Ukrainian Mathematical Bulletin
Volume 5 **(2008)**, № 4, 559 – 568

UMB

The motion of gravitating ellipsoidal masses of liquid with variable viscosity

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(*Presented by A. E. Shishkov*)

Abstract. This paper is devoted to the study of the classical problem of the motion of a gravitating ellipsoidal mass of liquid. The new element is the viscosity of liquid which is determined as a linear homogeneous function of the pressure. It is proved that the so determined viscosity does not destroy the homogeneous rotational flow of liquid.

2000 MSC. 76B47, 76D17, 76D27, 76E07, 76U05, 85A30.

Key words and phrases. Gravitating ellipsoidal mass of liquid, viscosity, Dirichlet ellipsoid, Maclaurin ellipsoid.

Introduction

For the first time, the problem on the rotation of a liquid gravitating ellipsoid was set up and solved by Newton (1686) [8] in order to investigate the Earth's shape. Later on, this problem was studied by Stirling (1735), Maclaurin (1742), Simpson (1743), d'Alembert (1773), Laplace (1778), Jacobi (1834), Mayer (1842), Liouville (1846), Dirichlet (1875), Dedekind, Riemann, Poincaré (1885), Cartan, Lyapunov, Roche (1850), Darwin (1906), Jeans (1916), Chandrasekhar (1969) and many others. The clear detailed presentation of these results can be found into monograph of P. Appell [1], article of L. N. Sretenskii [13], monograph of L. Lichtenstein [5], textbooks of H. Lamb [4] and M. F. Subbotin [14], and monograph of S. Chandrasekhar [3].

Firstly, the case of a oblate axisymmetric ellipsoid which rotates with permanent speed around of the symmetry axis (Maclaurin ellipsoid) was investigated. Jacobi discovered that a liquid figure of equilibrium can be a triaxial ellipsoid which rotates with permanent speed around of the minor axis (ellipsoid of Jacobi). In this case, liquid rotates without deformations

Received 4.12.2007

ISSN 1812 – 3309.

as a rigid body. This make it possible to omit any consideration of a viscosity of liquid. The stability of these figures was investigated by A. M. Lyapunov [7].

Dirichlet investigated the case of a pulsating rotating ellipsoid (Dirichlet ellipsoid), by assuming the liquid to be ideal. Riemann [12] investigated the case of a deformation of ideal ellipsoidal liquid. In those studies, a motion of liquid was homogeneous vortical, and liquid was ideal.

The investigation of a motion of several gravitating liquid masses is a difficult problem of celestial mechanics. The problem of the motion of two gravitating masses of liquid was posed by E.V. Pitkevich [10, 11]. The case where a motion of liquid is homogeneous vortical is a significant special case of this problem.

At the end of the XIXth – the beginning of the XXth century, the homogeneous vortical (rotational) motion of the ideal liquid was involves in the studies of the motion of the Earth's liquid core. The aim of these investigations was an adequate description of the motion of the Earth's pole. The results can be found in the monograph of H. Moritz and I.I. Mueller [8].

According to the article of V. V. Brazhkin [2], the viscosity of an iron melt increases very much with the pressure, and the Earth's liquid core consists of iron. Hence, the viscosity of the core must increase toward the Earth's center. If we assume that a similar effect is correct for ellipsoidal liquid celestial bodies, we must take into account this effect in mathematical models for the motion of a liquid celestial body. In the author's work [15], the problem of the motion of an ellipsoidal gravitating mass of a viscous liquid was studied. The viscosity of liquid was take into account as a function of coordinates and the lengths of the ellipsoid semiaxes. The function setting the viscosity of liquid was chosen so that the motion of ellipsoidal mass of liquid is homogeneous vortical.

In the present article, a similar problem is solved in the case where the viscosity is a linear function of the pressure.

