GROUP PROPERTIES OF OSP(2/1;C) GAUGE TRANSFORMATIONS

K. Ilyenko[1,2]

[1]*Institute for Radiophysics and Electronics of NAS of Ukraine, Kharkiv, Ukraine;*
e-mail: kost@ire.kharkov.ua;
[2]*Kharkiv Humanitarian Pedagogical Institute*

Given an explicit construction of the grade star hermitian adjoint representation of $\mathrm{osp}(2/1;\mathbf{C})$ graded Lie algebra, we consider the Baker-Campbell-Hausdorff formula and find reality conditions for the Grassmann-odd transformation parameters that multiply the pair of odd generators of the graded Lie algebra. Utilization of $\mathrm{su}(2)$ -spinors clarifies the nature of Grassmann-odd transformation parameters and allows one an investigation of the corresponding infinitesimal gauge transformations. We also explore the action of a corresponding group element on an appropriately graded representation space and find that a proper (graded) generalization of hermitian conjugation is consistent with a natural generalization of Dirac adjoint. A corresponding generalization of a unitary transformation is discussed.

PACS: 11.10.Ef, 11.15.-q

1. INTRODUCTION

A natural extension of the Lie algebras, which underlie the modern gauge theory, are graded Lie algebras introduced and studied to some extent in the articles [1-3]. In this paper we explore how one can utilize a Z_2-graded extension $\mathrm{osp}(2/1;\mathbf{C})$ of the compact Lie algebra $\mathrm{su}(2)$ for the purposes of defining a meaningful gauge theory of the Yang-Mills type (see, e.g., [4-6]). The form of $\mathrm{osp}(2/1;\mathbf{C})$ defining relations (proposed in [5] and refined in [6]) utilizes the Pauli matrices and strongly suggests a relation to spinors. Exponentiating the algebra, we observe the necessity of introduction of anticommuting (Grassmann-odd) spinors, which multiply the odd generators of the graded Lie algebra. Finally, we study some of infinitesimal properties of the composition law of group transformations and consider a generalization of the Dirac adjoint, thus, making preparations for an investigation of the gauge invariance of the proposed field strength for such a gauge theory, [5-6].

2. GRADED LIE ALGEBRA OSP(2/1;C)

The algebra $\mathrm{osp}(2/1;\mathbf{C})$ is a graded extension of $\mathrm{su}(2)$ algebra by a pair of odd generators, τ_A, which anticommute with one another and commute with the three even generators, T_a, of $\mathrm{su}(2)$. It is customary to assign a degree, $\deg T_\alpha$, to the even ($\deg T_a = 0$) and odd ($\deg \tau_A = 1$) generators. We use the square brackets to denote the commutator and the curly ones to denote the anticommutator. The defining relations have the form, [3,5-7]:

$$[T_a, T_b] = i\varepsilon_{abc}T_c; [T_a, \tau_A] = \frac{1}{2}(\sigma_a)_A{}^B \tau_B; \quad \{\tau_A, \tau_B\} = \frac{i}{2}(\sigma^a)_{AB} T_a. \tag{1}$$

Summation is assumed over all repeated indices. Lowercase Roman indices from the beginning of the alphabet run from 1 to 3; uppercase Roman indices run over 1 and 2; $\delta_{ab} = \delta^{ab}$ ($\delta_{ab} = \delta_{ba}$), ε_{abc} ($\varepsilon_{abc} = 1$) and $\in_{AB}$ ($\in_{12} = \in^{12} = 1$) are the three dimensional identity matrix and the Levi-Civita totally antisymmetric symbols in three and two dimensions, respectively; the matrices $(\sigma_a)_A{}^B$ $[(\sigma^a)_{AB} = (\sigma^a)_{BA} = \delta^{ab}(\sigma_a)_{AB} = \delta^{ab}(\sigma_a)_{AB} = \delta^{ab}(\sigma_b)_A{}^C \in_{CB}]$ are just the usual Pauli matrices:

$$(\sigma_a)_A{}^B = \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right];$$

$$(\sigma^a)_{AB} = \left[\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right].$$

We use the Levi-Civita symbols in two dimensions to raise and lower uppercase Roman indices paying attention to their antisymmetric properties:

$$\Sigma \equiv \|\in^{AB}\| = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \|\in_{AB}\| = -\Sigma^{-1}.$$

Note that, as concerned to these indices, we are working with two-component spinors and adopt conventions of the book [8]. We shall follow those conventions as more suitable for our purposes even when complex conjugation of spinor and Grassmann quantities is involved.

