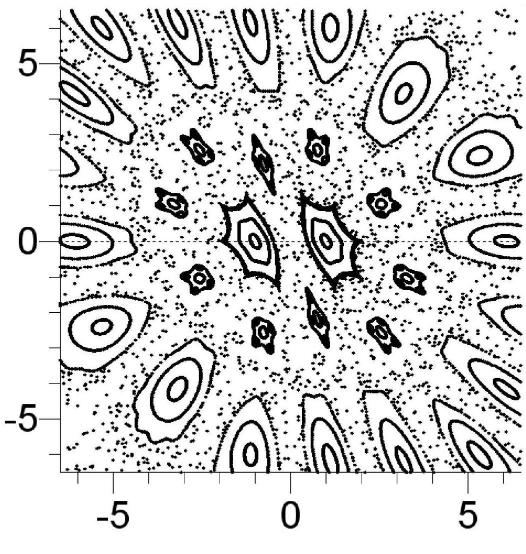

Fig. 1. *Phase portrait of the map*

The first region does not include points from preimages of the singularity. Therefore here we can observe the situation typical for Hamiltonian systems. This region consists of separate islands. It gathers around it fixed elliptical points of different orders. Each elliptical point is surrounded by invariant trajectories. On the border of such region we can observe classical situation of chaos rise in Hamiltonian systems. We can observe ordinary stochastic layers which are isolated from the outer stochastic sea and therefore do not interact with it. The most outer layer overlaps with surrounding it region of the overall chaos. Therefore transitions between them become possible. Such overlap becomes possible thanks to specific structure of the phase space (Fig. 4). Similar structure of the phase space leads to the restriction on the diffusion process. Leaving trajectories to the region of the overall chaos is possible only from the small neighborhood of unstable points and in the narrow range of the direction which is slightly differentt from the direction of the unstable manifold.

The second region contains preimages of the singularity line. Computer calculation of the large number of such preimages shows that region of the overall chaos is everywhere dense and is overlapped by points which run to the infinity for a finite number of iterations. Building of a set of preimages of the singularity line allows to find the overall chaos border (Fig. 2).

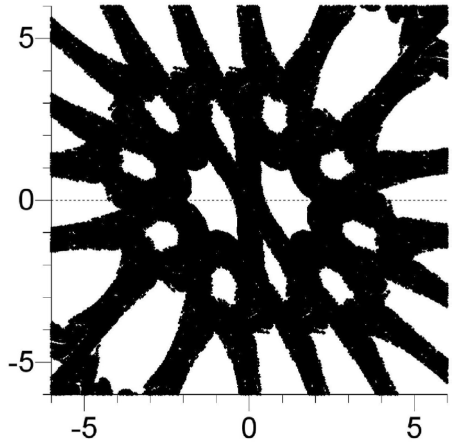

Fig. 2. *Stochastic sea region obtained by computer modeling*

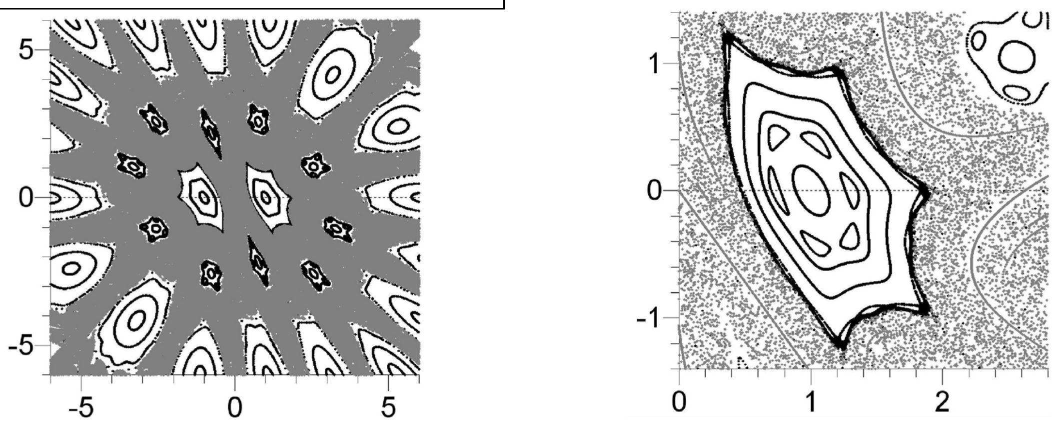

Fig. 3. *Phase portrait (black points) with superimposed preimages of the singularity line (gray points)*

Computer modeling shows that all trajectories in the region of overall stochasticity are unlocalizated. Diffusion of this trajectories to infinity has anomalous character.

