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Classification of symmetry properties of a system of chemotaxis equations

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Abstract. The full group classification of the systems of chemotaxis equations is performed.

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1. Introduction

In modern biophysical researches, the processes of symmetric propagation of bacterial population waves, when chemotaxis rings keep a sharply outlined form and move with a constant speed depending on the mobility of bacteria and their chemotaxis properties, are well described by the mathematical models based on the Keller–Segel's equations [14]

$$\begin{aligned} &S_t = D_S S_{xx} + k_1 g(S) b, \\ &b_t = -\nu \partial_x [b \chi(S) S_x] + D_b b_{xx} + k_2 g(S) b, \end{aligned} \tag{1.1}$$

where $S_t = \frac{\partial S}{\partial t}, S_x = \frac{\partial S}{\partial x}, b_t = \frac{\partial b}{\partial t}$, $S_{xx} = \frac{\partial^2 S}{\partial x^2}$, $b_{xx} = \frac{\partial^2 b}{\partial x^2}, \partial_x = \frac{\partial}{\partial x}$, and $S(t,x)$ is the concentration of a substrate-attractant which is consumed by bacteria, $b(t,x)$ is the density of bacteria, $g(S)$ is the specific growth rate of bacteria, $\chi(S)$ is a function of the chemotaxis answer, D_S and D_b are diffusion coefficients of a substrate and bacteria, respectively; ν, k_1, and k_2 are constants; and t and x are the time and spatial variables, respectively. The Keller–Segel's model and its some modifications

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describe the formation and propagation of Adler's chemotaxis rings [1] and different processes of structurization in bacterial colonies at their interaction [13]. We rewrite system (1.1) in the designations usual in mathematical researches, having generalized it as follows:

$$\begin{pmatrix} u^1 \\ u^2 \end{pmatrix}_0 = \partial_1 \left[\begin{pmatrix} \lambda_1 & 0 \\ f(u^1)u^2 & \lambda_2 \end{pmatrix} \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}_1 \right] + \begin{pmatrix} g^1(u^1, u^2) \\ g^2(u^1, u^2) \end{pmatrix}. \tag{1.2}$$

Here, $g^1(u^1, u^2)$, $g^2(u^1, u^2)$, $f(u^1)$ are arbitrary smooth functions of their arguments, and $f \neq 0, \lambda_1 > 0, \lambda_2 > 0$, $u^a = u^a(x_0, x_1)$, $a = \overline{1, 2}$, x_0 is the time variable, x_1 is the spatial variable, and the subscripts denote the differentiation with respect to the corresponding independent variable. We note that system (1.2) is a special case of the system of nonlinear equations of a diffusion reaction

$$\begin{pmatrix} u^1 \\ u^2 \end{pmatrix}_0 = \partial_1 \left[F(u^1, u^2) \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}_1 \right] + G(u^1, u^2), \tag{1.3}$$

where

$$F(u^1, u^2) = \begin{pmatrix} f^{11} & f^{12} \\ f^{21} & f^{22} \end{pmatrix}, \qquad G(u^1, u^2) = \begin{pmatrix} g^1 \\ g^2 \end{pmatrix},$$

$f^{ab} = f^{ab}(u^1, u^2)$, $g^a = g^a(u^1, u^2)$, $a; b = 1; 2$. The symmetry properties of the equation of a diffusion-convection reaction

$$u_0 = \partial_1(f(u)u_1) + g(u)u_1 + h(u) \tag{1.4}$$

were considered in a number of works. For example, the symmetry properties of Eq. (1.4) at $g(u) = h(u) = 0$ and $g(u) = 0$ were classified, respectively, in works [21] and [8]. The full description of symmetries at arbitrary values of the functions $f(u)$, $g(u)$, and $h(u)$ to within equivalence transformations was done in works [5] and [7]. The symmetry analysis of the second-order evolutionary equation of a general form

$$u_0 = F(x_0, x_1, u, u_1, u_{11}) \tag{1.5}$$

was performed in works [15, 16, 25, 26]. The Galilei invariance of system (1.3) was investigated in works [2,3,10]. The Lie and conditional symmetry of system (1.3) in the case of a diagonal matrix F was investigated in work [6].

In the given work, we will pose the following problem: to investigate the symmetry properties of system (1.2) depending on the values of the functions $f(u^1)$, $g^1(u^1,u^2)$, $g^2(u^1,u^2)$ and the constants λ_1, λ_2. We note that, at $f=0$, the symmetry properties of system (1.2) were investigated in works [4,17,18]; therefore, we consider further that $f\neq 0$.

2. Symmetry kernel and necessary conditions for its extension

To study the symmetry properties of system (1.2), we will use the Lie algorithm [9,11,19,22,23].

By acting with the infinitesimal operator extension

$$X=\xi^\mu\partial_\mu+\eta^a\partial_{u^a}, \tag{2.1}$$

where $\xi^\mu=\xi^\mu(x_0,x_1,u^1,u^2)$, $\eta^a=\eta^a(x_0,x_1,u^1,u^2)$, $\mu=\overline{0,1},\ a=\overline{1,2}$ on system (1.2), transiting to a manifold, and splitting the obtained system by the derivatives of the functions u^a, we obtain a determining system to find coordinates of the infinitesimal operator (2.1) and the functions f, g^1, and g^2. The determining system consists of three subsystems:

$$S_1(\xi,\eta)=0,\qquad S_2(\xi,\eta,f)=0,\qquad S_3(\xi,\eta,f,g^1,g^2)=0.$$

The system $S_1=0$ is a system of differential equations only for the functions ξ^μ and η^a

$$\begin{aligned}&\xi^0_1=\xi^\mu_{u^a}=\eta^1_{u^2}=\eta^a_{u^bu^c}=0,\qquad a,b,c=1;2,\\&\xi^0_0=2\xi^1_1,\qquad 2\lambda_1\eta^1_{1u^1}=-\xi^1_0.\end{aligned}\tag{2.2}$$

The system $S_2(\xi,\eta,f)=0$ connects the coordinates of the infinitesimal operator ξ^μ, η^a and the function $f(u^1)$ with one another and looks like

$$\begin{aligned}&\eta^1\dot f+\Big(\eta^1_{u^1}-\eta^2_{u^2}-\frac{1}{u^2}\eta^2\Big)f+\frac{1}{u^2}(\lambda_2-\lambda_1)\eta^2_{u^1}=0,\\&u^2\eta^1_1\dot f+\Big(u^2\eta^1_{1u^1}+\frac{1}{2}\eta^2_1\Big)f+\lambda_2\eta^2_{1u^1}=0,\\&\eta^1_1f+2\lambda_2\eta^2_{1u^2}=-\xi^1_0.\end{aligned}\tag{2.3}$$

The system $S_3(\xi,\eta,f,g^1,g^2)=0$ consists of two differential equations

$$\begin{aligned}&\eta^1 g^1_{u^1} + \eta^2 g^1_{u^2} = (\eta^1_{u^1} - \xi^0_0) g^1 + \eta^1_{u^2} g^2 + \eta^1_0 - \lambda_1 \eta^1_{11},\\ &\eta^1 g^2_{u^1} + \eta^2 g^2_{u^2} = (\eta^2_{u^2} - \xi^0_0) g^2 + \eta^2_{u^1} g^1 + \eta^2_0 - \lambda_2 \eta^2_{11} - u^2 f \eta^1_{11}\end{aligned} \tag{2.4}$$

which connect the functions g^1, g^2 and the functions ξ^μ, η^a, f with one another.

Remark 2.1. Considering $f, g^1, g^2, \lambda_1, \lambda_2$ as arbitrary in systems (2.2), (2.3), (2.4), we obtain

$$\xi^0 = d_0, \qquad \xi^1 = d_1, \qquad \eta^1 = \eta^2 = 0, \tag{2.5}$$

where d_0, d_1 are arbitrary constants. In this case, operator (2.1) looks like

$$X = d_0 \partial_0 + d_1 \partial_1. \tag{2.6}$$

Operator (2.6) generates the algebra

$$A_0 = \langle \partial_0, \partial_1 \rangle \tag{2.7}$$

named as *the symmetry kernel of system* (1.2).

Let us investigate, for which values of the functions f, g^1, and g^2, the symmetry of system (1.2) is wider than that of the algebra A_0. The necessary conditions for the symmetry extension are given by the following proposition.

Theorem 2.1. *If system* (1.2) *admits the extension of the symmetry kernel* A_0*, then the function* $f(u^1)$ *takes one of the following representations:*

1. $f = f(u^1)$,

2. $f = \lambda$,

3. $f = \frac{\lambda}{u^1}$,

4. $f = \frac{\lambda_1 - \lambda_2}{u^1}$,

5. $f = \frac{2\lambda_1}{u^1}$,

where $\varphi(u^1)$ *is an arbitrary smooth function, and* λ *is an arbitrary constant.*

Proof. To prove the theorem, we solve the system of determining equations which consists of the systems $S_1 = 0$ and $S_2 = 0$.

The general solution of system $S_1(\xi, \eta)$ is the functions

$$\begin{aligned}
&\xi^0 = 2A(x_0),\\
&\xi^1 = \dot{A}(x_0)x_1 + B(x_0),\\
&\eta^1 = -\frac{1}{2\lambda_1}\Big[\frac{1}{2}\ddot{A}(x_0)x_1^2 + \dot{B}(x_0)x_1 + C(x_0)\Big]u^1 + \beta^1(x_0, x_1),\\
&\eta^2 = \alpha^{21}(x_0, x_1)u^1 + \alpha^{22}(x_0, x_1)u^2 + \beta^2(x_0, x_1),
\end{aligned}$$

where A, B, C, α^{2a}, and β^a are arbitrary smooth functions of their arguments.

Due to the joint solution of the first and third equations of system (2.3), the conditions $\alpha_1^{21} = \alpha_1^{22} = \beta_1^2 = 0$ are obtained. Then the system $S_2 = 0$ takes the form

$$\begin{aligned}
&(\alpha^{11}u^1 + \beta^1)\dot{f} = -\alpha^{11}f,\\
&(\alpha_1^{11}u^1 + \beta_1^1)f = 2\lambda_1\alpha_1^{11},\\
&(\alpha^{21}u^1 + \beta^2)f = (\lambda_1 - \lambda_2)\alpha^{21}.
\end{aligned} \tag{2.8}$$

The solution of system (2.8) leads to the appearance of 5 nonequivalent representations of the function f which are given in the formulation of the theorem.