1. The equations of motion

Let $O\xi_1\xi_2\xi_3$ be the immovable Cartesian coordinate system, and let $Ox_1x_2x_3$ be the moving Cartesian coordinate system which can rotate with respect to its origin O. The boundary of the liquid ellipsoid is given in the coordinate system $Ox_1x_2x_3$ by the equation

$$x_1^2/c_1^2 + x_2^2/c_2^2 + x_3^2/c_3^2 = 1, \tag{1.1}$$

where c_1, c_2, c_3 are continuous differentiable functions of t. We assume that the condition

$$c_1 c_2 c_3 = R^3 = \text{const} \tag{1.2}$$

holds. The ellipsoid contain a viscous gravitating incompressible liquid without voids. The kinematic liquid viscosity ν is given by the formula

$$\nu = kp, \tag{1.3}$$

where k is a constant, and p is the pressure. The equations of motion of the liquid with respect to the moving axes $Ox_1x_2x_3$ have the form [6]

$$\begin{aligned}\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \nu(p)\Delta \mathbf{v} + 2\sigma \nabla \nu(p) \\ - \dot{\boldsymbol{\omega}} \times \mathbf{x} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{x}) - 2\boldsymbol{\omega} \times \mathbf{v} - \nabla \Phi,\end{aligned} \tag{1.4}$$

$$\operatorname{div} \mathbf{v} = 0, \tag{1.5}$$

where $\mathbf{v}$ is the liquid velocity vector with respect to the moving axes $Ox_1x_2x_3$; σ is the strain rate tensor of the liquid with components

$$\sigma_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \qquad i, j = 1, 2, 3; \tag{1.6}$$

$\boldsymbol{\omega}$ is the absolute angular velocity vector in the coordinate system $Ox_1x_2x_3$; Φ is the potential of gravitation forces [4],

$$\Phi = \pi\rho\gamma(\alpha_1 x_1^2 + \alpha_2 x_2^2 + \alpha_3 x_3^2 - \chi_0), \tag{1.7}$$

$$\alpha_i = c_1 c_2 c_3 \int_0^\infty \frac{d\lambda}{(c_i^2 + \lambda)D}, \quad i = 1, 2, 3, \qquad \chi_0 = c_1 c_2 c_3 \int_0^\infty \frac{d\,\lambda}{D},$$

$$D = [(c_1^2 + \lambda)(c_2^2 + \lambda)(c_3^2 + \lambda)]^{\frac{1}{2}};$$

and γ is the gravitational constant.

We assume that the pressure on the liquid boundary is equal to zero. Then the liquid viscosity (1.3) must be equal to zero on the boundary as well. Therefore, the boundary condition for Eqs. (1.4) and (1.5) must have the form

$$(\mathbf{v} - \mathbf{u}) \cdot \mathbf{n} \,|_S = 0, \tag{1.8}$$

where S is the liquid boundary (1.1), $\mathbf{n}$ is the unit vector of a normal to the boundary S, and $\mathbf{u}$ is the velocity of the liquid boundary in the moving coordinate system $Ox_1x_2x_3$.

The solution of Eqs. (1.4) and (1.5) is sought in the form

$$v_1 = c_1\left(\omega_2^*\frac{x_3}{c_3} - \omega_3^*\frac{x_2}{c_2}\right) + \frac{\dot{c}_1}{c_1}x_1 \quad (123), \tag{1.9}$$

$$p = -p_0(t)\left(\frac{x_1^2}{c_1^2} + \frac{x_2^2}{c_2^2} + \frac{x_2^2}{c_2^2} - 1\right), \tag{1.10}$$

where v_1, v_2, v_3 are the components of the velocity vector, p is the pressure, $\omega_1^*, \omega_2^*, \omega_3^*$, and $p_0(t)$ are the unknown functions of t. The symbol (123) means that the other expressions can be found by a cyclic permutation of indices. If we substitute expressions (1.9) and (1.10) in Eqs. (1.4), we have three equalities $k_{i1}x_1 + k_{i2}x_2 + k_{i3}x_3 = 0$, $i = 1, 2, 3$, which must be fulfilled for all x_1, x_2, x_3. It is possible only if the equalities $k_{ij} = 0$, $i, j = 1, 2, 3$ are correct. These equalities can be represented in the form

