DYNAMICAL CHAOS RISE IN THE SYSTEM OF LARGE NUMBER OF NONLINEAR COUPLED OSCILLATORS

V.A. Buts, I.K. Kovalchuk, and D.V. Tarasov

National Science Center "Kharkov Institute of Physics and Technology," Kharkov, Ukraine;
e-mail:kovalchuk_ik@kipt.kharkov.ua

The problem of dynamical chaos arising in distributed systems is considered. It was shown that in many cases it is possible to allocate relatively isolated subsystem which may be simpler for investigation. We suppose that chaos in this subsystem leads to chaotic behavior of all system. Besides, the allocated subsystem may be used for describing complex dynamics of nonlinear three-wave interaction, in particular, in plasma systems. The analytical criterion of arising dynamics chaos in distributed system was obtained. This criterion was confirmed by numerical simulation.

PACS: 05.45Ac, 52.35Hw

1. INTRODUCTION

The distributed physical systems are the most widely spread object in nature. For example there are hydrodynamic systems, astrophysical objects. They spread in applied physical investigation, technique and technology. As example of such systems there are different plasma ones. They may be used in nuclear fusion and in plasma electronics. The last case is interesting for us. The different electrodynamic structures filled by plasma are very interesting for designing electromagnetic generators in wide range of wave length and charged particles acceleration in different range of energy. Such systems are described by nonlinear differential equations in partial derivatives and so they are very difficult for investigation. In present time there are some number works in which a process of dynamic chaos arising is considered (see for example [1–3])

In plasma filled electrodynamics system there is large number of different oscillations kind which are often nonlinear and interact with each other. For studying these phenomena the Maxwell equations and hydrodynamics one (or kinetic equation) are used. In most cases the natural modes may be chosen in such systems for investigation. This allows to replace very complicated set of partial differential equations by ordinary differential ones. This essentially simplifies investigations of such systems. These equations describe nonlinear interaction of individual oscillators. Such model was considered in [1]. The left part of each equation in this case may be presented as equation for linear oscillation and right one is describing the nonlinear interaction between them.

In many cases it is possible to separate in distributed systems a relatively isolated subsystem which is described by means of the finite number of ordinary differential equations. We will suppose that chaos arising in such subsystem will cause chaos in all distributed system. It is necessary, of course, to note that this assumption will not be true always. Moreover, the cases are known when this assumption is not true. However discussion of this important theme we will consider in the next work.

We are interested in condition of arising of chaos in such dynamical system. Later we will follow to this mean of simplifying in distributed systems. As it will be seen later in some cases the analytical results may be obtained. As an example we considered a system in which it is possible to separate three connected nonlinear oscillators. For this case the analytical conditions are defined for arising of dynamic chaos. We note that considered system of three connected nonlinear oscillators may describe a chaos arising in three-wave interaction, in particular three-wave decay in plasma systems. In this case the HF wave decays into HF and LF waves.

Besides the present work we investigated relative role of quadratic and cubic nonlinearities. Besides analytical investigations we carried out numerical simulation. The spectra and correlation functions were obtained. It was shown that local instability is a reason of chaos arising in three-wave decay.

2. ALGORITHM OF SIMPLIFYING FOR DISTRIBUTED SYSTEM

The above considered approach may be realized particularly in the following way. Let us consider differential equation in partial derivatives. We shall restrict to consideration of the equations of second order with two independent variables x and t:

$$A\frac{\partial^2 E}{\partial t^2}+2B\frac{\partial^2 E}{\partial x\partial t}+C\frac{\partial^2 E}{\partial x^2} +D\frac{\partial E}{\partial t}+\varepsilon\,\frac{\partial E}{\partial x}+FE=\mu\,\Phi_{nonl} \qquad (1)$$

Moreover, we will consider the coefficients A, B, C, D that are constants and all nonlinearity and stationarity is taken into account by function Φ in right hand side of equation (1). We will consider that influence of nonlinearity and stationarity is small. This means that coefficient μ is small. Further we will be interested in a wave processes and so it means that coefficients A, B and C are connected by correlation (condition, inequality):

$$B^2-AC>0\,. \qquad (2)$$

This denotes that further we will be restricted to hyperbolic equations. In this case equation (1) may be reduced to the following:

$$\frac{\partial^2 E}{\partial t^2} - C^2 \frac{\partial^2 E}{\partial x^2} = \chi^2 E + \mu \Phi_{nonl} . \qquad (3)$$