Let us consider each of these cases separately. We will show that, at the specified values of the function $f(u^1)$, the extension of the symmetry of system (1.2) as compared with that of A_0 is possible.

1. Let $f = f(u^1)$ be an arbitrary smooth function. System (2.8) yields

$$\xi_0^1 = \alpha_1^a = \beta^a = 0. \tag{2.9}$$

In view of (2.9), we obtain

$$\begin{gathered}
\xi^0 = 2c_1x_0 + d_0, \qquad \xi^1 = c_1x_1 + d_1,\\
\eta^1 = 0, \qquad \eta^2 = \alpha^{22}(x_0)u^2,
\end{gathered} \tag{2.10}$$

where $\alpha^{22}(x_0)$ is an arbitrary smooth function, c_1, d_0, and d_1 are arbitrary constants. By comparing formulas (2.5) and (2.10), it is easy to see the possibility to extend symmetry (2.7).

In cases 2–5, the possibility to extend symmetry (2.7) is similarly proved. Not repeating these reasonings, we present the final form of coordinates of the infinitesimal operator for each of the indicated functions $f(u^1)$.

2. For $f = \lambda$, the coordinates of the infinitesimal operator (2.1) look like

$$\begin{aligned} \xi^0 &= 2c_1 x_0 + d_0, \qquad \xi^1 = c_1 x_1 + d_1, \\ \eta^1 &= \beta^1(x_0), \qquad \eta^2 = \alpha^{22}(x_0) u^2, \end{aligned} \tag{2.11}$$

where $\beta^1(x_0)$ is an arbitrary smooth function.

3. For $f = \frac{\lambda}{u^1}$ (λ is an arbitrary constant), system (2.8) yields

$$\begin{aligned} \xi^0 &= 2c_1 x_0 + d_0, \qquad \xi^1 = c_1 x_1 + d_1, \\ \eta^1 &= \alpha^1(x_0) u^1, \qquad \eta^2 = \alpha^{22}(x_0) u^2, \end{aligned} \tag{2.12}$$

where $\alpha^1(x_0)$ is an arbitrary smooth function.

4. For $f = \frac{\lambda_1 - \lambda_2}{u^1}$, we obtain

$$\begin{aligned} \xi^0 &= 2c_1 x_0 + d_0, \qquad \xi^1 = c_1 x_1 + d_1, \\ \eta^1 &= \alpha^1(x_0) u^1, \qquad \eta^2 = \alpha^{21}(x_0) u^1 + \alpha^{22}(x_0) u^2, \end{aligned} \tag{2.13}$$

where $\alpha^{21}(x_0)$ is an arbitrary smooth function.

5. For $f = \frac{2\lambda_1}{u^1}$, we have

$$\begin{aligned} \xi^0 &= 2A(x_0), \qquad \xi^1 = \dot{A}(x_0) x_1 + B(x_0), \\ \eta^1 &= \alpha^1(x_0) u^1, \qquad \eta^2 = \alpha^{22}(x_0) u^2, \end{aligned} \tag{2.14}$$

where $\alpha^1(x_0) = -\frac{1}{2\lambda_1}[\frac{1}{2}\ddot{A}(x_0) x_1^2 + \dot{B}(x_0) x_1 + C(x_0)]$, $A(x_0), B(x_0)$, and $C(x_0)$ are arbitrary smooth functions. The theorem is proved. □

Lemma 2.1. *System* (1.2) *has a group of continuous transformations of equivalence which are set by the following formulas for the coordinates of the equivalence operator E:*

$$\begin{aligned} \xi^0 &= 2c_1 x_0 + c_2, \qquad \xi^1 = c_1 x_1 + c_3, \\ \eta^1 &= c_4 u^1 + c_5, \qquad \eta^2 = c_6 u^1 + c_7 u^2. \end{aligned} \tag{2.15}$$

Here, $c_1, c_2, c_3, c_4, c_5, c_6$, *and* c_7 *are arbitrary constants which depend on the form of the function* f *and take the following values:*

1) *at* $f = f(u^1)$, $c_6 = 0$;

2) *at* $f = \lambda$, $c_4 = c_6 = 0$;

3) *at* $f = \frac{\lambda}{u^1}$, $c_5 = c_6 = 0$;

4) *at* $f = \frac{\lambda_1 - \lambda_2}{u^1}$, $c_5 = 0$;

5) *at* $f = \frac{2\lambda_1}{u^1}$, $c_5 = c_6 = 0$.

Proof. To proof the lemma, we will apply the algorithm of search for the equivalence transformations (see, e.g., [12, 16, 22]).

The form of the equivalence operator E depends on the form of the function f.

1. If $f = f(u^1)$ is an arbitrary smooth function, then we will search the operator E in the form

$$E = \xi^\mu \partial_\mu + \eta^a \partial_{u^a} + \zeta \partial_f + \tau^a \partial_{g^a}. \tag{2.16}$$

Acting by the operator E on system (1.2) and on the additional conditions

$$\frac{\partial f}{\partial x^\mu} = \frac{\partial f}{\partial u^2} = \frac{\partial g^a}{\partial x^\mu} = 0, \tag{2.17}$$

we obtain a system of determining equations for the coordinates of operator (2.16) ξ^μ, η^a, ζ, and τ^a:

$$\begin{aligned} &\xi^0_1 = \xi^1_0 = \xi^\mu_{u^a} = \eta^1_{u^2} = \eta^a_{u^b u^c} = \eta^a_\mu = \eta^2_{u^1} = 0, \qquad a, b, c = 1; 2, \\ &\xi^0_0 = 2\xi^1_1, \qquad u^2 \eta^2_{u^2} - \eta^2 = 0, \end{aligned} \tag{2.18}$$

$$\zeta = \eta^1_{u^1} f, \qquad \tau^1 = (\eta^1_{u^1} - \xi^0_0) g^1, \qquad \tau^2 = (\eta^2_{u^2} - \xi^0_0) g^2. \tag{2.19}$$

The general solution of system (2.18) looks like (2.15). Equalities (2.19) under conditions (2.15) can be written as follows:

$$\zeta = c_4 f, \qquad \tau^1 = (c_4 - 2c_1) g^1, \qquad \tau^2 = (c_6 - 2c_1) g^2.$$

2. If $f(u^1)$ is not arbitrary and is set by one of the formulas for f in cases 2)–5) in the lemma statement, we will search the infinitesimal operator of equivalence E in the form

$$E = \xi^\mu \partial_\mu + \eta^a \partial_{u^a} + \tau^a \partial_{g^a}. \tag{2.20}$$

Acting with the extension of the operator E on system (1.2) and on the additional conditions

$$\frac{\partial g^a}{\partial x^\mu} = 0 \tag{2.21}$$

and applying the algorithm [16], we find a system of determining equations for the coordinates of the operator (2.20) ξ^μ, η^a, and τ^a:

$$\begin{gathered}\xi^0_1 = \xi^1_0 = \xi^\mu_{u^a} = \eta^1_{u^2} = \eta^a_{u^b u^c} = \eta^a_\mu = \eta^1_{u^2} = 0, \\ \hfill a, b, c = 1, 2,\ \mu = 0, 1 \\ \xi^0_0 = 2\xi^1_1, \qquad u^2 \eta^2_{u^2} - \eta^2 = 0,\end{gathered} \tag{2.22}$$

$$\begin{gathered}\tau^1 = (\eta^1_{u^1} - \xi^0_0) g^1, \qquad \tau^2 = \eta^2_{u^1} g^1 + (\eta^2_{u^2} - \xi^0_0) g^2, \\ \eta^1 \dot{f} + f \eta^1_{u^1} = 0, \qquad (u^2 \eta^2_{u^2} - \eta^2) f = (\lambda_2 - \lambda_1) \eta^2_{u^1}.\end{gathered} \tag{2.23}$$

The general solution of Eqs. (2.22) are the functions

$$\begin{gathered}\xi^0 = 2c_1 x_0 + c_2, \qquad \xi^1 = c_1 x_1 + c_3, \\ \eta^1 = c_4 u^1 + c_5, \qquad \eta^2 = c_6 u^1 + c_7 u^2.\end{gathered} \tag{2.24}$$

By substituting (2.24) in (2.23), we obtain

$$\tau^1 = (c_4 - 2c_1) g^1, \qquad \tau^2 = c_6 g^1 + (c_7 - 2c_1) g^2, \tag{2.25}$$

$$(c_4 u^1 + c_5) \dot{f} + c_4 f = 0, \qquad c_6 (u^1 f - \lambda_1 + \lambda_2) = 0. \tag{2.26}$$

Solving Eq. (2.26), we come to cases 2)–5) of the lemma. The lemma is proved. □

Remark 2.2. Besides the equivalence transformations which are obtained in Lemma (2.1), the other equivalence transformations we name additional take place for more exactly specified functions f and g. Additional equivalence transformations will be presented in what follows for the function f of a specific form.

3. Classification of the symmetry properties of system (1.2) in the case of an arbitrary function $f(u^1)$

We now consider the system

$$\begin{pmatrix} u^1 \\ u^2 \end{pmatrix}_0 = \partial_1 \left[\begin{pmatrix} \lambda_1 & 0 \\ u^2 f(u^1) & \lambda_2 \end{pmatrix} \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}_1 \right] + \begin{pmatrix} g^1(u^1, u^2) \\ g^2(u^1, u^2) \end{pmatrix} \qquad (3.1)$$

and will classify its symmetry properties depending on the form of the functions $g^a(u^1; u^2)$ at any function $f(u^1)$.

Remark 3.1. It follows from Lemma 2.1 that the basic group of equivalence transformations of system (3.1) looks like

$$x_0 = te^{2\theta_2} + \theta_0, \qquad x_1 = xe^{\theta_2} + \theta_1,$$
$$u^1 = w^1 e^{\theta_3} + \theta_5, \qquad u^2 = w^2 e^{\theta_4}.$$

Besides the basic group of equivalence, system (3.1) for specific g admits some additional equivalence transformations, for example

$$x_0 = at, \qquad x_1 = bx, \qquad u^1 = w^1, \qquad u^2 = w^2 e^{mt},$$

where a, b, and m are arbitrary constants.

In view of Remark 3.1, we will formulate theorems on the maximal algebra of invariance of system (3.1) to within the indicated equivalence transformations.

The following proposition is true.