$$\ddot{c}_1 = c_1({\omega_2^*}^2 + {\omega_3^*}^2 + \omega_2^2 + \omega_3^2) + 2c_3\omega_2^*\omega_2 + 2c_2\omega_3^*\omega_3 \\ - 2\pi\rho\gamma\alpha_1 c_1 + \frac{2p_0}{c_1}\left(\frac{1}{\rho} + 2k\frac{\dot{c}_1}{c_1}\right) \quad (123), \tag{1.11}$$

$$\dot{\omega}_1^*\frac{c_2}{c_3} + \dot{\omega}_1 = -2\omega_1^*\frac{\dot{c}_2}{c_3} + \omega_2^*\omega_3^*\frac{c_2}{c_3} + 2kp_0\frac{\omega_1^*}{c_3^2}\left(\frac{c_3}{c_2} - \frac{c_2}{c_3}\right) \\ + \omega_2\omega_3 + 2\omega_2^*\omega_3\frac{c_2}{c_3} - 2\omega_1\frac{\dot{c}_3}{c_3} \quad (123), \tag{1.12}$$

$$\dot{\omega}_1^*\frac{c_3}{c_2} + \dot{\omega}_1 = -2\omega_1^*\frac{\dot{c}_3}{c_2} + \omega_2^*\omega_3^*\frac{c_3}{c_2} + 2kp_0\frac{\omega_1^*}{c_2^2}\left(\frac{c_3}{c_2} - \frac{c_2}{c_3}\right) \\ - \omega_2\omega_3 - 2\omega_3^*\omega_2\frac{c_1}{c_2} - 2\omega_1\frac{\dot{c}_2}{c_2} \quad (123), \tag{1.13}$$

where $\omega_1, \omega_2, \omega_3$ are the components of the angular velocity $\boldsymbol{\omega}$ in the coordinate system $Ox_1x_2x_3$.

The algebraic equation (1.2) and the system of ordinary differential equations (1.11)–(1.13) are the system of ten equations for ten unknowns $p_0, c_i, \omega_i^*, \omega_i$, $i = 1, 2, 3$.

2. Small oscillations of the axisymmetric ellipsoid

Let $c_1 = c_2$, $\omega_1 = \omega_2 = \omega_1^* = \omega_2^* = \omega_3^* = 0$. Then, if we exclude c_3 with the help of Eq. (1.2), the motion equations take the form

$$\ddot{c}_1 = c_1\omega_3^2 - 2\pi\rho\gamma\alpha_1 c_1 + \frac{2p_0}{c_1}\left(\frac{1}{\rho} - 2k\frac{\dot{c}_1}{c_1}\right),$$

$$-2R^3(c_1^{-3}\ddot{c}_1 - 3c_1^{-4}\dot{c}_1^2) = -2\pi\rho\gamma\alpha_3\frac{R^3}{c_1^2} + \frac{2p_0c_1^2}{R^3}\left(\frac{1}{\rho} + 4k\frac{\dot{c}_1}{c_1}\right), \tag{2.1}$$

$$\dot{\omega}_3 = -2\omega_3\frac{\dot{c}_1}{c_1}.$$

The last equation yields the equality

$$c_1^2\omega_3 = l, \quad l = \text{const} \tag{2.2}$$

which represents the conservation law of angular momentum. The variable ω_3 can be excluded from the first equation (2.1) by equality (2.2). Then we can exclude p_0 from two first equations (2.1) and introduce new variables