It turns out that not all of the $\mathrm{osp}(2/1;\mathbf{C})$ algebra generators are hermitian. A proper generalization of the hermitian conjugation is denoted by (‡): on the even generators the operation coincides with ordinary hermitian conjugation ($^+$) while the odd ones obey more complicated relations. Following the papers [9-10], we shall call them the grade star hermiticity conditions:

$$\tau_\pm^\ddagger = \pm\tau_\mp, \tag{2}$$

where $\tau_\pm = \tau_1 \pm i\tau_2$.

Let us consider complex-valued matrices:

$$M_{\text{even}} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \text{ and } M_{\text{odd}} = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix},$$

where, for the purposes of this paper, B and C are 2×3 rectangular blocks and A and D are 3×3 and 2×2 square blocks representing the division of a representation space into even and odd parts. On these matrices the grade star hermiticity condition reads

$$M^{\ddagger}_{\text{even}} = \begin{pmatrix} A^{+} & 0 \\ 0 & D^{+} \end{pmatrix} \text{ and } M^{\ddagger}_{\text{odd}} = \begin{pmatrix} 0 & -C^{+} \\ B^{+} & 0 \end{pmatrix}.$$

We shall also use multiplication of algebra generators by scalars. Such an operation must take into account that Grassmann-odd scalars anticommute with the odd algebra generators while commute with complex numbers and the even algebra generators, [11]. The following construction possesses all of these properties. Let a be a scalar and $\deg a$ be its degree (0 or 1 depending on whether it is Grassmann-even or Grassmann-odd, respectively). Then multiplication by a is defined as follows:

$$aM_{\text{even}} = M_{\text{even}}a \text{ and } aM_{\text{odd}} = (-1)^{\deg a} M_{\text{odd}}a .$$

3. THE GROUP PROPERTY

Given a Lie algebra one can turn over to a Lie group by exponentiating the generators multiplied by transformation parameters. This, in a usual fashion, gives us the gauge transformations. In the case of a graded Lie algebra we are faced with a problem: anticommutators seem to rule out the application of the Baker-Campbell-Hausdorff formula, which is necessary to prove that subsequent transformations do not leave the group manifold. This problem is solved via introduction of Grassmann-odd parameters (cf., [1]). In the case under consideration these are Grassmann-odd su(2) - spinors ξ^A, θ^A, etc., which multiply the odd generators. They are included on equal footing with ordinary (Grassmann-even) parameters ε^a multiplying the even generators (hopefully, there will not be confusion about use the same kernel letter, ε, to denote a Grassmann-even transformation parameter and the Levi-Civita totally antisymmetric symbol in three dimensions). By definition, ξ^A, θ^A, etc. satisfy

$$[\varepsilon^a,\theta^A]=0; \quad \{\xi^A,\xi^B\}=\{\theta^A,\theta^B\}=0;$$
$$[\xi^A,\theta^B]=0.$$

Then, the necessary relations can be given in terms of commutators only:

$$[\xi^A\tau_A,\theta^B\tau_B] = -\frac{i}{2}(\xi^{\{A}\theta^{B\}} + \xi^{[A}\theta^{B]})(\sigma^a)_{AB}T_a,$$

where $\xi^{\{A}\theta^{B\}} = (\xi^A\theta^B + \xi^B\theta^A)/2$ and $\xi^{[A}\theta^{B]} = (\xi^A\theta^B - \xi^B\theta^A)/2$ are convenient shorthand notations. This result was obtained using anticommutator for odd generators in definition (1). Using a fundamental fact of spinor algebra, $\in_{AB}\in_{CD} + \in_{AC}\in_{DB} + \in_{AD}\in_{BC} = 0$, one can calculate