We will note that this equation may be obtained, for example, from Maxwell equations to describe dynamical wave interaction in nonlinear matter, for example, in interaction of waves in plasma. For completeness of mathematical formulation it is needed to add boundary and initial conditions to equation (3). We will consider that waves interaction takes place in bounded space region $0 \le x \le L$. In this case one may use, for example, the following conditions:

$$\begin{aligned} & E(t,0) = E(t,L) = 0, \\ & E(0,x) = f(x), \\ & \left.\frac{\partial E}{\partial t}\right|_{t=0} = \Gamma(x). \end{aligned} \qquad (4)$$

The first condition in (4) denotes that amplitudes of oscillation processes at bounds of the interaction region are equal to zero. It may be, for example, a nonlinear string with fixed ends or a waves processes in long cavity. In the last case it is needed to consider transverse component of electric field as amplitude E . The solution of equation (2) we will present as following:

$$E(t,x,\mu) = \sum A_n(t,\mu) \sin(\frac{\pi n}{L} x) . \qquad (5)$$

It is needed to note that if nonlinearity is absent ($\mu = 0$) the coefficients $A_n(t)$ are

$$A_n(t) = a_n \exp(i\varpi_n t) , \qquad (6)$$

where $\varpi_n = c \sqrt{(\pi n / L)^2 - k_\perp^2}$ is the frequency of the corresponding Fourier component.

In the weakly nonlinear case equation (3) may be presented as an infinite set of connected nonlinear oscillators:

$$\frac{d^2 A_n}{dt^2} + \varpi_n^2 A_n = \mu \Phi\left(t, A_n, \dot{A}_n\right) . \qquad (7)$$

If boundary conditions (4) change then it is necessary to use solution in the form $E = \sum A_n(t) e^{i\frac{\pi n x}{L}}$ instead of solution (5). If waves interact in nonlinear matter placed in metallic cylinder with ideally conducting wall then one may write the following correlation:

$$k_{\perp m} R \equiv \lambda_m ; \; J_0(\lambda_m) = 0 , \qquad (8)$$

where $k_\perp$ defines transverse wave number and condition $J_0(\lambda_m) = 0$ defines boundary condition on cylindrical surface.

We will show that practically always one may separate two HF oscillators which will resonantly interact with one of the low frequency oscillators. Really, for HF oscillators $n >> 1$, so frequency difference of two HF oscillators may be presented in such form:

$$(\varpi_m - \varpi_n) = (\pi c / L)(m - n) . \qquad (9)$$

From expression (9) it follows that for long enough region of interaction ($L >> \lambda_{hf}$) distance between frequencies of HF oscillators may be practically of any value. It means that practically always the low frequency oscillator exists for which resonant condition is satisfied:

$$\varpi_m - \varpi_n = \Omega_{LF} . \qquad (10)$$

Thus we see that practically for any wave processes in distributed systems one may separate three resonantly interacting oscillators. Further we will suppose that if dynamics of these three oscillators is chaotic then dynamics of distributed system is chaotic also. Of course this is true not always, yet in many cases such correspondence takes place. Thus task about condition of a chaos arising in distributed systems may be reduced to simpler one: to define condition of chaos arising in system that consist of three connected nonlinear oscillators.

The analysis of dynamics of three nonlinear connected oscillators in our case may be essentially simplified. It is possible because we separated two HF oscillators. For these two oscillators one may use strict procedure of averaging. In this case the solution for HF oscillators may be presented in such form:

$$\begin{cases} A_n = x_n e^{i\varpi_n t} + x_{-n} e^{-i\varpi_n t} \\ \dot{A}_n = i\varpi_n \left[x_n e^{i\varpi_n t} - x_{-n} e^{-i\varpi_n t} \right], \end{cases} \qquad (11)$$

where new unknown functions are slowly changing. It is needed to note that instead one unknown function A_n we involve two unknown x_n and x_{-n} . For our boundary condition (4) and solution in form (5) the values A_n are real. This means that correlation $\varpi_{-n} = -\varpi_n$ takes place. Substituting solution (11) in equation (7) and carrying out averaging we obtain the following short-cut equations to define values of x_n :

$$\frac{dx_n}{dt} = \mu \frac{e^{-i\varpi_n t}}{2i\varpi_n} F_n(\ldots) , \qquad (12)$$

where in right part of (12) it is needed to keep slowly varying values. For LF oscillators we keep non short-cut equations:

$$\ddot{A}_{LF} + \Omega^2 A_{LF} = \mu F_{LF} . \qquad (13)$$

As example of using of described above algorithm for defining conditions of dynamical chaos in distributed system we will consider in following sections the interaction of two HF waves and one LF wave in plasma which may be reduced to three nonlinear connected oscillators.