Theorem 3.1. *If system* (3.1) *admits an extension of the symmetry kernel* A_0*, the functions* g^1 *and* g^2 *are set by one of the formulas*

$$g^1 = \varphi^1(u^1), \qquad g^2 = u^2[\varphi^2(u^1) + \lambda_3 \ln u^2]; \qquad (3.2)$$

$$g^1 = (u^2)^m \varphi^1(u^1), \qquad g^2 = (u^2)^{m+1} \varphi^2(u^1), \qquad (3.3)$$

where λ_3 *and* m *are arbitrary constants, and* $\varphi^1(u^1)$ *and* $\varphi^2(u^1)$ *are arbitrary smooth functions.*

Proof. Taking formulas (2.10) into account, system (2.4) can be written as follows:

$$\begin{aligned} &\alpha^{22}(x_0)u^2 g^1_{u^2} = -2c_1 g^1, \\ &\alpha^{22}(x_0)u^2 g^2_{u^2} = (\alpha^{22}(x_0) - 2c_1)g^2 + \dot{\alpha}^{22}(x_0)u^2. \end{aligned} \tag{3.4}$$

At arbitrary g^1 and g^2, system (3.4) implies that $\alpha^{22}(x_0) = c_1 = 0$. With regard for formulas (2.10), we obtain that, in this case, the maximal algebra of invariance of system (3.1) is the algebra A_0.

Let us determine now the functions g^1 and g^2, for which system (3.1) admits an extension of the symmetry kernel A_0. For this purpose, it is necessary, as follows from (3.4), that the functions g^1 and g^2 be solutions of the structural system (see [11])

$$\begin{aligned} &\text{æ} u^2 g^1_{u^2} = m g^1, \\ &\text{æ} u^2 g^2_{u^2} = (m + \text{æ})g^2 + \lambda_3 u^2, \end{aligned} \tag{3.5}$$

where $\text{æ} = \{0; 1\}$; m and λ_3 are arbitrary constants. System (3.5) at $\text{æ} = 1$ is connected with system (3.4) by the conditions

$$m\alpha^{22}(x_0) = -2c_1, \qquad \lambda_3 \alpha^{22} = \dot{\alpha}^{22}. \tag{3.6}$$

The solution of system (3.5) at $\text{æ} = 1$ depends on the constant m. Two essentially different cases are possible.

1. $m = 0$. The general solution of system (3.5) looks like (3.2), where $\lambda_3 \neq 0$, φ^1 and φ^2 are arbitrary smooth functions.

2. $m \neq 0$. It follows from the differential consequences of conditions (3.6) that $\dot{\alpha}^{22} = \lambda_3 = 0$. Then the general solution of system (3.5) looks like (3.3).

At $\text{æ} = 0$, we get from system (3.5) that the extension of a symmetry kernel A_0 occurs only at $g^1 = g^2 = 0$, which is a particular case of formulas (3.2), (3.3). The theorem is proved. □

Remark 3.2. Since formulas (3.2), (3.3) coincide at $\lambda_3 = m = 0$, we set $\lambda_3 \neq 0$ in formulas (3.2) in order to avoid their coincidence, while studying the symmetry properties of system (3.1).

The conditions of Theorem 3.1 are only necessary conditions for the extension of the symmetry kernel A_0 of system (3.1), but not sufficient. The classification of the symmetry properties of system (3.1) is presented by the following theorem.

Theorem 3.2. *If system* (3.1) *admits the extension of the symmetry kernel* A_0*, its maximal algebras of invariance depending on the functions* g^1 *and* g^2 *are given in Table* 1.

Table 1. Classification of the symmetry properties of system (3.1)

N n/n	Kind of functions g^1, g^2	Operators of maximal algebra of invariance
1.	$g^1 = \varphi^1(u^1)$, $g^2 = u^2(\varphi^2(u^1) + \lambda_3 \ln u^2)$	$\partial_0, \partial_1,\ Q_1 = e^{\lambda_3 x_0} u^2 \partial_{u^2}$
2.	$g^1 = (u^2)^m \varphi^1(u^1)$, $g^2 = (u^2)^{m+1} \varphi^2(u^1)$	$\partial_0, \partial_1, D = m(2x_0\partial_0 + x_1\partial_1) - 2u^2\partial_{u^2}$
3.	$g^1 = 0$, $g^2 = 0$	$\partial_0, \partial_1, D = 2x_0\partial_0 + x_1\partial_1$, $I = u^2\partial_{u^2}$

In Table 1, $\varphi^1(u^1)$ and $\varphi^2(u^1)$ are arbitrary smooth functions, and $\lambda_3 \neq 0$, λ_4, m are arbitrary constants.

Proof. In Theorem 3.1, it is shown that the extension of the symmetry kernel A_0 of system (3.1) is possible only in the case where the functions g^a are set by formulas (3.2) or (3.3). We will consider each of these formulas separately.

A. We set that the functions g^1, g^2 look like (3.2). Substituting (3.2) in system (2.4), we obtain

$$c_1 = 0, \qquad \dot{\alpha}^{22} - \lambda_3 \alpha^{22} = 0,$$

whence $\alpha^{22} = c_2 e^{\lambda_3 x_0}$, where c_2 is an arbitrary constant.

From formulas (2.10), we obtain the algebra presented in the first point of Table 1.

B. If the functions g^a are set by formulas (3.3), system (2.4) yields

$$\begin{aligned}(m\alpha^{22} + 2c_1)\varphi^1 &= 0,\\ (m\alpha^{22} + 2c_1)\varphi^2 &= \dot{\alpha}^{22}(u^2)^{-m}.\end{aligned} \tag{3.7}$$

In the case where φ^1, φ^2 are arbitrary smooth functions, system (3.7) yields

$$\dot{\alpha}^{22} = 0, \qquad m\alpha^{22} + 2c_1 = 0,$$

that is

$$\alpha^{22} = -2c_2, \qquad c_1 = mc_2, \tag{3.8}$$

where c_2 is an arbitrary constant. From formulas (2.10) and (3.8), we obtain the algebra which is presented in the second point of Table 1. The symmetry extensions of the second point of Table 1 is possible only at

$$m = 0, \qquad \varphi^1 = 0, \qquad \varphi^2 = \lambda_4.$$

In this case,

$$\alpha^{22} = 2\lambda_4 c_1 x_0 + c_2, \tag{3.9}$$

where λ_4 and c_2 are arbitrary constants. By applying the equivalence transformations presented in Remark 3.1, we obtain the third point of Table 1 from formulas (2.10) and (3.9). The theorem is proved. ☐

4. Symmetry properties of system (1.2) at $f = \lambda$

We mow consider the system

$$\begin{pmatrix} u^1 \\ u^2 \end{pmatrix}_0 = \partial_1 \left[\begin{pmatrix} \lambda_1 & 0 \\ \lambda u^2 & \lambda_2 \end{pmatrix} \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}_1 \right] + \begin{pmatrix} g^1(u^1, u^2) \\ g^2(u^1, u^2) \end{pmatrix} \tag{4.1}$$

and will perform the classification of its symmetry properties depending on the form of the functions $g^a(u^1; u^2)$.

Remark 4.1. It follows from Lemma 2.1 that the basic group of equivalence transformations of system (4.1) looks like

$$\begin{gathered} x_0 = te^{2\theta_2} + \theta_0, \qquad x_1 = xe^{\theta_2} + \theta_1, \\ u^1 = w^1 + \theta_3, \qquad u^2 = w^2 e^{\theta_4}. \end{gathered} \tag{4.2}$$

Besides the basic group of equivalence, system (4.1) admits additional equivalence transformations at specific g. For example,

$$x_0 = at, \quad x_1 = bx, \quad u^1 = w^1 + kt, \quad u^2 = w^2 e^{mt}, \tag{4.3}$$

where a, b, k, m are arbitrary constants.

In view of Remark 4.1, we will formulate the theorems on the maximal invariance algebras of system (4.1) to within the transformations of equivalence (4.2) and (4.3).

The necessary condition for the extension of the symmetry kernel A_0 of system (4.1) is given by the following proposition.

Theorem 4.1. *If system* (4.1) *admits the extension of the symmetry kernel* A_0*, the functions* g^1, g^2 *are set by formulas* (3.2), (3.3) *or one of the formulas*

$$g^1 = \varphi^1(\omega) + \lambda_3 u^1, \qquad g^2 = u^2[\varphi^2(\omega) + \lambda_4 u^1]; \tag{4.4}$$

$$g^1 = e^{mu^1}\varphi^1(\omega), \qquad g^2 = u^2 e^{mu^1}\varphi^2(\omega), \tag{4.5}$$

where $\omega = ku^1 + \ln u^2$*,* $m, k, \lambda_3, \lambda_4$ *are arbitrary constants, and* $\varphi^1(\omega)$*,* $\varphi^2(\omega)$ *are arbitrary smooth functions.*

Proof. Substituting formulas (2.11) in system (2.4), we obtain

$$\begin{aligned} &\beta^1(x_0)g^1_{u^1} + \alpha^{22}(x_0)u^2 g^1_{u^2} = -2c_1 g^1 + \dot\beta^1(x_0),\\ &\beta^1(x_0)g^2_{u^1} + \alpha^{22}(x_0)u^2 g^2_{u^2} = (\alpha^{22}(x_0) - 2c_1)g^2 + \dot\alpha^{22}(x_0)u^2. \end{aligned} \tag{4.6}$$

It is obvious that, at arbitrary functions g^1, g^2, system (4.6) does not admit the extension of the symmetry kernel A_0.

It follows from (4.6) that the functions g^a should satisfy the structural system

$$\begin{aligned} &æ g^1_{u^1} - ku^2 g^1_{u^2} = mg^1 + \lambda_3,\\ &æ g^2_{u^1} - ku^2 g^2_{u^2} = (m-k)g^2 + \lambda_4 u^2, \end{aligned} \tag{4.7}$$

where $æ = \{0; 1\}$, $k, m, \lambda_3, \lambda_4$ are arbitrary constants. If $æ = 0$, system (4.7) will coincide with system (3.5), which has been analyzed in Theorem 3.1, according to which the functions g^a are set by formulas (3.2) and (3.3).

If $æ = 1$, then system (4.7) is connected with system (4.6) by the conditions

$$\alpha^{22} + k\beta^1 = 0, \quad m\beta^1 = -2c_1, \quad \lambda_3\beta^1 = \dot\beta^1, \quad \lambda_4\alpha^{22} = \dot\alpha^{22}. \tag{4.8}$$

The solution of system (4.7) at $æ = 1$ depends on the constant m. Two essentially different cases are possible.

