TRUE QUANTUM CHAOS

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It is shown, that the dynamic chaos is inherent for quantum systems not only in semi-classical approximation. As an example the especially quantum three-level system is considered. The value of external perturbation is analytically found, at which the regimes with dynamic chaos is realized. The possible consequences of regimes with dynamic chaos in quantum systems are discussed.

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1. INTRODUCTION

Now there is paradigm in which have been formulated in an obvious kind, that the evolutionary operator describing dynamics systems with a regime with dynamic chaos, should have two indisputable properties: 1) To be stretching. 2) To be nonlinear. Certainly, these two features are necessary for realization of dynamic chaos. However concerning the second property (to be nonlinear) it is necessary to give some explanations. Really, for example, it is known, that the equations of the quantum mechanics and Maxwell equations are the linear equations. However at transition from the quantum equations to the classical equations, and also at transition from the Maxwell equation to the equations of geometrical optics we have got systems of the nonlinear equations. Such equations can describe regimes with dynamic chaos. Thus, now are known at least two examples, when in linear systems at the certain meanings of their parameters (which allow to pass to classical consideration) the regimes with dynamic chaos are possible. In work [1] is shown, that this situation is considerably more widespread, that the regimes with dynamic chaos is internally inherent in huge number of linear systems. The results of the analysis of these features for the quantum systems are presented in this article.

2. STATEMENT OF A PROBLEM. THE BASIC EQUATIONS

Let's consider quantum system, which is described by Hamiltonian:

$$\hat{H} = \hat{H}_0 + \hat{H}_1(t) . \quad (1)$$

Second item in the right part describes perturbation. The wave function of system (1) obey to the Schrödinger equation which decision we shall search as a series of own functions of the unperturbed task for:

$$\psi(t) = \sum_n A_n(t) \cdot \varphi_n \cdot \exp(i\omega_n t) , \quad (2)$$

where $\omega_n = E_n / \hbar$; $i\hbar \frac{\partial \varphi_n}{\partial t} = \hat{H}_0 \varphi_n = E_n \cdot \varphi_n$.

Let's substitute (2) in the equation Schrödinger and in the ordinary way we shall get system of the connected equations for finding of complex amplitudes:

$$i\hbar \cdot \dot{A}_n = \sum_m U_{nm}(t) \cdot A_m , \quad (3)$$

where $U_{nm} = \int \varphi_m^* \cdot \hat{H}_1(t) \cdot \varphi_n \cdot \exp[i \cdot t \cdot (E_n - E_m)/\hbar] \cdot dq$.

Let's consider the most simple case of harmonic perturbation: $\hat{H}_1(t) = \hat{U} \cdot \exp(i\Omega t)$. Then the matrix elements of interaction will get the following expression:

$$U_{nm} = V_{nm} \exp\{i \cdot t \cdot [(E_n - E_m)/\hbar + \Omega]\} ,$$

$$V_{nm} = \int \varphi_n^* \cdot \hat{U} \cdot \varphi_m dq \quad (4)$$

Let's consider dynamics of three-level system ($|0\rangle, |1\rangle, |2\rangle$) (see Fig.1). We'll consider that frequency of external perturbation and the own meanings of energy of these levels satisfy to the ratio:

$$m = 1,\ n = 0 , \quad \hbar\Omega = E_1 - E_0 , \qquad m = 2,\ n = 0 ,$$

$$\hbar(\Omega + \delta) = E_2 - E_0 , \quad |\delta| << \Omega . \quad (5)$$

These ratios show on that fact, that the frequency of external perturbation is resonant for transitions between zero and first levels, and the energy of the third level is slightly differs from energy second one. Using these ratios in system (3), it is possible to left only three equations:

$$i \cdot \hbar \cdot \dot{A}_0 = V_{01} A_1 + V_{02} A_2 \cdot \exp(i \cdot \delta \cdot t) ; \qquad i \cdot \hbar \cdot \dot{A}_1 = V_{10} A_0 ;$$

$$i \cdot \hbar \cdot \dot{A}_2 = V_{20} A_0 \cdot \exp(-i \cdot \delta \cdot t) . \quad (6)$$

Let matrix elements of interaction for direct and opposite transitions are equal ($V_{i0} = V_{0i}$, $(i = 1; 2)$). Then from (6) we find the following connection between squares of complex amplitudes A_n :

$$\frac{d}{d\tau}\left[A_0^2 - A_1^2 - A_2^2 \right] = 2 \cdot \mu_2 \cdot A_0 A_2 \sin(\delta\tau) , \quad (7)$$

where $\tau = \Omega \cdot t$, $\delta \equiv \delta / \Omega$, $\mu_i = 2 \cdot V_{0i} / \hbar \cdot \Omega$. From this ratio follows, that if the third level coincides with second (two-level system, $\delta = 0$), the system (6) has only one degree of freedom. The development of dynamic chaos in such system is impossible. Below we shall see, that this difference in energy between second and third levels $(\hbar\delta)$ define the distance between nonlinear resonances. For further analysis of dynamics of complex amplitudes $A_i(\tau)$ it is convenient present them as:

$$A_i(\tau) = a_i(\tau) \exp(i\varphi(\tau)) . \quad (8)$$

Here a_i , φ_i - real amplitudes and real phases. Substituting (8) in (6) for a finding of the real amplitudes and phases, we shall get the following system of the equations:

$$\dot{a}_0 = \mu_1 \cdot a_1 \cdot \sin(\Phi) + \mu_2 \cdot a_2 \cdot \sin(\Phi_1) ,$$

$$\dot{a}_1 = -\mu_1 \cdot a_0 \cdot \sin(\Phi) , \quad \dot{a}_2 = -\mu_2 \cdot a_0 \cdot \sin(\Phi_1) ,$$

$$\dot{\Phi} = -\mu_1 \left(\frac{a_0}{a_1} - \frac{a_1}{a_0} \right) \cos(\Phi) + \mu_2 \left(\frac{a_2}{a_0} \right) \cos(\Phi_1) , \quad (9)$$

$$\dot{\Phi}_1 = -\mu_2 \left(\frac{a_0}{a_2} - \frac{a_2}{a_0} \right) \cos(\Phi_1) + \mu_1 \left(\frac{a_1}{a_0} \right) \cos(\Phi) + \delta ,$$

where $\Phi \equiv \varphi_1 - \varphi_0$, $\Phi_1 \equiv \varphi_2 - \varphi_0 + \delta\tau$.

From first three equations of this system follows such integral: $a_0^2 + a_1^2 + a_2^2 = const$. The system of the equations (9) is nonlinear. In general, dynamics of such system can be chaotic.

3. CRITERION OF DYNAMIC CHAOS OCCURENCE

It is significant to find analytical conditions, at which fulfilled the dynamics of quantum system (9) will be chaotic. For this purpose in the beginning we shall assume, that there are only two levels - zero and first. Third level is absent ($A_2 = 0$). In this case from system (9) it is possible to get the following equation for the phase Φ :

$$\ddot{\Phi} = -\frac{\mu_1^2}{2}\left[\frac{\left(a_0^2 + a_1^2\right)^2 + \left(a_0^2 - a_1^2\right)^2}{a_0^2 a_1^2}\right]\sin\left(2\Phi\right). \quad (10)$$

The equation (10) represents the equation of a mathematical pendulum. Minimal width of a nonlinear resonance can be estimated by value $\Delta \sim \mu_1$. Let's consider now situation, when the first level is absent ($A_1 = 0$). Similarly to the previous case, from system (9) it is possible to get the equation for a phase Φ_1. This equation also represents the equation of nonlinear oscillator. The analysis of this equation gives the following estimation of minimal width of a nonlinear resonance: $\Delta_1 \sim \mu_2$. It is natural to expect, that when the nonlinear resonances will be overlapped, i.e. when the condition $\left(\mu_1 + \mu_2\right) > \delta$ will be executed, dynamics of system (9) will be chaotic. Condition of occurrence of local instability is convenient to rewrite as:

$$V_{01} > \hbar\delta \text{ or } K \equiv V_{01} / \delta\hbar > 1. \quad (11)$$

We shall assume now, that the conditions for realization of dynamic chaos are executed. In this case the investigated system will wander on three power levels. It is interesting to give estimation for transition time, which is necessary for the system to pass from one level to another. For an estimation of time of transition in a stochastic regime we shall assume, that in this regime any correlation are absent. Then, for example, for value of an average square of the real amplitude it is possible to get the following estimation: $\left\langle a_1^2 \right\rangle \sim \mu^2 \cdot \left\langle a_0^2 \right\rangle \cdot \tau$. Thus, the average time of transitions between levels in a stochastic regime appears about a square of time of transition in a regular regime: $\tau_{ch} \sim (\tau_r)^2 \sim \left(\hbar \cdot \Omega / U_{01}\right)^2$. In a stochastic regime it is possible diffuse of a quantum system along energetic levels. Thus the time of diffuse in energy space on value ΔE can be estimated by value: $\tau_D \sim (\Delta E / \hbar\Omega)\left(\hbar\Omega / U_{01}\right)^2$. Let's note, that the time of excitation of energy levels from a zero level on levels with energy in a vicinity $E_0 + \Delta E$ by manyphoton excitation will be inverse proportional to a square of a compound matrix element: $\tau \sim 1/\left|H\right|^2$. Here H - compound matrix element, which is equal to the sum of products of separate matrix elements determining transitions between intermediate (often virtual) levels. Each matrix element is small value. Therefore, practically always the time of transitions caused by stochastic instability much less of the time of transitions, induced by manyphoton processes. Thus, as soon as the conditions for development of stochastic instability are executed, the processes connected to her, will determine transitions between levels, when the frequency of external perturbation is much less than distance between levels ($\Delta E >> \hbar\Omega$).