3. STOCHASTIC INSTABILITY OF A WEAKLY NONLINEAR DYNAMIC WAVE-WAVE INTERACTION PROCESS

The conditions under which the waves excited in plasma have sufficiently large amplitudes can be favorable for their efficient nonlinear interaction with other eigenwaves of an electrodynamic plasma structure. The dynamics of this interaction can be either regular or stochastic. We will be interested in stochastic regimes that can occur in various nonlinear wave-wave interaction schemes. The simplest of such regimes is the modified decay process. The stochastic instability develops only when the amplitude of the decaying wave (the pump wave) exceeds a certain threshold value.

Let a wave with the amplitude a_1, wave vector k_1, and frequency ω_1 decay into two waves with parameters (a_2, k_2, ω_2) and (a_3, k_3, ω_3). Let there also be a forth wave with parameters (a_4, k_4, ω_4) such that $k_4 = k_3$ and $\omega_3 - \omega_4 \ll \omega_1$. We first assume that fourth wave does not participate in the decay process. In this case the time evolution of the amplitudes of three interaction wave is described by equations

$$\begin{aligned} \dot{a}_1 &= iV_1^* a_2 a_3, \\ \dot{a}_2 &= iV_1 a_1 a_3^*, \\ \dot{a}_1 &= iV_1 a_1 a_2^*, \end{aligned} \tag{14}$$

where $V_1 = |V_1| \exp(i\Phi_0)$ is interaction matrix element and $a_j = |a_j| \exp(i\Phi_j)$. The phase $\Phi = 2(\Phi_1 - \Phi_2 - \Phi_3 + \Phi_0)$ varies in accordance with the mathematical pendulum equations

$$\ddot{\Phi} + (2|a_1||V_1|)^2 \sin\Phi = 0 . \tag{15}$$

This equation implies that half-width of nonlinear resonance is equal to $4G$, $G = |a_1||V_1|$. The interaction between first, second, and fourth (instead of third) waves is described by the following set of equations:

$$\begin{aligned} \dot{a}_1 &= iV_2^* a_2 a_4 \exp(-i\delta t), \\ \dot{a}_2 &= iV_2 a_1 a_4^* \exp(-i\delta t), \\ \dot{a}_1 &= iV_2 a_1 a_2^* \exp(-i\delta t), \end{aligned} \tag{16}$$

where $\delta = \omega_1 - \omega_2 - \omega_4$.

In this case the phase $\Psi = 2(\Phi_1 - \Phi_2 - \Phi_4 + \Phi_0 + \delta t)$ also satisfies the mathematical pendulum equations. In this case, the half-width of nonlinear resonance is equal to $4G_2$, $G_2 = |a_1||V_2|$. The frequency separation between two nonlinear resonances is 2δ. Assuming that, in the case of interaction involving the fourth wave, the width of the nonlinear resonance is small ($G \gg G_2$), we obtain the following resonance overlap condition, or the stochastic instability criterion:

$$K = 2G/\delta > 1 . \tag{17}$$

Let us consider (as in [4]) a high-frequency (HF) electromagnetic wave with the frequency ω_i and wave vector k_i, that propagates in infinite plasma and decays into HF electromagnetic wave with parameters (ω_s, k_s) and LF Langmuir wave with the parameters (ω_p, k_p). We describe this decay process by Maxwell's equations for the electromagnetic field and the hydrodynamic equations for the plasma electrons against the neutralizing back-ground of immobile ions. Averaging over time $t_0 \gg t_{fst} \approx 1/\omega_i$ the following set of equations for slowly varying dimensionless amplitudes of HF decaying electromagnetic wave ε_i, scattered HF wave ε_s and LF plasma wave ρ was obtained:

$$\begin{aligned} i\frac{d\varepsilon_i}{d\tau} &= \varepsilon_s \rho \exp(i\Delta\tau) \\ i\frac{d\varepsilon_s}{d\tau} &= \varepsilon_s \rho^* \exp(-i\Delta\tau) \\ \frac{d^2\rho}{dx^2} + \Omega^2\rho &= \varepsilon_i \varepsilon_s^* \exp(-i\Delta\tau), \end{aligned} \tag{18}$$