1. $m = 0$. In this case, the general solution of system (4.7) is set by functions (4.4).

2. $m \neq 0$. From the differential consequences of the first and second conditions (4.8), we get

$$\dot{\alpha}^{22} = \dot{\beta}^1 = 0, \qquad \lambda_3 = \lambda_4 = 0. \tag{4.9}$$

Under conditions (4.9), the general solution of system (4.7) is functions (4.5). The theorem is proved. ☐

Remark 4.2. As formulas (4.4) and (4.5) at $\lambda_3 = \lambda_4 = m = 0$ coincide, then, in order to avoid their coincidence, we will consider $|\lambda_3| + |\lambda_4| \neq 0$ in formulas (4.4), while studying the symmetry properties of system (4.1).

We will classify now the symmetry properties of system (4.1), using the results of Theorem 4.1.

Theorem 4.2. *If system* (4.1) *admits the extension of the symmetry kernel* A_0*, its maximal algebras of invariance depending on a kind of the functions* g^1, g^2 *are presented in Tables* 1 *and* 2.

Таблиця 2. Classification of symmetry properties of system (4.1)

№ n/n	Kind of functions g^1, g^2	Operators of maximal algebra of invariance
1.	$g^1 = e^{mu^1}\varphi^1(\omega)$, $g^2 = u^2 e^{mu^1}\varphi^2(\omega)$	$\partial_0, \partial_1, D = m(2x_0\partial_0 + x_1\partial_1)$ $+2(-\partial_{u^1} + ku^2\partial_{u^2})$
2.	$g^1 = \varphi^1(\omega) + \lambda_3 u^1$, $g^2 = u^2(\varphi^2(\omega) - k\lambda_3 u^1)$	$\partial_0, \partial_1, Q = e^{\lambda_3 x_0}(\partial_{u^1} - ku^2\partial_{u^2})$
3.	$g^1 = \lambda_5 e^{u^1}$, $g^2 = \lambda_6 e^{u^1} u^2$	$\partial_0, \partial_1, Q = u^2\partial_{u^2}$, $D = 2x_0\partial_0 + x_1\partial_1 - 2\partial_{u^1}$
4.	$g^1 = \lambda_5 (u^2)^n e^{mu^1} - m\lambda_9$, $g^2 = u^2(\lambda_6 (u^2)^n e^{mu^1} + n\lambda_9)$	$\partial_0, \partial_1, Q = n\partial_{u^1} - mu^2\partial_{u^2}$, $D = n(2x_0\partial_0 + x_1\partial_1)$ $+ 2n\lambda_9 x_0 Q - 2u^2\partial_{u^2}$
5.	$g^1 = \lambda_3 u^1 + \lambda_5 \ln u^2 + \lambda_7$, $g^2 = u^2(\lambda_4 u^1 + \lambda_6 \ln u^2 + \lambda_8)$, $D > 0$	∂_0, ∂_1, $Q_1 = e^{m_1 x_0}(\lambda_5\partial_{u^1} + (m_1 - \lambda_3)u^2\partial_{u^2})$, $Q_2 = e^{m_2 x_0}((m_2 - \lambda_6)\partial_{u^1} + \lambda_4 u^2\partial_{u^2})$
6.	$g^1 = \lambda_3 u^1 + \lambda_5 \ln u^2 + \lambda_7$, $g^2 = u^2(\lambda_4 u^1 + \lambda_6 \ln u^2 + \lambda_8)$, $D = 0, \quad |\lambda_3| + |\lambda_5| + |\lambda_6| \neq 0$	$\partial_0, \partial_1, Q_1 = e^{\alpha x_0}[x_0(2\lambda_5\partial_{u^1}$ $+ (\lambda_6 - \lambda_3)u^2\partial_{u^2}) + u^2\partial_{u^2}]$, $Q_2 = e^{\alpha x_0}[2\lambda_5\partial_{u^1} + (\lambda_6 - \lambda_3)u^2\partial_{u^2}]$

7.	$g^1 = \lambda_3 u^1 + \lambda_5 \ln u^2 + \lambda_7$, $g^2 = u^2(\lambda_4 u^1 + \lambda_6 \ln u^2 + \lambda_8)$, $D < 0$	$\partial_0, \partial_1, Q_1 = e^{\alpha x_0}[2\lambda_5 \cos\beta x_0 \partial_{u^1}$ $+ ((\lambda_6 - \lambda_3)\cos\beta x_0 - 2\beta \sin\beta x_0) u^2 \partial_{u^2}]$, $Q_2 = e^{\alpha x_0}[2\lambda_5 \sin\beta x_0 \partial_{u^1}$ $+ (2\beta\cos\beta x_0 + (\lambda_6 - \lambda_3)\sin\beta x_0) u^2 \partial_{u^2}]$
8.	$g^1 = 0$, $g^2 = u^1 u^2$	$\partial_0, \partial_1, Q_1 = \partial_{u^1} + x_0 Q_2, \quad Q_2 = u^2 \partial_{u^2}$
9.	$g^1 = 0$, $g^2 = 0$	$\partial_0, \partial_1, Q_1 = \partial_{u^1}, \quad Q_2 = u^2\partial_{u^2}$, $D = 2x_0\partial_0 + x_1\partial_1$

In Table 2, $m, n, \lambda_i, i = \overline{1;9}$ are arbitrary constants, $\omega = \ln u^2 + ku^1$, $\varphi^1(\omega), \varphi^2(\omega)$ are arbitrary smooth functions; $D = (\lambda_3 - \lambda_6)^2 + 4\lambda_4\lambda_5$ is a discriminant, and m_1, m_2 are roots of the characteristic equation $m^2 - (\lambda_3 + \lambda_6)m + \lambda_3\lambda_6 - \lambda_4\lambda_5 = 0$, $\alpha = \frac{\lambda_3+\lambda_6}{2}$, $\beta = \frac{1}{2}\sqrt{|(\lambda_3 - \lambda_6)^2 + 4\lambda_4\lambda_5|}$.

5. Symmetry properties of system (1.2) at $f = \frac{\lambda}{u^1}$

In this subsection, we consider the system

$$\begin{pmatrix} u^1 \\ u^2 \end{pmatrix}_0 = \partial_1 \left[\begin{pmatrix} \lambda_1 & 0 \\ \frac{\lambda}{u^1} u^2 & \lambda_2 \end{pmatrix} \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}_1 \right] + \begin{pmatrix} g^1(u^1, u^2) \\ g^2(u^1, u^2) \end{pmatrix}, \qquad (5.1)$$

where λ is an arbitrary constant, and we will perform the classification of its symmetry properties depending on a kind of the functions $g^a(u^1; u^2)$.

Remark 5.1. It follows from Lemma 2.1 that the basic group of equivalence transformations of system (5.1) looks like

$$\begin{aligned} x_0 &= te^{2\theta_2} + \theta_0, \qquad & x_1 &= xe^{\theta_2} + \theta_1, \\ u^1 &= w^1 e^{\theta_3}, \qquad & u^2 &= w^2 e^{\theta_4}. \end{aligned} \qquad (5.2)$$

In addition to the basic group of equivalence, system (5.1) at specific g admits additional equivalence transformations, for example,

$$x_0 = at, \quad x_1 = bx, \quad u^1 = w^1 e^{kt}, \quad u^2 = w^2 e^{mt}, \qquad (5.3)$$

where a, b, k, m are arbitrary constants. Therefore, we will formulate the theorems on the maximal algebra of invariance of system (5.1) to within the transformations of equivalence (5.2) and (5.3).

The necessary condition for the extension of the symmetry kernel A_0 of system (5.1) is given by the following proposition.

Theorem 5.1. *If system* (5.1) *admits the extension of the symmetry kernel* A_0*, then the functions* g^1, g^2 *are set by formulas* (3.2)*,* (3.3) *or one of the following formulas:*

$$g^1 = u^1(\varphi^1(\omega) + \lambda_3 \ln u^1), \qquad g^2 = u^2(\varphi^2(\omega) + \lambda_4 \ln u^2); \tag{5.4}$$

$$g^1 = (u^1)^{m+1}\varphi^1(\omega), \qquad g^2 = u^2(u^1)^m\varphi^2(\omega), \tag{5.5}$$

where $\omega = \frac{u^2}{(u^1)^k}$*;* $\varphi^1(\omega), \varphi^2(\omega)$ *are arbitrary smooth functions,* m, k, λ_3*,* λ_4 *are arbitrary constants.*

Proof. Substituting formulas (2.12) which set the coordinates of the infinitesimal operator (2.1) for system (5.1) in system (2.4), we obtain

$$\begin{aligned} \alpha^1(x_0)u^1 g^1_{u^1} + \alpha^{22}(x_0)u^2 g^1_{u^2} &= (\alpha^1(x_0) - 2c_1)g^1 + \dot\alpha^1(x_0)u^1, \\ \alpha^1(x_0)u^1 g^2_{u^1} + \alpha^{22}(x_0)u^2 g^2_{u^2} &= (\alpha^{22}(x_0) - 2c_1)g^2 + \dot\alpha^{22}(x_0)u^2. \end{aligned} \tag{5.6}$$

It is obvious that, at arbitrary functions g^1, g^2, system (5.6) does not admits the extension of the symmetry kernel A_0.

System (5.6) admits the widest class of functions g^a, at which the extension of the symmetry kernel A_0 is possible, if they satisfy the structural system

$$\begin{aligned} æu^1 g^1_{u^1} + ku^2 g^1_{u^2} &= (m + æ)g^1 + k_1 u^1, \\ æu^1 g^2_{u^1} + ku^2 g^2_{u^2} &= (m + k)g^2 + k_2 u^2, \end{aligned} \tag{5.7}$$

where $æ = \{0; 1\}$; k, m, k_1, k_2 are arbitrary constants. The general solution of system (5.7) at $æ = 0$ is set by formulas (3.2), (3.3). System (5.7) at $æ = 1$ is connected with system (5.6) by the conditions

$$\alpha^{22} - k\alpha^1 = 0, \quad m\alpha^1 = -2c_1, \quad k_1\alpha^1 = \dot\alpha^1, \quad k_2\alpha^1 = \dot\alpha^{22}.$$

The solution of system (5.7) depends on the constant m.