$$\zeta = c_1/R, \qquad \tau = T^{-1}t, \qquad \eta = \frac{d\zeta}{d\tau},$$

where T is the characteristic time. Then the motion equations take the form

$$\frac{d\zeta}{d\tau} = \eta,$$

$$\frac{d\eta}{d\tau} = \left\{L^2\zeta^3 - 2\pi\rho\gamma T^2\zeta(\alpha_1\zeta^6 - \alpha_3) + 6\frac{\eta^2}{\zeta} - \frac{4k\rho}{T}\eta\left[3\frac{\eta^2}{\zeta^2} - L^2\zeta^2 + \pi\rho\gamma T^2(2\alpha_1\zeta^6 + \alpha_3)\right]\right\} \times \left[\zeta^6 + 2 + \frac{4k\rho}{T}\frac{\eta}{\zeta}(\zeta^6 - 1)\right]^{-1}, \tag{2.3}$$

where $L = lT/R^2$. For the oblate axisymmetric ellipsoid, the quantities α_1 and α_3 can be expressed by the formulas [4]

$$\begin{aligned} \alpha_1 = \alpha_2 &= (\xi^2 + 1)\xi \operatorname{arcctg}\xi - \xi^2, \\ \alpha_3 &= 2(\xi^2 + 1)(1 - \xi \operatorname{arcctg}\xi), \\ \xi &= (\zeta^6 - 1)^{-\frac{1}{2}}. \end{aligned} \tag{2.4}$$

The parameters of the stationary solutions of system (2.3) are connected by the relation

$$L^2 = 2\pi\rho\gamma T^2(\alpha_1\zeta^4 - \alpha_3\zeta^{-2}), \tag{2.5}$$

where $1 \le \zeta \le \infty$. We denote, by L_0 and ζ_0, the value of L and ζ which satisfy relation (2.5). Then every solution of Eqs. (2.3) can be represented in the form

$$\zeta = \zeta_0 + \delta, \tag{2.6}$$

where δ is the unknown function of the dimensionless time τ. If we substitute (2.6) into Eqs. (2.3) and make the linearization by η and δ assuming them small, the motion equations take the form

$$\frac{d\delta}{d\tau} = \eta,$$
$$\frac{d\eta}{d\tau} = a\delta + b\eta, \tag{2.7}$$

where

$$a = -2\pi\rho\gamma T^2(4\zeta_0^6\alpha_{10} + \zeta_0^7\alpha_{11} + 2\alpha_{30} - \zeta_0\alpha_{31})(\zeta_0^6 + 2)^{-1},$$
$$b = -4k\pi\rho^2\gamma T(2\alpha_{10}\zeta_0^6 + \alpha_{30})(\zeta_0^6 + 2)^{-1},$$
$$\alpha_{10} = \zeta_0^6(\zeta_0^6 - 1)^{-3/2}\text{arcctg}(\zeta_0^6 - 1)^{-1/2} - (\zeta_0^6 - 1)^{-1},$$
$$\alpha_{11} = -3\zeta_0^5(\zeta_0^6 - 1)^{-2}\big[(\zeta_0^6 + 2)(\zeta_0^6 - 1)^{-1/2}\text{arcctg}(\zeta_0^6 - 1)^{-1/2} - 3\big],$$
$$\alpha_{30} = 2\zeta_0^6(\zeta_0^6 - 1)^{-1}\big[1 - (\zeta_0^6 - 1)^{-1/2}\text{arcctg}(\zeta_0^6 - 1)^{-1/2}\big],$$
$$\alpha_{31} = 6\zeta_0^5\big[(\zeta_0^6 + 2)(\zeta_0^6 - 1)^{-5/2}\text{arcctg}(\zeta_0^6 - 1)^{-1/2} - 3(\zeta_0^6 - 1)^{-2}\big].$$

The characteristic equation of system (2.7)

$$\begin{vmatrix} -\lambda & 1 \\ a & b - \lambda \end{vmatrix} = 0$$

can be written as

$$\lambda^2 - b\lambda - a = 0.$$

This equation has the solutions

$$\lambda_{1,2} = \frac{1}{2}(b \pm \sqrt{b^2 + 4a}).$$

The coefficient a is independent of k. If $\zeta_0 > 1$, then $a < 0$. The coefficient $b < 0$, and $|b|$ is directly proportional to k. Then if $k = 0$ (the case of the ideal liquid), we have $b = 0$, and roots of the characteristic equation are purely imaginary $\lambda = \pm\sqrt{|a|}$. In this case, the nonlinear equations of motion describe the undamped periodic oscillations of a Dirichlet ellipsoid [4].