where τ – is dimensionless time, Δ – dimensionless difference between ω_s and ω_s, Ω – dimensionless frequency of LF wave. In deriving equations (18) it is assumed that there is spatial synchronization between the interacting waves, $k_i - k_s = k_p$. If the growth rate of the decay instability described by (18) is much lower than frequency Ω of the LF wave, then (18) can be simplified by replacing the third of them, which is a second order differential equation, with a first-order equation for for the LF wave amplitude. Under the assumption $\omega_1 - \omega_2 = \Omega$, the set of Eqs. (18) is analogous to the set of Eqs. (14). For the case of interaction of these two HF waves with a backward LF wave (having the frequency $-\Omega$) Eqs. (5) can be reduced to (16) in which by a_4 should be meant the amplitude of the backward LF wave. Eqs. (18) refer to a situation that was described by two set of equations (14) and (16). In such system can exist chaotic regimes. For the large value amplitude of the incident wave the dynamic of this system is chaotic.

The set of equations (18) was investigated numerically. The typical shapes of ε_i, ε_s and is presented in Fig. 1. As it is seen three wave decay have chaotic character.

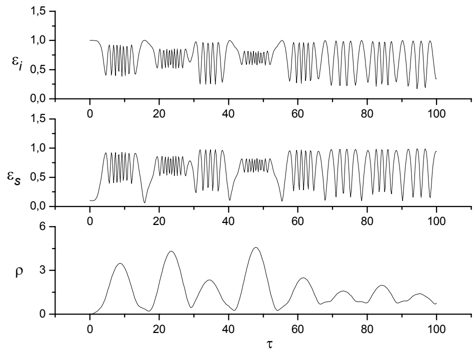

***Fig. 1**. Results of numerical simulation of equations set (18)*

4. ARISING CHAOS IN A SYSTEM OF THREE NONLINEAR CONNECTED OSCILLATORS

Later we will consider more some general system then in previous section. This is a system of three connected nonlinear oscillators that was investigated numerically. We do not make averaging in this case. Such system may be described by means a Hamiltonian that is as following:

$$\begin{aligned} H = {} & \dot{y}_1\dot{y}_1^* + \omega_1^2 y_1 y_1^* + \dot{y}_2\dot{y}_2^* + \omega_2^2 y_2 y_2^* + \dot{y}_3\dot{y}_3^* \\ & + \omega_3^2 y_3 y_3^* + V y_1 y_2^* y_3 + V^* y_2^* y_2 y_3^* \\ & + V_{11} y_1 y_1^* y_1 y_1^* + V_{12} y_1 y_1^* y_2 y_2^* + V_{13} y_1 y_1^* y_3 y_3^* \\ & + V_{22} y_2 y_2^* y_2 y_2^* + V_{23} y_2 y_2^* y_3 y_3^* + V_3 y_3 y_3^* y_3 y_3^*, \end{aligned} \tag{19}$$

where y_i is the generalized coordinate describing i-th oscillator, point above character y_i denotes derivative with respect to time, ω_i is the natural frequency of i-th oscillator, V is the coefficient describing quadratic nonlinearity and V_{ij} is the coefficient responsible for cubic nonlinear interaction between i-th and j-th oscillators. From Hamiltonian (19) we obtained equations of motion for oscillators which do not give here. One may describe the processes similar to one that we considered in previous section. Using the Hamiltonian is more general way of solving this problem We suppose that second oscillator corresponds to the high frequency wave with frequency ω_2 which decays into HF wave with frequency ω_1 and low frequency with ω_3. Thus the condition $\omega_2 = \omega_1 + \omega_3$ is satisfied, and $\omega_{1,2} \gg \omega_3$. In [5] the criterion for arising of chaos conditioned by quadratic nonlinearity was obtained and looks as following:

$$K = \frac{V y_{20}}{\omega_3} > 1, \tag{20}$$

where y_{20} is the initial amplitude of decaying wave. A system described by Hamiltonian (19) was investigated numerically separately for quadratic and cubic nonlinearity. A spectrum and correlation function was calculated. Lyapunov exponent and characteristic number of Jacobian were calculated for quadratic nonlinearity. It is needed to note that when the condition (20) was satisfied among characteristic numbers there were such whose real part was positive. In this case the realization of y_1, y_2 and y_3 looks as no more regular as demonstrated in Fig. 2.