$$\xi^{[A}\theta^{B]} = \frac{1}{2}(\xi_C\theta^C)\in^{AB}.$$

From symmetry of $(\sigma^a)_{AB}$ in the uppercase indices, it then follows that

$$\begin{aligned}[\xi^A\tau_A,\theta^B\tau_B] &= -\frac{i}{2}\xi^{\{A}\theta^{B\}}(\sigma^a)_{AB}T_a \\ &\equiv -\frac{i}{2}[\xi^A,\theta^B](\sigma^a)_{AB}T_a\end{aligned} \tag{3}$$

and, in particular, the commutator $[\theta^A\tau_A,\theta^B\tau_B]$ vanishes identically. One can also calculate

$$[\kappa^aT_a,\varepsilon^bT_b] = i\kappa^{[a}\varepsilon^{b]}\varepsilon_{abc}T^c;$$
$$[\varepsilon^aT_a,\theta^A\tau_A] = \frac{1}{2}\varepsilon^a\theta^A(\sigma_a)_A{}^B\tau_B,$$

where $\varepsilon^a\theta^A(\sigma_a)_A{}^B$ is again a Grassmann-odd transformation parameter.

Group elements are obtained by exponentiating the algebra

$$U(\varepsilon,\theta) = \exp[i(\varepsilon^aT_a + \theta^A\tau_A)] \tag{4}$$

and the Baker-Campbell-Hausdorff formula,

$$\exp M \exp N = \exp(M + N + \frac{1}{2}[M,N] + \ldots) \tag{5}$$

may be applied to determine motion in the parameter space under a (left) multiplication with a group element $U(\kappa,\xi)$:

$$U(\varepsilon',\theta') = U(\kappa,\xi)U(\varepsilon,\theta). \tag{6}$$

4. INFINITESIMAL TRANSFORMATIONS AND REALITY CONDITIONS

Let us examine expression (5) restricting ourselves by taking into account the first non-trivial contribution – the two-fold commutator $[M,N]$. Writing $M = i(\varepsilon^aT_a + \theta^A\tau_A)$ and $N = i(\varepsilon^aT_a + \theta^A\tau_A)$, we have

$$i(\varepsilon^aT_a + \theta^A\tau_A) = M + N + \frac{1}{2}[M,N] + \ldots,$$

where dots denote the sum of linear combinations of k-fold ($k > 2$, $k \in \mathbf{Z}$) commutators of M and N [12]. Substituting expressions for M, N and using (3), we obtain after some algebra

$$\varepsilon'^a = \varepsilon^a + \kappa^a - \frac{1}{2}\kappa_b\varepsilon_c\varepsilon^{bca} + \frac{1}{4}[\xi^A,\theta^B](\sigma^a)_{AB}\ldots;$$

$$\theta'^A = \theta^A + \xi^A + \frac{i}{4}(\kappa_b\theta^B - \varepsilon_b\xi^B)(\sigma^b)_B{}^A \ldots \tag{7}$$

Here again dots denote the contribution from the sum of linear combinations of k-fold ($k > 2$, $k \in \mathbf{Z}$) commutators. The first three summands in the first row of formula (7) reflect the non-commutative character of the proper Lie subalgebra, su(2), of osp(2/1;**C**) the last one being contribution from the odd part of the algebra. The last summand in the second row of the formula is obviously a Grassmann-odd quantity, and it reflects the non-commutative property of the even and odd parts of the graded Lie algebra.