If $m = 0$, the general solution of system (5.7) looks like (5.4). At $m \neq 0$, it is set by formulas (5.5). The theorem is proved. □

Remark 5.2. If we set $\lambda_3 = \lambda_4 = 0$ in the representations of the functions g^a given by formulas (5.4) the obtained form of the functions g^a will be a special case of the representation of functions g^a given by formulas (5.5) under the condition $m = 0$. Hence, the classes of systems (5.1), (5.4) and (5.1), (5.5) will have a nonempty crossing. To avoid the consideration of equivalent systems in the subsequent researches of symmetry properties, we impose restrictions on the parameters of representations of the functions g^a in formulas (5.4): $|\lambda_3| + |\lambda_4| \neq 0$.

Let's classify now the symmetry properties of system (5.1), by using Theorem 5.1.

Theorem 5.2. *If system* (5.1) *admits the extension of the symmetry kernel* A_0*, its maximal algebras of invariance depending on a kind of the functions* g^1, g^2 *are presented in Tables* 1 *and* 3.

Таблиця 3. Classification of the symmetry properties of system (5.1)

№ n/n	Kind of functions g^1, g^2	Operators of maximal algebra of invariance
1.	$g^1 = u^1(\varphi^1(u^2) + \lambda_3 \ln u^1)$, $g^2 = u^2(\varphi^2(u^2) + \lambda_4 \ln u^2)$	$\partial_0, \partial_1, Q = e^{\lambda_3 x_0} u^1 \partial_{u^1}$
2.	$g^1 = u^1(\varphi^1(\omega) + \lambda_3 \ln u^1)$, $g^2 = u^2(\varphi^2(\omega) + \lambda_3 \ln u^2)$	$\partial_0, \partial_1,\ Q = e^{\lambda_3 x_0}(u^1 \partial_{u^1} + k u^2 \partial_{u^2})$
3.	$g^1 = (u^1)^{m+1} \varphi^1(\omega)$, $g^2 = u^2 (u^1)^m \varphi^2(\omega)$	$\partial_0, \partial_1, D = m(2x_0\partial_0 + x_1\partial_1)$ $- 2(u^1\partial_{u^1} + k u^2 \partial_{u^2})$
4.	$g^1 = u^1(\lambda_5 (u^1)^n (u^2)^m + m\lambda_7)$, $g^2 = u^2(\lambda_6 (u^1)^n (u^2)^m - n\lambda_7)$	$\partial_0, \partial_1, Q = m u^1 \partial_{u^1} - n u^2 \partial_{u^2}$, $D = m(2x_0\partial_0 + x_1\partial_1 + 2\lambda_7 x_0 Q)$ $- 2u^2\partial_{u^2}$
5.	$g^1 = \lambda_5 (u^1)^{n+1}$, $g^2 = \lambda_6 (u^1)^n u^2$	$\partial_0, \partial_1, D = 2x_0\partial_0 + x_1\partial_1 - \frac{2}{n} u^1 \partial_{u^1}$, $Q = u^2 \partial_{u^2}$
6.	$g^1 = u^1(\lambda_3 \ln u^1 + \lambda_5 \ln u^2 + \lambda_7)$, $g^2 = u^2(\lambda_4 \ln u^1 + \lambda_6 \ln u^2 + \lambda_8)$, $D > 0$	$\partial_0, \partial_1, Q_1 = e^{m_1 x_0}(\lambda_5 u^1 \partial_{u^1}$ $+ (m_1 - \lambda_3) u^2 \partial_{u^2})$, $Q_2 = e^{m_2 x_0}((m_2 - \lambda_6) u^1 \partial_{u^1}$ $+ \lambda_4 u^2 \partial_{u^2})$
7.	$g^1 = u^1(\lambda_3 \ln u^1 + \lambda_5 \ln u^2 + \lambda_7)$, $g^2 = u^2(\lambda_4 \ln u^1 + \lambda_6 \ln u^2 + \lambda_8)$, $D = 0,\ |\lambda_3| + |\lambda_5| + |\lambda_6| \neq 0$	$\partial_0, \partial_1, Q_1 = e^{\alpha x_0}[x_0(\lambda_5 u^1 \partial_{u^1}$ $+ (\alpha - \lambda_3) u^2 \partial_{u^2}) + u^2 \partial_{u^2}]$, $Q_2 = e^{\alpha x_0}[\lambda_5 u^1 \partial_{u^1} + (\alpha - \lambda_3) u^2 \partial_{u^2}]$
8.	$g^1 = \lambda_7 u^1$, $g^2 = \lambda_4 u^2 \ln u^1$	$\partial_0, \partial_1, Q_1 = u^2 \partial_{u^2}$, $Q_2 = u^1 \partial_{u^1} + \lambda_4 x_0 u^2 \partial_{u^2}$
9.	$g^1 = u^1(\lambda_3 \ln u^1 + \lambda_5 \ln u^2 + \lambda_7)$, $g^2 = u^2(\lambda_4 \ln u^1 + \lambda_6 \ln u^2 + \lambda_8)$, $D < 0$	$\partial_0, \partial_1, Q_1 = e^{\alpha x_0}[\lambda_5 \cos\beta x_0 u^1 \partial_{u^1}$ $+ ((\alpha - \lambda_3)\cos\beta x_0 - \beta \sin\beta x_0) u^2 \partial_{u^2}]$, $Q_2 = e^{\alpha x_0}[\lambda_5 \sin\beta x_0 u^1 \partial_{u^1}$ $+ (\beta \cos\beta x_0 + (\alpha - \lambda_3)\sin\beta x_0) u^2 \partial_{u^2}]$
10.	$g^1 = 0$, $g^2 = 0$	$\partial_0, \partial_1, Q_1 = u^1 \partial_{u^1}, \quad Q_2 = u^2 \partial_{u^2}$, $D = 2x_0\partial_0 + x_1\partial_1 + 2x_0(\lambda_7 Q_1 + \lambda_8 Q_2)$

In Table 3, $m, n, \lambda_i,\ i=\overline{1;9}$ are arbitrary constants, $\omega=\frac{u^2}{(u^1)^k}$, $\varphi^1(\omega)$, $\varphi^2(\omega)$ are arbitrary smooth functions, $D=(\lambda_3-\lambda_6)^2+4\lambda_4\lambda_5$ is a discriminant, and m_1, m_2 are roots of the characteristic equation $m^2-(\lambda_3+\lambda_6)m+\lambda_3\lambda_6-\lambda_4\lambda_5=0$, $\alpha=\frac{\lambda_3+\lambda_6}{2}$, $\beta=\frac{1}{2}\sqrt{|D|}$.

6. Symmetry properties of system (1.2) at $f=\frac{\lambda_1-\lambda_2}{u^1}$

We now consider the system

$$\begin{pmatrix} u^1 \\ u^2 \end{pmatrix}_0 = \partial_1\left[\begin{pmatrix} \lambda_1 & 0 \\ \frac{\lambda_1-\lambda_2}{u^1}u^2 & \lambda_2 \end{pmatrix}\begin{pmatrix} u^1 \\ u^2 \end{pmatrix}_1\right]+\begin{pmatrix} g^1(u^1,u^2) \\ g^2(u^1,u^2) \end{pmatrix} \tag{6.1}$$

and will perform the classification of its symmetry properties depending on a kind of the functions $g^a(u^1;u^2)$.

Remark 6.1. It follows from Lemma 2.1 that the basic group of equivalence transformations of system (6.1) looks like

$$\begin{aligned} x_0 &= te^{2\theta_2}+\theta_0, \qquad x_1=xe^{\theta_2}+\theta_1, \\ u^1 &= w^1e^{\theta_3}, \qquad u^2=w^2e^{\theta_4}+\theta_5w^1. \end{aligned} \tag{6.2}$$

Besides the basic group of equivalence transformations (6.2), system (6.1) at specific g^a admits additional transformations of equivalence of the form (5.3). Therefore, we will formulate theorems on the maximal algebras of invariance of system (6.1) to within the equivalence transformations (5.3) and (6.2).

The necessary condition for the extension of the symmetry kernel A_0 of system (6.1) is given the following proposition.

Theorem 6.1. *If the system* (6.1) *admits the extension of the symmetry kernel* A_0*, then the functions* g^1, g^2 *are set by formulas* (3.2), (3.3), (5.4), (5.5) *or, to within the equivalence transformations* (5.3) *and* (6.2) *look like*

$$g^1=u^1(\varphi^1(\omega)+\lambda_3), \qquad g^2=u^1\varphi^2(\omega)+u^2\varphi^1(\omega), \tag{6.3}$$

where $\omega=u^1$, λ_3 *is an arbitrary constant;*

$$g^1=u^1e^{\frac{u^2}{u^1}}\varphi^1(\omega), \qquad g^2=e^{\frac{u^2}{u^1}}[u^1\varphi^2(\omega)+u^2\varphi^1(\omega)], \tag{6.4}$$

where $\omega=u^1$*;*

$$g^1 = (u^1)^{m+1}\varphi^1(\omega), \qquad g^2 = (u^1)^m[u^1\varphi^2(\omega) + u^2\varphi^1(\omega)], \tag{6.5}$$

where $\omega = \frac{u^2}{u^1} + k \ln u^1$, $k \neq 0$ *and* m *are arbitrary constants;*

$$\begin{aligned} g^1 &= u^1(\varphi^1(\omega) + \lambda_3) + \lambda_4 u^2, \\ g^2 &= u^1\varphi^2(\omega) + u^2\varphi^1(\omega) + \lambda_4\frac{(u^2)^2}{u^1}, \end{aligned} \tag{6.6}$$

where $\omega = \frac{u^2}{u^1} + k \ln u^1$, λ_3, λ_4, k *are arbitrary constants,* $k \neq 0$, $|\lambda_3| + |\lambda_4| \neq 0$, *and, in formulas* (6.3)–(6.6), $\varphi^1(\omega), \varphi^2(\omega)$ *are arbitrary smooth functions.*

Proof. As was already mentioned at the beginning of the present article, the solution of determining systems $S_1(\xi, \eta) = 0$ and $S_2(\xi, \eta, f) = 0$ at $f = \frac{\lambda_1 - \lambda_2}{u^1}$ are the coordinates of operator (2.1) given by formulas (2.13). In view of these formulas, the determining system $S_3(\xi, \eta, f, g) = 0$ can be written as follows:

$$\begin{aligned} &\alpha^1(x_0)u^1 g^1_{u^1} + (\alpha^{21}(x_0)u^1 + \alpha^{22}u^2)g^1_{u^2} \\ &\qquad\qquad = (\alpha^1(x_0) - 2c_1)g^1 + \dot{\alpha}^1 u^1, \\ &\alpha^1(x_0)u^1 g^2_{u^1} + (\alpha^{21}(x_0)u^1 + \alpha^{22}u^2)g^2_{u^2} \\ &\qquad = (\alpha^{22}(x_0) - 2c_1)g^2 + \alpha^{21}(x_0)g^1 + \dot{\alpha}^{21}u^1 + \dot{\alpha}^{22}u^2. \end{aligned} \tag{6.7}$$

It is obvious that, at arbitrary functions g^1, g^2, system (6.7) does not admits the extension of the symmetry kernel A_0.