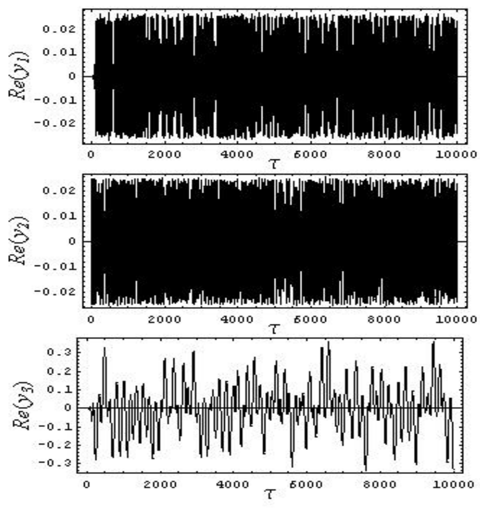

***Fig. 2**. Realization of y_1, y_2 and y_3 for system described by Hamiltonian (19) without cubic addendum*

The Lyapunov exponent was calculated by means of Benettin algorithm and was positive. It indicates local instability in phase space and chaotic behavior of oscillations. The spectrum and correlation function for $\mathrm{Re}(y_1)$ is given in Fig. 3 and 4 correspondingly.

As it is seen the correlation function decreases. When inequality (20) is inverse the chaotic behaviors disappear gradually, thus it does not have a threshold. In this case the spectrum converges gradually and correlation time increases gradually too when y_{20} decreases and oscillations become regular.

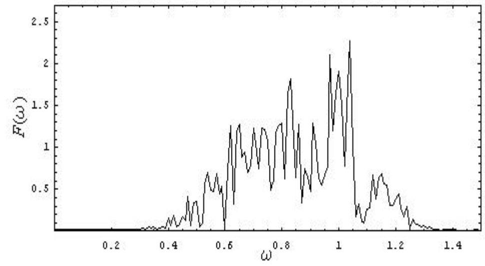

***Fig. 3**. Spectrum for $\mathrm{Re}(y_1)$ presented in Fig. 2.*

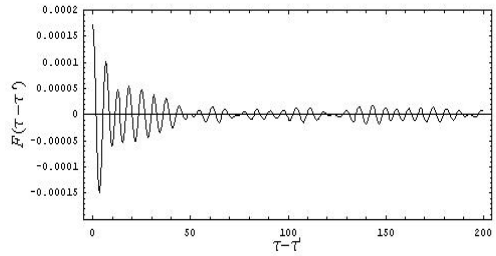

***Fig. 4**. Correlation function for $\mathrm{Re}(y_1)$ presented in Fig. 2.*

Numerical simulation for cubic nonlinearity showed that chaos in this case arises at large initial value of generalized coordinates. As it is seen from (19) the cubic addendums are responsible for nonlinear frequency shift.

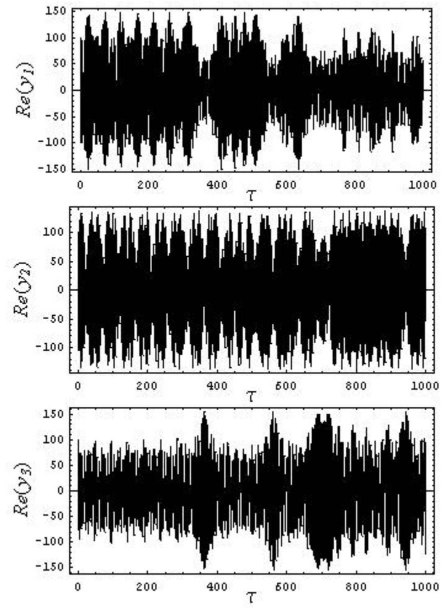

***Fig. 5**. Realization of y_1, y_2 and y_3 for system described by Hamiltonian (19) with cubic addendum*

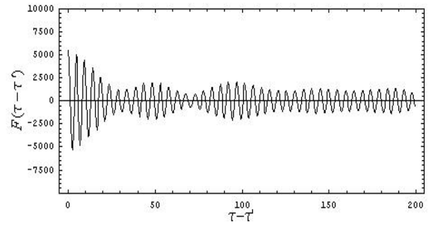

***Fig. 6.** Correlation function for realizations presented in Fig. 5*

Thus the quadratic nonlinearity plays essential role in chaos arising in system of connected nonlinear oscillators. Chaos in this case arises at lower level of initial amplitude then for cubic nonlinearity. The most characteristic realizations for this case are presented in Fig. 5 and correlation function in Fig. 6.