Number and relative position of islands of the classical Hamiltonian behavior in the phase space defines its coarse structure. Such structure remains invariant in some interval of parameter a values. With its change the system bifurcates.

For systems with singularity exists criterion of its integrability [3]. One of such criterion is Painlevé test for checking singularity confinement property. In our system such test is successful only for $a = 0$.

$$x_n = x,$$

$$x_{n+1} = \varepsilon,$$

$$x_{n+2} = a^2/\varepsilon - x.$$

The set of the parameter a values which exists under that singularity disappears at the some step. But as a whole the system is not integrable.

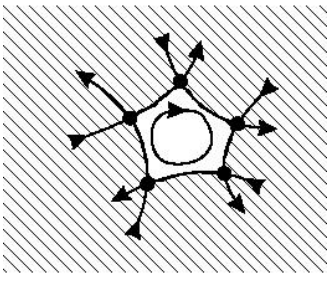

Fig. 4. *Structure of the phase space on the border between the stochastic sea and the region of the classical Hamiltonian behavior*

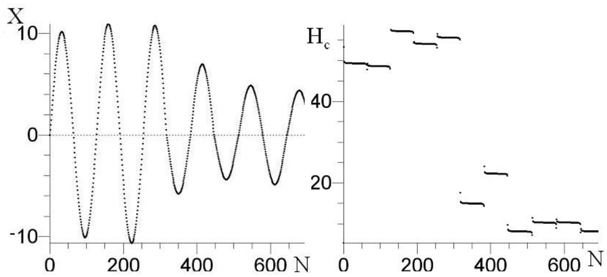

Fig. 5. *Intermittency behavior of the map trajectory*

3. SCENARIO OF CHAOTISATION

The presence of the singularity in the map phase space can bring to the appearance of not typical dynamics of Hamiltonian systems. A classical scenario of the chaos rise in Hamiltonian systems [4,5] consists in the formation of stochastic layers in the neighborhood of separatrixes. These layers overlap and form large region of phase space where system has a chaotic behavior. Usually such region is called a stochastic sea. Regular trajectories are saved only in the small neighborhood of elliptical points. Phase portrait of such system is stochastic sea with stability islands around the elliptical points.

In the map with singularity in addition to general chaotisation mechanism another mechanism of stochasticity rises. It is based on the phase flow break in the map phase space. This break corresponds to singularity line. Phase drop which is crossed by singularity line will be divided into a few not connected drops. The map dynamics can be reduced to rotation and some deformation connected with the singularity. Therefore any drop which was divided once would be divided unlimited number of times under further iterations. At the same time intermixing in single parts of the drop is not appreciable. Thus the trajectories complication is happened thanks to the step-by-step fragmentation of the phase drop. Such chaotisation mechanism strongly differs from the classical mechanism of resonances destruction and stochastic layers overlapping.

Jump process of the map dynamics complication brings to concentration of trajectory chaotic regions into short chaotic bursts. System behavior is regular on intervals between these bursts. As a whole we can consider such behavior like intermittency (Fig. 5). Any chaotic phases in such regime are reduced to the single iteration. It is a very important feature of such regime. At the same time we can find that any trajectory from the stochastic sea consists only from laminar phases. Any phase differs from a previous one by the value of the some parameter, which is invariant during the whole laminar phase. Under the transition from one phase to another this parameter changes stepwise.

3. MEASURE OF CHAOS

The Lyapunov exponent is a general measure of chaos in maps. But results of the computer modelling showed that both Lyapunov exponents are vanishing. It indicates that the sensitivity to starting conditions is lower then exponential. It is the result of the specific for the map intermixing process.