The widest class of functions g^a such that they satisfy system (6.7) and allow the symmetry kernel A_0 to be extended is possible under the conditions

$$\begin{aligned} &\alpha^1 = k_1\varphi(x_0), \quad \alpha^{21} = k_0\varphi(x_0), \quad \alpha^{22} = k_2\varphi(x_0), \\ &\dot{\alpha}^1 = k_4\varphi(x_0), \quad \dot{\alpha}^{21} = k_5\varphi(x_0), \quad \dot{\alpha}^{22} = k_6\varphi(x_0), \\ &\qquad\qquad -2c_1 = k_3\varphi(x_0), \end{aligned} \tag{6.8}$$

where $\varphi(x_0)$ are arbitrary smooth functions, $k_0, k_1, \ldots, k_6$ are arbitrary constants. With regard for (6.8) and (6.7), we obtain the structural system for the functions g^a:

$$\begin{aligned} k_1 u^1 g^1_{u^1} + (k_0 u^1 + k_2 u^2)g^1_{u^2} &= (k_1 + k_3)g^1 + k_4 u^1, \\ k_1 u^1 g^2_{u^1} + (k_0 u^1 + k_2 u^2)g^2_{u^2} &= (k_2 + k_3)g^2 + k_0 g^1 + k_5 u^1 + k_6 u^2. \end{aligned} \tag{6.9}$$

Let us analyze this system and its influence on solutions of system (6.7). The solution of system (6.9) essentially depends on the ratios between the constants k_0, k_1, k_2. If we set $k_0 = 0$ in the structural system (6.9), then system (6.9) coincides with system (5.7). If $k_0 \neq 0$, and $k_1 \neq k_2$, we can use the equivalence transformations (6.2) at $\theta_5 = -\frac{k_0}{k_2}$ and $\theta_i = 0, i = \overline{0,4}$ and reduce system (6.9) to system (5.7) investigated in Theorem 5.1, according to which the functions g^a are set by formulas (5.5), (5.4) or (3.2), (3.3).

If $k_0 \neq 0$ (without loss generality, it is possible to consider $k_0 = 1$) and $k_1 = k_2$, then formulas (6.8) yield $k_4 = k_6$. Then system (6.9) takes the form

$$\begin{gathered}
k_1 u^1 g^1_{u^1} + (u^1 + k_1 u^2) g^1_{u^2} = (k_1 + k_3) g^1 + k_4 u^1, \\
k_1 u^1 g^2_{u^1} + (u^1 + k_1 u^2) g^2_{u^2} = (k_1 + k_3) g^2 + g^1 + k_4 u^2 + k_5 u^1.
\end{gathered} \tag{6.10}$$

The solution of system (6.10) depends on the parameters k_1, k_3. We obtain 4 nonequivalent cases:

1) $k_1 = 0, \quad k_3 = 0,$

2) $k_1 = 0, \quad k_3 \neq 0,$

3) $k_1 \neq 0, \quad k_3 = 0,$

4) $k_1 \neq 0, \quad k_3 \neq 0.$

1) Let $k_1 = 0,\ k_3 = 0$. If $k_1 = 0$, Eqs. (6.8) imply that $k_4 = 0$. Then system (6.10) takes the form

$$u^1 g^1_{u^2} = 0, \qquad u^1 g^2_{u^2} = g^1 + k_5 u^1. \tag{6.11}$$

By solving Eq. (6.11), we obtain the representation of functions g^a of form (6.3), where $\lambda_3 = -k_5$.

2) Consider the case where $k_1 = 0, k_3 \neq 0$. Without loss of generality, it is possible to consider that $k_3 = 1$. It follows from Eqs. (6.8) that $k_4 = k_5 = 0$. Then system (6.10) takes the form

$$u^1 g^1_{u^2} = g^1, \qquad u^1 g^2_{u^2} = g^2 + g^1,$$

whose general solution is functions (6.4).

3) If $k_1 \neq 0, k_3 = 0$, system (6.10) takes the form

$$\begin{aligned} k_1 u^1 g^1_{u^1} + (u^1 + k_1 u^2) g^1_{u^2} &= k_1 g^1 + k_4 u^1, \\ k_1 u^1 g^2_{u^1} + (u^1 + k_1 u^2) g^2_{u^2} &= k_1 g^2 + g^1 + k_5 u^1 + k_4 u^2. \end{aligned} \tag{6.12}$$

The general solution of system (6.12) looks like (6.6), where $k = -\frac{1}{k_1}$, $\lambda_3 = -k_5, \lambda_5 = k_4$.

4) If $k_1 \neq 0$, $k_3 \neq 0$, it follows from (6.8) that $k_4 = k_5 = 0$. In this case, system (6.10) becomes

$$\begin{aligned} k_1 u^1 g^1_{u^1} + (u^1 + k_1 u^2) g^1_{u^2} &= (k_1 + k_3) g^1, \\ k_1 u^1 g^2_{u^1} + (u^1 + k_1 u^2) g^2_{u^2} &= (k_1 + k_3) g^2 + g^1. \end{aligned} \tag{6.13}$$

By solving system (6.13), we obtain the representation of the functions g^a which is set by formulas (6.5), where $m = \frac{k_3}{k_1}$. The theorem is proved. ☐

Let us classify the symmetry properties of system (6.1), by using Theorem 6.1.

Remark 6.2. In formulas (6.5) and (6.6), the restrictions are imposed to avoid their crossing.

Theorem 6.2. *If system* (6.1) *admits the extension of the symmetry kernel* A_0*, its maximal algebras of invariance depending on a kind of the functions* g^1*,* g^2 *are given in Tables* 1*,* 3*, and* 4.

Таблиця 4. Classification of the symmetry properties of system (6.1)

№ n/n	Kind of functions g^1, g^2	Operators of maximal algebra of invariance
1.	$g^1 = u^1(\varphi^1(u^1) + \lambda_3)$, $g^2 = u^1\varphi^2(u^1) + u^2\varphi^1(u^1)$	$\partial_0, \partial_1, Q_1 = e^{-\lambda_3 x_0} Q$
2.	$g^1 = u^1 e^{\frac{u^2}{u^1}} \varphi^1(u^1)$, $g^2 = e^{\frac{u^2}{u^1}}(u^1\varphi^2(u^1) + u^2\varphi^1(u^1))$	$\partial_0, \partial_1, Q$
3.	$g^1 = (u^1)^{m+1}\varphi^1(\omega)$, $g^2 = (u^1)^m(u^1\varphi^2(\omega) + u^2\varphi^1(\omega))$, $\omega = \dfrac{u^2}{u^1} + k\ln u^1$	$\partial_0, \partial_1, D = m(2x_0\partial_0 + x_1\partial_1) - 2I + 2kQ$
4.	$g^1 = u^1(\varphi^1(\omega) + k) + u^2$, $g^2 = u^1\varphi^2(\omega) + u^2\varphi^1(\omega) + \dfrac{(u^2)^2}{u^1}$, $\omega = \dfrac{u^2}{u^1} + k\ln u^1, \quad k \neq 0$	$\partial_0, \partial_1, Q = e^{-kx_0}(I - kQ)$