5. CONCLUSIONS

In this report the problem of chaos arising in complex distributed system is considered. The algorithm of reducing of investigation is proposed and used for concrete physical system. Such system may be used for investigation of different physical processes. In particular it may describe weakly nonlinear interaction waves in plasma electrodynamics system. It may be a nonlinear decay of a high frequency electromagnetic wave into new one and a low frequency wave. At certain conditions which were presented above this process is chaotic. It was shown that in such system the chaotic regimes may exist.

The results of numerical investigation of chaos arising in system which contains three nonlinear connected oscillators are presented. The numerical simulation showed the existence of chaotic regimes for quadratic nonlinearity. It was confirmed by calculation of spectrum, correlation function and Lyapunov exponents. When process is chaotic the spectrum is spread, correlation function decreases, and Lyapunov exponents have positive real part. It points on the local instability.

It is needed to note that investigation of dynamical chaos in distributed systems with cubic nonlinearity was carried out in [2]. The analytical condition of appearance of local instability were obtained. Comparing this conditions with (20) shows that dynamical chaos conditioned by quadratic nonlinearity develops at smaller amplitude values of interacting wave.

REFERENCES

1. F.M. Izrailev, B.V.Chirikov. Statistical properties of nonlinear string //*DAN USSR*. 1966, v. 166, N1, p. 57–59 (in Russian).
2. G.M. Zaslavsky, R.Z. Sagdeev. *Introduction to nonlinear physics: from pendulum to turbulence*. M.: "Nauka", 1988 (in Russian).
3. T.V. Dmitrieva, N.M. Rybkin. Complex dynamics and chaos at parametric interaction of encountering waves //*JETP*. 1999, v. 116, N5(11), p. 1871-1881 (in Russian).
4. V.A. Buts, I.K. Koval'chuk, E.A. Kornilov, D.V. Tarasov. Stabilization of Beam Instability by a Local Instability Developing due to a Wave-wave Interaction //*Fizika Plasmy.* 2006, v. 1, N7, p. 613-622.
5. V.A. Buts, O.V. Manuilenko, A.P. Tolstoluzhskii, Yu.A. Turkin. Stochastic instability of the modified decay //*International Congress on Plasma Physics combines with the 25-th EPS conference on Controlled Fusion and Plasma Physics. Prague, Czech Rep., June 29-July 3, 1998*. Conf. Proc., p. 252-255. (ECA vol. 22C (1998) 252-255).

РАЗВИТИЕ ДИНАМИЧЕСКОГО ХАОСА В СИСТЕМЕ БОЛЬШОГО ЧИСЛА НЕЛИНЕЙНЫХ СВЯЗАННЫХ ОСЦИЛЛЯТОРОВ

В.А. Буц, И.К. Ковальчук, Д.В. Тарасов

Рассматривается проблема возникновения динамического хаоса в распределенных системах. Показано, что во многих случаях возможно выделить относительно изолированные подсистемы в распределенных системах, которые могут быть значительно проще для исследования. Мы полагаем, что хаос в этих подсистемах является источником хаотического поведения всей системы. Кроме того, выделенные системы могут быть использованы для описания сложной динамики нелинейного трехволнового взаимодействия, в частности, в плазменных системах. Получен аналитический критерий возникновения динамического хаоса для выделенных подсистем. Этот критерий подтвержден численными исследованиями.

РОЗВИТОК ДИНАМІЧНОГО ХАОСУ В СИСТЕМІ ВЕЛИКОЇ КІЛЬКОСТІ НЕЛІНІЙНИХ ЗВ'ЯЗАНИХ ОСЦИЛЯТОРІВ

В.О. Буц, І.К. Ковальчук, Д.В. Тарасов

Розглядається проблема виникнення динамічного хаосу в розподілених системах. Показано, що в багатьох випадках можливо виділити відносно ізольовані підсистеми в розподілених системах, які можуть бути значно простіше для дослідження. Ми вважаємо, що хаос у цих підсистемах є джерелом хаотичної поведінки всієї системи. Крім того, виділені системи можуть бути використані, для опису складної динаміки нелінійної трьох-хвильової взаємодії, зокрема в плазмових системах. Отримано аналітичний критерій виникнення динамічного хаосу для виділених підсистем. Цей критерій підтверджений чисельними дослідженнями.