If k is enough small and $b^2+4a < 0$, then the roots of the characteristic equation are complex conjugate, and their real parts are negative. In this case, the Dirichlet ellipsoid approaches the Maclaurin ellipsoid by means of decaying oscillations.

If k is enough large and $b^2+4a \geq 0$, then the roots of the characteristic equation are real and negative. In this case, the Dirichlet ellipsoid asymptotically approaches the Maclaurin ellipsoid.

3. Oscillations of a nonrotating spherical mass of liquid

If we assume that

$$\omega_1^* = \omega_2^* = \omega_3^* = \omega_1 = \omega_2 = \omega_3 = 0,$$

then Eqs. (1.12) and (1.13) are satisfied identically, and Eqs. (1.11) takes the form

$$\ddot{c}_1 = -2\pi\rho\gamma\alpha_1 c_1 + \frac{2p_0}{c_1}\left(\frac{1}{\rho} - 2k\frac{\dot{c}_1}{c_1}\right) \quad (123). \tag{3.1}$$

In Eqs. (3.1), we exclude c_3 by using Eqs. (1.2). Then we exclude p_0 and introduce the new variables $\zeta_i = c_i/R$, $i = 1, 2$, $\tau = T^{-1}t$. Thus, we obtain the system of equations

$$\frac{d\zeta_i}{d\tau} = \eta_i, \quad a_{i1}\frac{d\eta_i}{d\tau} + a_{i2}\frac{d\eta_2}{d\tau} = f_i, \quad i = 1, 2, \tag{3.2}$$

where

$$a_{11} = \zeta_1 + 2k\rho T^{-1}(\eta_1 + \zeta_1\zeta_2^{-1}\eta_2) + \zeta_1^{-3}\zeta_2^{-2}(1 - 2k\rho T^{-1}\zeta_1^{-1}\eta_1),$$
$$a_{12} = \zeta_1^{-2}\zeta_2^{-3}(1 - 2k\rho T^{-1}\zeta_1^{-1}\eta_1),$$
$$a_{21} = \zeta_1^{-3}\zeta_2^{-2}(1 - 2k\rho T^{-1}\zeta_2^{-1}\eta_2),$$
$$a_{22} = \zeta_2 + 2k\rho T^{-1}(\zeta_1^{-1}\zeta_2\eta_1 + \eta_2) + \zeta_1^{-2}\zeta_2^{-3}(1 - 2k\rho T^{-1}\zeta_2^{-1}\eta_2),$$

$$\begin{aligned} f_1 = -2\pi\rho\gamma T^2\alpha_1\zeta_1^2[1 + 2k\rho T^{-1}(\zeta_1^{-1}\eta_1 + \zeta_2^{-1}\eta_2)] \\ + [2\zeta_1^{-2}\zeta_2^{-2}(\zeta_1^{-2}\eta_1^2 + \zeta_1^{-1}\zeta_2^{-1}\eta_1\eta_2 + \zeta_2^{-2}\eta_2^2) \\ + 2\pi\rho\gamma T^2\alpha_3\zeta_1^{-2}\zeta_2^{-2}](1 - 2k\rho T^{-1}\zeta_1^{-1}\eta_1), \end{aligned}$$

$$f_2 = -2\pi\rho\gamma T^2\alpha_2\zeta_2^2[1 + 2k\rho T^{-1}(\zeta_1^{-1}\eta_1 + \zeta_2^{-1}\eta_2)] \\ + [2\zeta_1^{-2}\zeta_2^{-2}(\zeta_1^{-2}\eta_1^2 + \zeta_1^{-1}\zeta_2^{-1}\eta_1\eta_2 + \zeta_2^{-2}\eta_2^2) \\ + 2\pi\rho\gamma T^2\alpha_3\zeta_1^{-2}\zeta_2^{-2}](1 - 2k\rho T^{-1}\zeta_2^{-1}\eta_2).$$