5.	$g^1=(u^1)^{m+1}$, $g^2=u^1((u^1)^n+\lambda_8)+u^2(u^1)^m$, $m\neq 0,\quad n\neq 0$	∂_0,∂_1,Q, $D=m(2x_0\partial_0+x_1\partial_1-2u^1\partial_{u^1})$ $+2n(\lambda_8x_0Q-u^2\partial_{u^2})$
6.	$g^1=\lambda_5(u^1)^{m+1}$, $g^2=(u^1)^m(\lambda_6u^2+u^1)$, $\lvert\lambda_5\rvert+\lvert\lambda_6\rvert\neq 0$	$\partial_0,\partial_1,D=m(2x_0\partial_0+x_1\partial_1)-2I$, $Q_1=Q+(\lambda_6-\lambda_5)u^2\partial_{u^2}$
7.	$g^1=u^1((u^1)^m+\lambda_6)$, $g^2=u^2((u^1)^m+\lambda_7)+\lambda_8u^1$, $m\neq 0,\ \lambda_6\neq 0,\ \lambda_7\neq\lambda_6$	$\partial_0,\partial_1,Q_1=e^{(\lambda_7-\lambda_6)x_0}Q$, $Q_2=\lambda_8Q+(\lambda_7-\lambda_6)u^2\partial_{u^2}$
8.	$g^1=u^1((u^1)^m+\lambda_7)$, $g^2=u^2((u^1)^m+\lambda_7)+\lambda_8u^1$, $m\neq 0,\ \lambda_7\neq 0$	$\partial_0,\partial_1,Q,Q_1=\lambda_8x_0Q-u^2\partial_{u^2}$
9.	$g^1=(u^1)^{m+1}$, $g^2=(u^1)^m(\lambda_8(u^1)^{m+1}+u^2)$, $m\neq 0$	∂_0,∂_1,Q, $D=m(2x_0\partial_0+x_1\partial_1)-2(I+mu^2\partial_{u^2})$
10.	$g^1=\lambda_5u^1$, $g^2=(u^1)^{n+1}+\lambda_7u^2$, $n\neq 0,\ \lambda_7\neq\lambda_5$	$\partial_0,\partial_1,Q_1=e^{(\lambda_7-\lambda_5)x_0}Q$, $Q_2=I+nu^2\partial_{u^2}$
11.	$g^1=u^1$, $g^2=u^1((u^1)^n+\lambda_8)+u^2$, $n\neq 0$	$\partial_0,\partial_1,Q,Q_1=I+nu^2\partial_{u^2}-n\lambda_8x_0Q$
12.	$g^1=0$, $g^2=(u^1)^n+\lambda_6u^1+\lambda_7$, $n\neq 1,\ \lambda_7\neq 0$	$\partial_0,\partial_1,Q,D=2x_0\partial_0+x_1\partial_1+2u^2\partial_{u^2}$
13.	$g^1=\lambda_3u^2+\lambda_5u^1$, $g^2=u^1\Big(\lambda_4\Big(\dfrac{u^2}{u^1}\Big)^2+\dfrac{(\lambda_6-\lambda_5)^2}{4(\lambda_4-\lambda_3)}\Big)$ $+\lambda_6u^2$, $\lvert\lambda_3\rvert+\lvert\lambda_4\rvert\neq 0,\ \lambda_4\neq\lambda_3,\ \lambda_6\neq\lambda_5$	$\partial_0,\partial_1,I,D=(\lambda_4-\lambda_3)$ $\times[(2x_0\partial_0+x_1\partial_1)-2u^2\partial_{u^2}]$ $-(\lambda_6-\lambda_5)Q+x_0(2\lambda_5(\lambda_4-\lambda_3)$ $-\lambda_3(\lambda_6-\lambda_5))I$
14.	$g^1=u^2$, $g^2=u^1\Big(\Big(\dfrac{u^2}{u^1}\Big)^2+\lambda_8\Big)\pm u^2$	$\partial_0,\partial_1,\ I,\ Q_1=e^{\pm x_0}(I\pm Q)$
15.	$g^1=e^{n\frac{u^2}{u^1}}(u^1)^{p+1}$, $g^2=e^{n\frac{u^2}{u^1}}(u^1)^p(u^2+\lambda_4u^1)$, $p\neq 0,\ n\neq 0$	$\partial_0,\partial_1,Q_1=nI-pQ$, $D=n(2x_0\partial_0+x_1\partial_1)-2Q$
16.	$g^1=[\lambda_3e^{n\frac{u^2}{u^1}}(u^1)^p+\lambda_5]u^1$, $g^2=e^{n\frac{u^2}{u^1}}(u^1)^p(\lambda_3u^2+\lambda_4u^1)$ $+\lambda_5u^2-\dfrac{p}{n}\lambda_5u^1$, $n\neq 0,\ \lvert\lambda_3\rvert+\lvert\lambda_4\rvert\neq 0$	$\partial_0,\partial_1,Q_1=I-\dfrac{p}{n}Q$, $D=2x_0\partial_0+x_1\partial_1+2\lambda_5x_0Q_1-\dfrac{2}{n}Q$
17.	$g^1=(u^1)^{m+1}$, $g^2=u^1(\ln u^1+\lambda_8)+u^2(u^1)^m$, $m\neq 0$	∂_0,∂_1,Q, $D=m(2x_0\partial_0+x_1\partial_1)-2I-2x_0Q$

18.	$g^1 = u^1(\lambda_5 \ln u^1 + \lambda_3) + \lambda_4 u^2$, $g^2 = (\lambda_6 u^1 + \lambda_5 u^2)\ln u^1$ $+ \lambda_8 u^1 + \lambda_7 u^2 + \lambda_4 \dfrac{(u^2)^2}{u^1}$, $\lvert\lambda_5\rvert + \lvert\lambda_6\rvert \neq 0,\ \Delta > 0$	$\partial_0, \partial_1, Q_1 = e^{m_1 x_0}[\lambda_4 I + (m_1 - \lambda_5)Q]$, $Q_2 = e^{m_2 x_0}[\lambda_4 I + (m_2 - \lambda_5)Q]$
19.	$g^1 = u^1(\lambda_5 \ln u^1 + \lambda_3) + \lambda_4 u^2$, $g^2 = (\lambda_6 u^1 + \lambda_5 u^2)\ln u^1$ $+ \lambda_8 u^1 + \lambda_7 u^2 + \lambda_4 \dfrac{(u^2)^2}{u^1}$, $\lvert\lambda_3\rvert + \lvert\lambda_5\rvert + \lvert\lambda_6\rvert \neq 0,\ \Delta = 0$	$\partial_0, \partial_1, Q_1 = e^{\alpha x_0}[\lambda_4 I + (\alpha - \lambda_5)Q]$, $Q_2 = e^{\alpha x_0} Q + x_0 Q_1$
20.	$g^1 = u^1(\lambda_5 \ln u^1 + \lambda_3) + \lambda_4 u^2$, $g^2 = (\lambda_6 u^1 + \lambda_5 u^2)\ln u^1$ $+ \lambda_8 u^1 + \lambda_7 u^2 + \lambda_4 \dfrac{(u^2)^2}{u^1}$, $\Delta < 0$	$\partial_0, \partial_1, Q_1 = e^{\alpha x_0}[\lambda_4 \cos\beta x_0 I$ $+ ((\alpha - \lambda_5)\cos\beta x_0 - \beta \sin\beta x_0)Q]$, $Q_2 = e^{\alpha x_0}[\lambda_4 \sin\beta x_0 I$ $+ ((\alpha - \lambda_5)\sin\beta x_0 + \beta\cos\beta x_0)Q]$
21.	$g^1 = u^1(\lambda_5 \ln u^1 + \lambda_3)$, $g^2 = u^1 \ln u^1 + u^2(\lambda_5 \ln u^1 + \lambda_7)$, $\lambda_3 \neq \lambda_7 - \lambda_5$	$\partial_0, \partial_1, Q_1 = e^{(\lambda_7 - \lambda_3)x_0} Q$, $Q_2 = e^{\lambda_5 x_0}[(\lambda_3 + \lambda_5 - \lambda_7)I + Q]$
22.	$g^1 = u^1(\lambda_5 \ln u^1 + \lambda_7 - \lambda_5)$, $g^2 = u^1 \ln u^1 + u^2(\lambda_5 \ln u^1 + \lambda_7)$	$\partial_0, \partial_1, Q_1 = e^{\lambda_5 x_0} Q$, $Q_2 = e^{\lambda_5 x_0} I + x_0 Q_1$
23.	$g^1 = 0$, $g^2 = \ln u^1$	$\partial_0, \partial_1, Q$, $D = 2x_0\partial_0 + x_1\partial_1 + 2u^2\partial_{u^2}$
24.	$g^1 = (u^1)^{m+1}$, $g^2 = u^2(u^1)^m + \lambda_8 u^1$, $m \neq 0$	∂_0, ∂_1, $D = m(2x_0\partial_0 + x_1\partial_1) + 2\lambda_8 x_0 Q - 2I$, $Q, Q_1 = \lambda_8 x_0 Q - u^2\partial_{u^2}$
25.	$g^1 = \lambda_5 u^1$, $g^2 = u^1((u^1)^n + \lambda_8)$ $+ (n+1)\lambda_5 u^2$, $n \neq 0$	∂_0, ∂_1, $D = 2x_0\partial_0 + x_1\partial_1 + 2x_0 Q_2 - \dfrac{2}{n} I$, $Q_1 = e^{n\lambda_5 x_0} Q$, $Q_2 = \lambda_5(I + nu^2\partial_{u^2}) + \lambda_8 Q$
26.	$g^1 = 0$, $g^2 = u^1((u^1)^n + \lambda_8)$, $n \neq 0$	$\partial_0, \partial_1, Q$, $D = 2x_0\partial_0 + x_1\partial_1 - \dfrac{2}{n} I + 2\lambda_8 x_0 Q$, $Q_1 = I + nu^2\partial_{u^2} - n\lambda_8 x_0 Q$
27.	$g^1 = u^2$, $g^2 = u^1\Big(\Big(\dfrac{u^2}{u^1}\Big)^2 + \lambda_8\Big)$	$\partial_0, \partial_1,\ I,\ Q_1 = x_0 I + Q$, $D = 2x_0\partial_0 + x_1\partial_1 + 2\lambda_8 x_0^2 I$ $+ 4\lambda_8 x_0 Q - 2u^2\partial_{u^2}$
28.	$g^1 = 0$, $g^2 = \lambda_4 u^1 + \lambda_5$	$\partial_0, \partial_1, D = 2x_0\partial_0 + x_1\partial_1 + 2u^2\partial_{u^2}$, $Q, Q_1 = u^1\partial_{u^1} + \lambda_4 x_0 Q$
29.	$g^1 = u^1 \ln u^1$, $g^2 = u^2 \ln u^1 + \lambda_8 u^1$	$\partial_0, \partial_1, Q, Q_1 = e^{x_0} I$, $Q_2 = u^2\partial_{u^2} - \lambda_8 x_0 Q$

30.	$g^1 = \lambda_7 u^1$, $g^2 = u^1 \ln u^1 + \lambda_7 u^2$	$\partial_0, \partial_1, Q, Q_1 = I + x_0 Q$, $D = 2x_0\partial_0 + x_1\partial_1 + 2\lambda_7 x_0 Q_1 - \lambda_7 x_0^2 Q + 2u^2\partial_{u^2}$
31.	$g^1 = 0$, $g^2 = u^1 \ln u^1$	$\partial_0, \partial_1,\ D = 2x_0\partial_0 + x_1\partial_1 + 2u^2\partial_{u^2}$, $Q, Q_1 = I + x_0 Q$
32.	$g^1 = 0$, $g^2 = u^2$	$\partial_0, \partial_1, I, Q_1 = e^{x_0} Q, Q_2 = u^1\partial_{u^1}$, $D = 2x_0\partial_0 + x_1\partial_1 + 2x_0 u^2\partial_{u^2}$
33.	$g^1 = 0$, $g^2 = \lambda_8 u^1$	$\partial_0, \partial_1, I, Q, D = 2x_0\partial_0 + x_1\partial_1 - 2u^1\partial_{u^1}$, $Q_3 = u^1\partial_{u^1} + \lambda_8 x_0 Q$

In Table 4, $m, n, \lambda_i,\ i = \overline{1;9}$ are arbitrary constants, $\omega = \frac{u^2}{u^1} + k\ln u^1$, $\varphi^1(\omega)$, $\varphi^2(\omega)$ are arbitrary smooth functions, $I = u^1\partial_{u^1} + u^2\partial_{u^2}$, $Q = u^1\partial_{u^2}$, $\Delta = (\lambda_3 + \lambda_5 - \lambda_7)^2 + 4\lambda_4\lambda_6$ is a discriminant, and m_1, m_2 are roots of the characteristic equation

$$\begin{vmatrix} \lambda_5 - m & \lambda_4 \\ \lambda_6 & \lambda_7 - \lambda_3 - m \end{vmatrix} = 0,$$

$\alpha = \frac{\lambda_7 + \lambda_5 - \lambda_3}{2}$, $\beta = \frac{1}{2}\sqrt{|\Delta|}$.