In the view of intended applications, contribution from Grassmann-odd part of the algebra into the law of composition of Grassmann-even parameters needs to be investigated in more detail. First, let us calculate that

$$\begin{aligned} &2[\xi^A,\theta^B](\sigma^a)_{AB} \\ &= \xi_A \in^{AB} (\sigma^a)_B{}^C \theta_C - \theta_A \in^{AB} (\sigma^a)_B{}^C \xi_C \\ &= \xi^T \Sigma \sigma^a \theta - \theta^T \Sigma \sigma^a \xi, \end{aligned} \tag{8}$$

where we employed some self-evident matrix notations; the superscript (^{T}) denotes transposition. Comparing the result (8) and a description of su(2)-spinors of 3D Euclidean space in the book [13, p. 48], one immediately realizes that the last term of the first equation in system (7) is, in general, a *complex* vector of 3D Euclidean space, e.g. it transforms like a vector under SO(3) transformations. Second, the representation (8) tells us that components of this vector vanish if $\xi_A = \theta_A$ as required by a property of a one-parameter subgroup of transformations (6). Finally, this vector also has all components equal to zero if $\xi_A = -\theta_A$. This shows that the inverse of the group element $U(\varepsilon,\theta)$ has the form

$$U^{-1}(\varepsilon,\theta) = \exp[-i(\varepsilon^a T_a + \theta^A \tau_A)].$$

If one intends, as customarily done in a meaningful Yang-Mills theory, to treat ε^a, κ^a, etc. as real-valued Grassmann-even transformation parameters, then it is necessary to impose some conditions on the su(2)-spinors ξ_A, θ_A, etc. in order to ensure that (8) will be a *real* 3D Euclidean vector. Such a condition must be compatible with transformation properties of the corresponding space of su(2)-spinors, ξ_A, and take into account that its members are also Grassmann-odd quantities. In fact, this condition should involve a passage from an su(2)-spinor to its conjugate and, thus, rely on the definition of an anti-involution in the space of spinors (see, e.g. [13, p. 100]). Let us observe first that for a Grassmann algebra on one generator the last term in the first relation in (7) vanishes identically. This is a somewhat trivial situation. The next non-trivial one arises when all su(2)-spinors under consideration take values in a Grassmann algebra on two odd generators, β_1 and β_2: $\beta_1^2 = \beta_2^2 = 0$, $\beta_1\beta_2 = -\beta_2\beta_1$ (see, e.g. [14, p. 7]). We shall employ lowercase Roman indices from the middle of the alphabet running over 1 and 2 to enumerate the decompositions of various quantities in the corresponding basis of the Grassmann algebra. Decomposing ξ_A and θ_B into this basis one obtains

$$\xi_A = \underset{i}{\xi}{}_A \beta_i \text{ and } \theta_B = \underset{j}{\theta}{}_B \beta_j,$$

where $\underset{i}{\xi}{}_A$ and $\underset{j}{\theta}{}_B$ are ordinary (commuting), su(2)-spinors of 3D Euclidean space, and summation over repeated indices is assumed. In this case we can write

$$[\xi^A,\theta^B](\sigma^a)_{AB} = 2\beta_1\beta_2(\underset{1}{\xi}{}^T \Sigma\sigma^a \underset{2}{\theta} - \underset{1}{\theta}{}^T \Sigma \sigma^a \underset{2}{\xi}). \tag{9}$$

Now we shall impose some additional conditions on su(2)-spinors $\underset{i}{\xi}{}_A$ and $\underset{j}{\theta}{}_A$, etc. to ensure that (9) gives a *real* Grassmann-even 3D Euclidean vector. One way of doing so in a manner preserving all the spinor transformations properties is to define

$$\underset{1}{\xi}{}_A = iC_A{}^{B'} \bar{\underset{2}{\xi}}{}_{B'},\ \underset{1}{\theta}{}_A = iC_A{}^{B'} \bar{\underset{2}{\theta}}{}_{B'}, \text{ etc.}, \tag{10}$$

where the 'charge conjugation' matrix C ($C\bar{C} = -I$) is given by

$$C = \| C_A{}^{B'} \| = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \| \bar{C}_{A'}{}^B \| = \bar{C}. \tag{11}$$