System (3.2) can be written in the form

$$\begin{aligned} \frac{d\zeta_1}{d\tau} &= \eta_1, \quad \frac{d\eta_1}{d\tau} = (f_1a_{22} - f_2a_{12})(a_{11}a_{22} - a_{12}a_{21})^{-1}, \\ \frac{d\zeta_2}{d\tau} &= \eta_2, \quad \frac{d\eta_2}{d\tau} = (f_2a_{11} - f_1a_{21})(a_{11}a_{22} - a_{12}a_{21})^{-1}. \end{aligned} \tag{3.3}$$

System (3.3) has a stationary solution $\zeta_1 = \zeta_2 = 1, \quad \eta_1 = \eta_2 = 0$. This solution describes the equilibrium of the spherical mass of liquid. Then, every another solution of system (3.3) can be represented in the form

$$\zeta_1 = 1 + \delta_1, \quad \zeta_2 = 1 + \delta_2, \quad \eta_1, \quad \eta_2, \tag{3.4}$$

where δ_1 and δ_2 are new unknown functions of τ. We substitute (3.4) in system (3.3) and assume that $\delta_1, \delta_2, \eta_1, \eta_2$ are small. The linearization in $\delta_1, \delta_2, \eta_1, \eta_2$ gives the system

$$\begin{aligned} \frac{d\delta_i}{d\tau} &= \eta_i, \\ \frac{d\eta_i}{d\tau} &= -\frac{8}{3}\pi\rho\gamma T^2\left(\frac{2}{5}\delta_i + k\rho T^{-1}\eta_i\right), \end{aligned} \qquad i = 1, 2. \tag{3.5}$$

Thus, the systems of linearized equations for the variables ζ_1, η_1 and ζ_2, η_2 are independent of each other, and both can be solved separately. Taking into account that the systems differ from each other only by the unknown variables and the initial conditions, they can be solved similarly. The solution of system (3.5) is sought as

$$(\delta_i,\ \eta_i)^T = (b_1,\ b_2)^T \exp(\lambda\tau),$$

where b_1, b_2, λ are the unknown constants. The characteristic equation has the form

$$\begin{vmatrix} -\lambda & 1 \\ -\frac{16}{15}\pi\rho\gamma T^2 & -\frac{8}{3}k\pi\rho^2\gamma T - \lambda \end{vmatrix} = 0.$$

This equation can be written as the equation for λ,

$$\lambda^2 + \frac{8}{3}k\pi\rho^2\gamma T\lambda + \frac{16}{15}\pi\rho\gamma T^2 = 0.$$

The solution of this equation is

$$\lambda_{1,2} = \frac{4}{3}\left(-k\pi\rho^2\gamma T \pm \sqrt{(k\pi\rho^2\gamma T)^2 - \frac{3}{5}\pi\rho\gamma T^2}\right).$$

For $k = 0$, the system becomes conservative and will perform the undamped oscillations in the vicinity of the stable equilibrium state, i.e. the sphere. In this case, the roots of the characteristic equation are purely imaginary $\lambda_{1,2} = \pm\frac{4}{3}\sqrt{\frac{3}{5}\pi\rho\gamma T^2}\, i$.

In the case of $0 < k < \sqrt{3/(5\pi\rho^3\gamma)}$, the roots of the characteristic equation are complex conjugate with a negative real part. Into this case, the ellipsoidal mass of liquid will approach a sphere by performing the damped oscillations.

In the case of $k > \sqrt{3/(5\pi\rho^3\gamma)}$, the ellipsoidal mass of liquid, the kinetic energy of which is sufficiently small and the shape is enough close to a spherical one, will asymptotically approach the sphere.

The author is grateful to the reviewer for the careful review and the useful remarks that make possible to find new approaches to the problem.

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