For another argument of chaotic nature of the map we can use the correlation function. In our case the general correlation function diverges. Therefore, it is practical to calculate the correlation function by lower fractional moments

$$K(\tau)=\left(\frac{1}{T}\int_0^T\left(\langle x\rangle-x(t+\tau)\right)^{1/\alpha}\left(\langle x\rangle-x(t)\right)^{1/\alpha}dt\right)^{\alpha}.$$

The deposit of large bursts decreases with the decrease of α. Therefore its divergence becomes better. The results of calculation of the correlation function under different values of the parameter α are shown in Fig. 6.

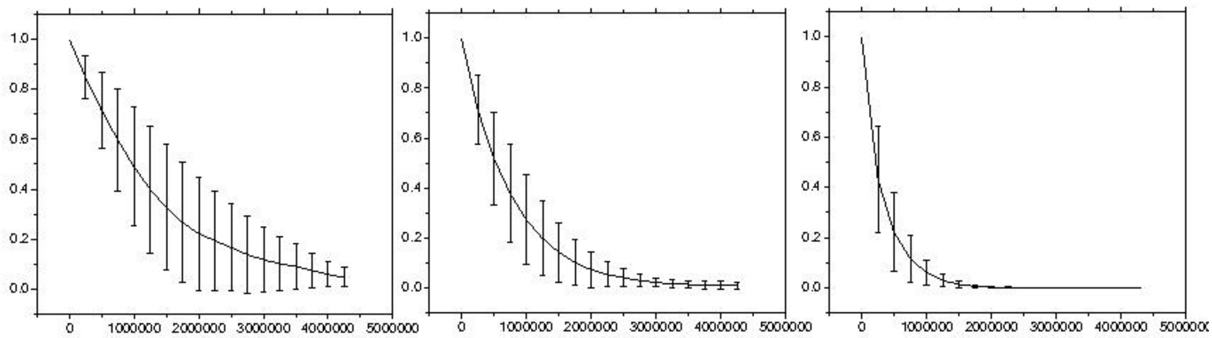

Fig. 6. *Correlation function computed with different values of parameter* α. *Left to right:* $\alpha=0.5$, $\alpha=0.25$, $\alpha=0.1$. *Fluctuations decrease with decreasing* α

These results can be approximated by the expression

$$K\sim e^{-cx^{d}}.$$

Value $d<1$. It depends on α. As well known, in the system with such expression of the correlation function Lyapunov exponent vanishes. Chaotic nature of the system behavior becomes apparent in the decrease of the correlation function, in other words, in time system loses the information of its initial conditions.

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ПЕРЕМЕЖАЕМОСТЬ В ГАМИЛЬТОНОВЫХ СИСТЕМАХ

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Представлено исследование свойств двумерного отображения с особенностью. В таком отображении наблюдается перемежаемость. Такая система может возникать двумя способами. Во-первых, она может рассматриваться как результат дискретизации непрерывной гамильтоновой системы с одной степенью свободы. Во-вторых, мы можем рассматривать такое отображение как сечение Пуанкаре некоторой двумерной гамильтоновой системы. При этом поведение сечения Пуанкаре определяет поведение системы в целом. Исследовался механизм возникновения перемежаемости вблизи особенности. Показано, что особенность приводит к возникновению стохастического моря в гамильтоновых системах при любых значениях возмущения. Возникающие при этом режимы имеют перемежаемую структуру.

ПЕРЕМІЖНІСТЬ В ГАМІЛЬТОНОВИХ СИСТЕМАХ

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Подано дослідження властивостей двомірного відображення з сингулярністю. У такому відображенні спостерігається переміжність. Така система може виникати двома способами. По-перше, вона може розглядатися як результат дискретизації безперервної гамільтонової системи з одним ступенем свободи. По-друге, ми+ можемо розглядати таке відображення як переріз Пуанкаре деякої двовимірної гамільтонової системи. При цьому поведінка перерізу Пуанкаре в цілому визначає поведінку всієї системи. Досліджувался механізм виникнення переміжності поблизу сингулярності. Показано, що сингулярність призводить до виникнення стохастичного моря у гамільтонових системах за будь-яких значень збурення. Режими, що виникають при цьому, мають переміжну структуру.