In (10) and (11) a bar over the spinors in the left-hand sides of the relations and primes over the indices denote complex conjugation. We again adhere to Penrose's notations whenever spinors are concerned, [8]. The charge conjugation matrix, $C_A{}^{B'}$, is responsible for invariant preservation of spinor properties (for details see, e.g. the review article [15, p. 72-73]; also compare with the treatment in [13, p. 100]). Note that the definitions (10) are essentially *the proper generalization of reality conditions* from numbers to spinors. As also seen from those definitions, each Grassmann-odd su(2)-spinor ξ_A, θ_A, etc. is defined by a single ordinary, i.e. complex-valued Grassmann-even, su(2)-spinor. For the sake of notations denoting

$$\eta_A = \underset{2}{\xi}{}_A \text{ and } \vartheta_B = \underset{2}{\theta}{}_B,$$

respectively, we write

$$\begin{aligned} \upsilon^a &\equiv \underset{1}{\xi}{}^T \Sigma\sigma^a \underset{2}{\theta} - \underset{1}{\theta}{}^T \Sigma\sigma^a \underset{2}{\xi} \\ &= i(\bar{\eta}^T C^T \Sigma \sigma^a \vartheta - \bar{\vartheta}^T C^T \Sigma \sigma^a \eta). \end{aligned}$$

On comparison with [13, p. 50], one can check that υ^a is indeed a *real* 3D Euclidean vector. In components it reads:

$$\begin{aligned} \upsilon^1 &= i(\bar{\eta}_{1'}\vartheta_2 - \bar{\vartheta}_{2'}\eta_1 + \bar{\eta}_{2'}\vartheta_1 - \bar{\vartheta}_{1'}\eta_2); \\ \upsilon^2 &= \bar{\eta}_{2'}\vartheta_1 + \bar{\vartheta}_{1'}\eta_2 - \bar{\eta}_{1'}\vartheta_2 - \bar{\vartheta}_{2'}\eta_1; \\ \upsilon^3 &= i(\bar{\eta}_{1'}\vartheta_1 - \bar{\vartheta}_{1'}\eta_1 - \bar{\eta}_{2'}\vartheta_2 + \bar{\vartheta}_{2'}\eta_2). \end{aligned}$$

These are obviously real quantities and the vector υ^a vanishes if and only if $\eta_A = \pm\vartheta_A$ as expected.

5. ACTION ON A REPRESENTATION SPACE

Having formulated meaningful reality conditions, we are in position to explore action of the group element (4) on a suitable vector space.

First, let us observe that because of definition of the matrix U by its Taylor's expansion, the fact that the generators T_a and τ_A are 'block' and 'off-block' diagonal (see [6]), respectively, their multiplication properties and those of the Grassmann-even and Grassmann-odd transformation parameters, it is easy to see that any matrix U has a specific decomposition

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \qquad (12)$$

where A is a ($r \times p$) sub-matrix, B is a ($s \times p$) sub-matrix, C is a ($r \times q$) sub-matrix and D is a ($s \times q$) sub-matrix. Following nomenclature of the book [14], we shall call the matrix U a ($p/q \times r/s$) super-matrix; in the adjoint representation under consideration of matrices T_a and τ_A, A is a (3×3) sub-matrix, B is a (2×3) sub-matrix, C is a (3×2) sub-matrix and D is a (2×2) sub-matrix. Moreover, the sub-matrices A and D have contributions only from an even number of τ s' multipliers and, hence, only even multipliers of Grassmann-odd transformation parameters θ s' are present there. Thus, elements of those sub-matrices are in the even subspace, $\mathbf{C}B_{L0}$, of the complex Grassmann algebra (for more details see the book [14, p. 10-11]). The sub-matrices B and C by an analogues argument include an odd number of τ s' and θ s' multipliers and, hence, are in the odd subspace, $\mathbf{C}B_{L1}$, of the complex Grassmann algebra. Therefore, any such a super-matrix U is an *even* super-matrix and by the results of the previous section such matrices form a supergroup. Furthermore, by construction any such a super-matrix is invertible.