5. NUMERICAL RESULTS

System of the equations (6) and system of the equations (9) were investigated numerically. It is naturally, that dynamics of real and imaginary parts of complex amplitudes A_n was regular (system (6)). Dynamics of the real amplitudes a_i and phases Φ and Φ_1 at performance of conditions for overlapping of nonlinear resonances was chaotic: the spectra were wide, the correlation functions quickly fell down, the main Lyapunov parameters were positive. For an illustration of chaotic regimes in figures 2-5 are represented: dependence on time of amplitude a_1 and phase Φ (fig. 2 and fig. 3), correlation function for a phase Φ (fig. 4), and also distribution of the main Lyapunov parameters on a phase plane (a_1, Φ). The dependences submitted in these figures, are received at $\delta = 0.1$, $\mu_1 = 0.2$, $\mu_2 = 0.2$. Let's notice that despite of that fact, that dynamics of functions a_i and φ_i are chaotic, the dynamics of function $a_i \cdot \cos\varphi_i$ - is regular.

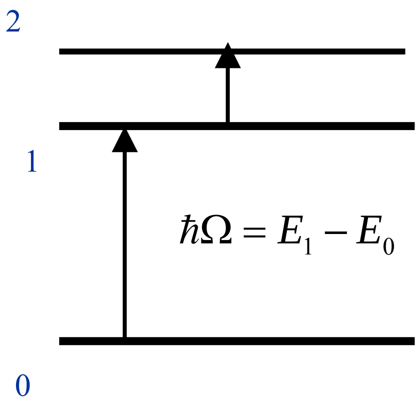

Fig.1. Three-level system

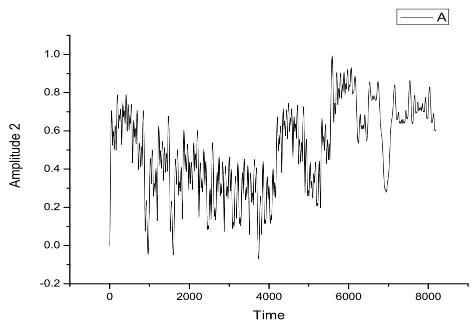

Fig.2. Time evolution of amplitude a_1

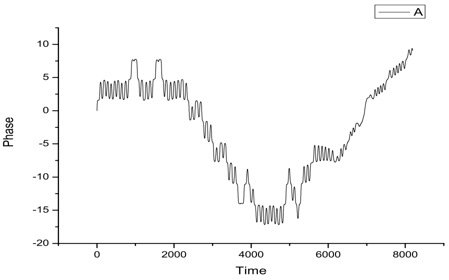

Fig.3. Time evolution of phase Φ_1

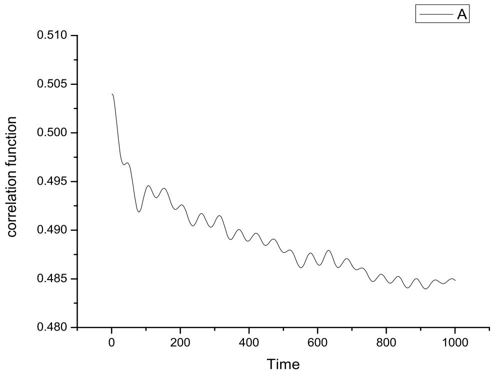

Fig.4. Correlation function for Φ

5. CONCLUSIONS

Thus, in quantum systems the regimes with dynamic chaos can be realized. At that, it is necessary to note, that the phenomenon of quantum chaos for a long time was intensively studied (see, for example, [2,3]). But in this cases, however, all authors have emphasized, that the quantum chaos is not true chaos, that in quantum chaos

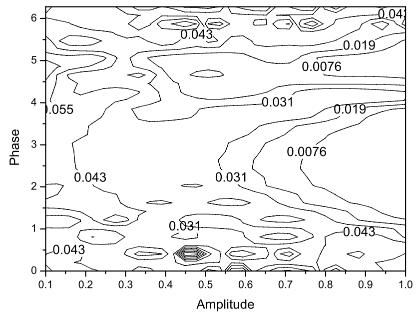

Fig.5. Main Lyapunov index

those quantum systems are studied, which parameters allow the semi-classical description and which in a classical limit have a regimes with dynamic chaos.

For this reason many authors take the name of quantum chaos in inverted commas. In this work is investigated the true quantum system. Its parameters are those that do not allow semi-classical consideration. For this reason it is possible to name a circle of such phenomena as true quantum chaos.

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ИСТИННЫЙ КВАНТОВЫЙ ХАОС

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Показано, что динамический хаос характерен для квантовых систем не только в квазиклассическом приближении. В качестве примера рассмотрена сугубо квантовая трехуровневая система. Аналитически найдена величина внешнего возмущения, при котором реализуется режим с динамическим хаосом. Обсуждаются возможные следствия режимов с динамическим хаосом в квантовых системах.

ІСТИННИЙ КВАНТОВИЙ ХАОС

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Показано, що динамічний хаос є характерним для квантових систем не тільки в квазікласичному наближенні. Як приклад розглянута сугубо квантова трьохрівнева система. Аналітично знайдена величина зовнішнього збурення, при якому реалізується режим з динамічним хаосом. Обговорюються можливі наслідки режимів з динамічним хаосом в квантових системах.