Second, consider *even* super-column Ψ [($p/q \times 0/1$) super-matrices] and super-row Φ [($1/0 \times r/s$) super-matrices] vectors:

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \text{ and } \Phi = (\Phi_1, \Phi_2), \qquad (13)$$

where Ψ_1 and Φ_1 are ($1 \times p$) and ($r \times 1$) sub-matrices, Ψ_2 and Φ_2 are ($1 \times q$) and ($s \times 1$) sub-matrices, respectively. The elements of Ψ_1 and Φ_1 are Grassmann-even and those of Ψ_2 and Φ_2 are Grassmann-odd entities. *Action of even super-matrices U on such even super-column(-row) vectors transform them again into even super-column(-row) vectors*.

For the sake of argument let U be ($1/1 \times 1/1$) matrices [see (12)], the actual size can be easily treated the same way, and let also Ψ be a ($1/1 \times 0/1$) even super-column vector as regarded to the linear transformations defined below. The entries Ψ_1 and Ψ_2 themselves could be, for example, Dirac bispinors. Consider a linear transformation

$$\Psi' \equiv \begin{pmatrix} \Psi_1' \\ \Psi_2' \end{pmatrix} = \begin{pmatrix} A\Psi_1 + B\Psi_2 \\ C\Psi_1 + D\Psi_2 \end{pmatrix} \equiv U\Psi . \qquad (14)$$

Taking transposition of each line in (14) (it acts on Ψ_i 's) and complex conjugate as well as denoting the Dirac conjugates as $\bar{\Psi}_i'$, we obtain:

$$(\bar{\Psi}_1', \bar{\Psi}_2') = (\bar{\Psi}_1 A^* - \bar{\Psi}_2 B^*, \bar{\Psi}_1 C^* + \bar{\Psi}_2 D^*)$$
$$= (\bar{\Psi}_1, \bar{\Psi}_2) \begin{pmatrix} A^* & C^* \\ -B^* & D^* \end{pmatrix},$$

where the Grassmann character of the involved quantities has been taken into account. Recall that for any super-matrix U partitioned as in (12) the super-transpose is defined by

$$U^{st} = \begin{pmatrix} A^T & (-1)^{\deg U} C^T \\ -(-1)^{\deg U} B^T & D^T \end{pmatrix}; \qquad (15)$$

for *even* super-column(-row) vectors this implies:

$$\Psi^{st} = (\Psi_1^T, \Psi_2^T) \text{ and } \Phi^{st} = \begin{pmatrix} \Phi_1^T \\ -\Phi_2^T \end{pmatrix}. \qquad (16)$$

Given definitions imply

$$\begin{pmatrix} A^* & C^* \\ -B^* & D^* \end{pmatrix} = \begin{pmatrix} A & C \\ -B & D \end{pmatrix}^* = \left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{st} \right)^*, \qquad (17)$$

i.e. if, as in (14), $\Psi' = U\Psi$ then

$$\bar{\Psi}' = \bar{\Psi} U^{\ddagger}, \qquad (18)$$

thus, generalizing the corresponding result in the Yang-Mills gauge theory.

It also follows from (17) that grade star hermitian conjugation, denoted earlier by $\ddagger$, can be interpreted as *complex conjugated of super-transpose*.

6. DISCUSSION AND OUTLOOK

The result (18) calls for a study of an analogue of 'unitary' property for matrices U given by (4). A first suggestion would be to have $U^{\ddagger} = U^{-1}$, however, a direct calculation shows that this does not hold. A direct calculation shows that

$$(T_a)^{\ddagger} = T_a \text{ and } (\tau_A)^{\ddagger} = -i\bar{C}_{A'}{}^{B} \tau_B . \qquad (19)$$

The later equation in (19) is just a re-statement of hermiticity of the even generators of the graded Lie algebra $\mathrm{osp}(2/1;\mathbf{C})$ while the former one, to the best of the author's knowledge, for the first time exhibits a strong connection between compact graded Lie algebras on one side and Euclidean spinors on the other (cf. (2)). This same property of τ_A 's prevents one from having $U^{\ddagger} = U^{-1}$. Traced back to basic definitions, the problem lies in the very definition of the *super-transpose*, in particular, $(\Psi^{st})^{st} \neq \Psi$ but

$$(((\Psi^{st})^{st})^{st})^{st} = \Psi ,$$

which also motivates our spinor approach to the treatment of graded Lie algebras. In the view of this spinor connection, it is understandable that a generalization of unitary property for a matrix U given by (4) cannot have the form $U^{\ddagger} = U^{-1}$. In the theory of the Dirac equation we have $\bar{\varphi}\varphi \equiv \varphi^{+}\Pi\varphi$, where the matrix Π, which numerically coincides with γ_0 in the chiral and Dirac representations of γ-matrices but not in the Majorana representation, defines the *invariant real-valued internal product* on the (*symplectic*) space of Dirac bispinors φ, [15, p. 49-50; 16, p. 85]. Then, for a linear transformation, S, $\varphi' = S\varphi$ preserving the internal product, we have $S^{+} = \Pi S^{-1}\Pi^{-1}$ (cf., e.g., [17, p. 32]) instead of unitary property for S.

It is, therefore, natural to define an internal *unitary-symplectic* product on the space given by even supervectors (13), where the adjoint representation of the graded Lie algebra $\mathrm{osp}(2/1;\mathbf{C})$ acts, as

$$\Psi^{\ddagger}\Pi\Psi = \mathrm{inv} \tag{20}$$

with the matrices Π given by

$$\Pi = \begin{pmatrix} I & 0 \\ 0 & D \end{pmatrix}. \tag{21}$$

Here I is the square three-dimensional identity matrix while hermiticity property of the square two-dimensional matrix D remains to be determined. To clarify this issue, let us consider the grade star hermitian conjugated of (20) with account for (15):

$$(\Psi^{\ddagger}\Pi\Psi)^{\ddagger} = \Psi^{\ddagger}\Pi^{\ddagger}(\Psi^{\ddagger})^{\ddagger} = \mathrm{inv}\,,$$

where

$$\Pi^{\ddagger} = \begin{pmatrix} I & 0 \\ 0 & D^{+} \end{pmatrix}.$$

Using (13) and (16), we infer

$$(\Psi^{\ddagger})^{\ddagger} = \begin{pmatrix} \Psi_1 \\ -\Psi_2 \end{pmatrix},$$

where Ψ_1 is (1×3) sub-matrix and Ψ_2 is (1×2) sub-matrix in terms of definition (13). Denoting $\tilde{\Psi} = (\Psi^{\ddagger})^{\ddagger}$, we can preserve the invariant property of the internal product (20) by requiring

$$\Pi^{\ddagger}\tilde{\Psi} = \Pi\Psi\,.$$

This can be accomplished if the matrix D is *anti-hermitian*

$$D^{+} = -D\,.$$

Taking into account the described analogy with the Dirac bispinors, one is lead to the following choice:

$$D = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{22}$$

where the matrix D numerically coincides with $i\sigma_1$.

Thus, it is necessary to check that for a matrix U given by (4)

$$U^{\ddagger} = \Pi U^{-1}\Pi^{-1}$$

holds, where (21) and (22) define the matrix Π.

ACKNOWLEDGMENTS

I am grateful to Drs. T.S. Tsou and V. Pidstrigach for an interest to this work and to Dr. V. Gorkavyi and Prof. Yu.P. Stepanovsky for numerous helpful discussions.

REFERENCES

1. F.A. Berezin, G.I. Kac. Lie groups with commuting and anticommuting parameters //*Math. USSR – Sb*. 1971, v. 11, p. 311-325.
2. L. Corwin, Y. Ne'eman, S. Stenberg. Graded Lie algebras in mathematics and physics (Bose-Fermi symmetry) //*Rev. Mod. Phys*. 1975, v. 47, p. 573-603.
3. V. Kac. Representations of classical Lie superalgebras //*LMS Lecture Notes*. 1977, v. 676, p. 597-626.
4. Y. Ne'eman. Irreducible gauge theory of a consolidated Salam-Weinberg model //*Phys. Lett. B*. 1979, v. 81, p. 190-194.
5. R. Brooks, A. Lue. The monopole equations in topological Yang-Mills //*J. Math. Phys*. 1996, v. 37, p. 1100-1105.
6. K. Ilyenko. Field strength for graded Yang-Mills theory //*Problems of Atomic Science and Technology*. 2001, v. 6, p. 74-75 (hep-th/0307230).
7. J.W. Hughes. Representations of $\mathrm{osp}(2/1;\mathbf{C})$ and the metaplectic representation //*J. Math. Phys*. 1981, v. 22, p. 245-250.
8. J. Stewart. *Advanced general relativity*. Cambridge: "Cambridge University Press", 1996, 228 p.
9. M. Scheunert, W. Nahm, Y. Rittenberg. Graded Lie algebras: generalization of hermitian representations //*J. Math. Phys*. 1977, v. 18, p. 146-154.
10. M. Scheunert, W. Nahm, Y. Rittenberg. Irreducible representations of the $\mathrm{osp}(2,1)$ and $\mathrm{spl}(2,1)$ graded Lie algebras //*J. Math. Phys*. 1977, v. 18, p. 155-162.
11. L.E. Gendenshtein, I.V. Krive. Supersymmetry in quantum mechanics //*Sov. Phys. – Usp*. 1985, v. 28, p. 645-666.
12. W. Magnus, A. Karras, D. Solitar. *Combinatorial group theory*. London: "Interscience", 1966, 578 p.
13. E. Cartan. *The theory of spinors*. Paris: "Hermann", 1966, 157 p.
14. J.F. Cornwell. *Group theory in physics*. London: "Academic Press", 1989, v. 3, 628 p.
15. P.K. Rashevskij. The theory of spinors //*Transl. Am. Math. Soc. II* (*Ser.* 6) 1957, p. 3-77.
16. H.K. Dreiner, H.E. Haber, S.P. Martin. *Supersymmetry*. Cambridge: "CUP", 2004, 271 p.
17. J.D. Bjorken, S.D. Drell. *Relativistic quantum mechanics*. New York: "McGraw Hill", 1975, 732 p.

ГРУППОВЫЕ СВОЙСТВА OSP(2/1;C) КАЛИБРОВОЧНЫХ ПРЕОБРАЗОВАНИЙ

К. Ильенко

На основе явной конструкции градуированного обобщенно-эрмитового присоединенного представления $\mathrm{osp}(2/1;\mathbf{C})$ градуированной алгебры Ли рассмотрена формула Бейкера-Кэмпбелла-Хаусдорффа и найдены условия вещественности, налагаемые на грассманово-нечетные параметры, которые являются множителями пары нечетных генераторов градуированной алгебры Ли при экспоненцировании. Использование формализма $\mathrm{su}(2)$-спиноров поясняет природу грассманово-нечетных параметров и существенно облегчает исследование соответствующих инфинитезимальных калибровочных преобразований. Также изучено действие общего группового элемента на подходящем пространстве представления и проверено, что соответствующее (градуированное) обобщение эрмитового сопряжения согласуется с естественным обобщением дираковского сопряжения. Обсуждается подходящее обобщение унитарного преобразования соответствующего векторного пространства.

ГРУПОВІ ВЛАСТИВОСТІ OSP(2/1;C) КАЛІБРУВАЛЬНИХ ПЕРЕТВОРЕНЬ

К. Ільєнко

Виходячи з явної конструкції градуйованого узагальнено-ермітового приєднаного уявлення $\mathrm{osp}(2/1;\mathbf{C})$ градуйованої алгебри Лі, розглянуто формулу Бейкера-Кемпбелла-Хаусдорффа та знайдено умови дійсності, що накладаються на грасманово-непарні параметри, які є множниками пари непарних генераторів градуйованої алгебри Лі при експоненціюванні. Використання формалізму $\mathrm{su}(2)$-спінорів прояснює природу грасманово-непарних параметрів та суттєво полегшує дослідження відповідних інфінітезимальних калібрувальних перетворень. Також вивчено дію загального групового елементу на придатному просторі уявлення та перевірено, що відповідне (градуйоване) узагальнення ермітівського спряження погоджується із природним узагальненням дираківського спряження. Обговорюються придатні узагальнення унітарного перетворення відповідного векторного простору.