7. Invariance under the Galilei algebra

In the present subsection, we will comprehensively study the symmetry properties of system (1.2) for $f(u^1) = \frac{2\lambda_1}{u^1}$.

So, we consider the system

$$\begin{pmatrix} u^1 \\ u^2 \end{pmatrix}_0 = \partial_1 \left[\begin{pmatrix} \lambda_1 & 0 \\ 2\lambda_1 \frac{u^2}{u^1} & \lambda_2 \end{pmatrix} \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}_1 \right] + \begin{pmatrix} g^1(u^1, u^2) \\ g^2(u^1, u^2) \end{pmatrix}. \tag{7.1}$$

Remark 7.1. It follows from Lemma 2.1 that the basic group of equivalence transformations of system (7.1) looks like

$$\begin{aligned} x_0 &= te^{2\theta_2} + \theta_0, \qquad & x_1 &= xe^{\theta_2} + \theta_1, \\ u^1 &= w^1 e^{\theta_3}, \qquad & u^2 &= w^2 e^{\theta_4}. \end{aligned} \tag{7.2}$$

Besides the basic group of equivalence transformations (7.2), system (7.1) admits the additional equivalence transformations of form (5.3) at specific g. Therefore, we will formulate the theorems on the maximal invariance algebras of system (7.1) to within the specified equivalence transformations (5.3), (7.2).

The following proposition is valid.

Theorem 7.1. *If system* (7.1) *admits the extension of the symmetry kernel* A_0*, the functions* g^1, g^2 *are set by formulas* (3.2), (3.3), (5.4), *or by the formulas*

$$g^1 = u^1[(u^1)^m\varphi^1(\omega) + \lambda_3], \qquad g^2 = u^2[(u^1)^m\varphi^2(\omega) + \lambda_4], \tag{7.3}$$

where $\omega = \frac{u^2}{(u^1)^k}$*,* $m, \lambda_3, \lambda_4, k$ *are arbitrary constants, and* $\varphi^a(\omega)$ *are arbitrary smooth functions.*

Proof. As has been shown in the proof of Theorem 2.1, the solution of systems $S_1(\xi,\eta) = 0$ and $S_2(\xi,\eta,f) = 0$ for $f = \frac{2\lambda_1}{u^1}$ are the coordinates of the infinitesimal operator X set by formulas (2.14).

In view of values of ξ^0, ξ^1, η^a in formulas (2.14), system (2.4) can be written as

$$\begin{aligned} \alpha^1 u^1 g^1_{u^1} + \alpha^{22} u^2 g^1_{u^2} &= (\alpha^1 - 2\dot{A})g^1 + (\alpha^1_0 - \lambda_1\alpha^1_{11})u^1, \\ \alpha^1 u^1 g^2_{u^1} + \alpha^{22} u^2 g^2_{u^2} &= (\alpha^{22} - 2\dot{A})g^2 + (\alpha^{22}_0 - 2\lambda_1\alpha^1_{11})u^2. \end{aligned} \tag{7.4}$$

It is obvious that, at arbitrary functions g^1, g^2, system (7.4) does not admits the extension of the symmetry kernel A_0.

Since the functions α^1, α^{22}, A depend only on the variables x_0, x_1, and the functions g^a do on the variables u^1, u^2, the widest class of the functions g^1, g^2 such that they satisfy system (7.4) and allow the symmetry kernel A_0 to be extended is a solution of the structural system

$$\begin{aligned} æu^1 g^1_{u^1} + ku^2 g^1_{u^2} &= (m + æ)g^1 + k_1 u^1, \\ æu^1 g^2_{u^1} + ku^2 g^2_{u^2} &= (m + k)g^2 + k_2 u^2. \end{aligned} \tag{7.5}$$

Moreover, $\alpha^1 = æ\psi(x_0,x_1)$, $\alpha^{22} = k\psi(x_0,x_1)$, $-2\dot{A} = m\psi(x_0,x_1)$, $\alpha^1_0 - \lambda_1\alpha^1_{11} = k_1\psi(x_0,x_1)$, $\alpha^{22}_0 - 2\lambda_1\alpha^1_{11} = k_2\psi(x_0,x_1)$, where $æ = \{0,1\}$; k, m, k_1, k_2 are arbitrary constants which we will call *structural constants* for the functions g^a, and $\psi(x_0,x_1)$ is an arbitrary smooth function.

1. If $æ = 0$, system (7.5) takes form (3.4), whose solutions are formulas (3.2) and (3.3), as it has been shown in Theorem 3.1.

2. If $æ = 1$, the general solution of system (7.5) is expressed through the first integrals of the system of ordinary differential equations

$$\frac{du^1}{u^1} = \frac{du^2}{ku^2} = \frac{dg^1}{(m+1)g^1 + k_1 u^1} = \frac{dg^2}{(k+m)g^2 + k_2 u^2}. \qquad (7.6)$$

One of the first integrals of system (7.6) is $J_1 = \omega = \frac{u^2}{(u^1)^k}$, and two other ones, J_2, J_3, depend on the constant m.

The following nonequivalent cases are possible:

2.1) $m = 0$. In this case,

$$J_2 = \frac{g^1}{u^1} - k_1 \ln u^1, \qquad J_3 = \frac{g^2}{u^2} - k_2 \ln u^1.$$

By constructing the general solution of system (7.5) in the standard way (see, for example, [24]), we obtain formulas (5.4), where $\lambda_3 = -k_1$, $\lambda_4 = -k_2$ are arbitrary constants, and $\varphi^a(\omega)$ are arbitrary smooth functions, $\omega = \frac{u^2}{(u^1)^k}$.

2.2) $m \neq 0$. In this case, by calculating the first integrals of system (7.6),

$$J_2 = (u^1)^{-m}(\frac{g^1}{u^1} + \frac{k_1}{m}), \qquad J_3 = (u^1)^{-m}(\frac{g^2}{u^2} + \frac{k_2}{m}),$$

we obtain the general solution of system (7.5) which looks like (7.3), where $\lambda_3 = -\frac{k_1}{m}$, $\lambda_4 = -\frac{k_2}{m}$ are arbitrary constants, and $\varphi^a(\omega)$ are arbitrary smooth functions, $\omega = \frac{u^2}{(u^1)^k}$.

So, by solving system $S_3(\xi, \eta, f, g) = 0$ for g^1, g^2 at $f = \frac{2\lambda_1}{u^1}$, we have obtained the nonequivalent forms (3.2), (3.3), (5.4), (7.3) of the functions g^1, g^2, what proves the theorem. □

Conditions of Theorem 7.1, as well as those of Theorem 2.1, are only necessary conditions for the extension of the symmetry kernel of system (1.2). To obtain sufficient conditions, it is necessary to substitute each representation of the functions g^a of forms (3.2), (3.3), (5.4), (7.3) in the system $S_3 = 0$ and to solve the obtained system for functions $A(x_0)$, $B(x_0)$, $C(x_0)$, $\alpha(x_0)$ with regard for a kind of the functions $\varphi^a(\omega)$ and values of the constants $m, k, \lambda_3, \lambda_4$. The following statement is a result of such researches.

Theorem 7.2. *The maximal invariance algebras of system* (7.1) *depending on values of the functions* g^1, g^2 *are given in Tables* 1, 3, *and* 5.

Таблиця 5. Classification of the symmetry properties of system (7.1)

№ n/n	Kind of functions g^1, g^2	Operators of maximal algebra of invariance
1.	$g^1 = u^1\varphi^1(u^2)$, $g^2 = u^2\varphi^2(u^2)$	$\partial_0, \partial_1, G = x_0 - \frac{x_1}{2\lambda_1}u^1\partial_{u^1}, \quad I_1 = u^1\partial_{u^1}$
2.	$g^1 = \lambda_6 u^1 \ln u^2$, $g^2 = \lambda_8 u^2 \ln u^2$	$\partial_0, \partial_1, G, I_1,\ Q_1 = e^{\lambda_8 x_0}(\lambda_6 I_1 + \lambda_8 I_2)$
3.	$g^1 = \lambda_6 u^1 \ln u^2$, $g^2 = \lambda_9 u^2$	$\partial_0, \partial_1, G, I_1, \lambda_6 x_0 I_1 + I_2$
4.	$g^1 = u^1[\lambda_6(u^2)^n + \lambda_3]$, $g^2 = \lambda_5(u^2)^{n+1}$	$\partial_0, \partial_1, G, I_1,$ $D_1 = 2x_0\partial_0 + x_1\partial_1 + 2\lambda_3 x_0 I_1 - \frac{2}{n}I_2$
5.	$g^1 = u^1[\lambda_6(u^2)^2 + \lambda_3]$, $g^2 = \lambda_5(u^2)^3$	$\partial_0, \partial_1, G, I_1,$ $D_2 = 2x_0\partial_0 + x_1\partial_1 + 2\lambda_3 x_0 I_1 - I_2,$ $\Pi_1 = x_0^2\partial_0 + x_0 x_1\partial_1$ $- \frac{1}{2\lambda_1}\Big(\frac{x_1^2}{2} - 2\lambda_1\lambda_3 x_0^3 - \lambda_1 x_0\Big)I_1 - x_0 I_2$
6.	$g^1 = 0$, $g^2 = 0$	$\partial_0, \partial_1, G, I_1, I_2,$ $D_3 = 2x_0\partial_0 + x_1\partial_1 - \frac{1}{2}I_1 - I_2,$ $\Pi_2 = x_0^2\partial_0 + x_0 x_1\partial_1 - \Big(\frac{x_1^2}{2} + \lambda x_0\Big)I_1 - x_0 I_2$
7.	$g^1 = u^1(\varphi^1(u^2) + \lambda_3 \ln u^1)$, $g^2 = u^2\varphi^2(u^2)$	$\partial_0, \partial_1, \mathcal{G} = e^{\lambda_3 x_0}\Big(\partial_1 - \frac{\lambda_3}{2\lambda_1}x_1 I_1\Big),$ $M = e^{\lambda_3 x_0} I_1$
8.	$g^1 = u^1(\lambda_3 \ln u^1 + \lambda_6 \ln u^2)$ $g^2 = \lambda_8 u^2 \ln u^2$, $\lambda_8 \neq \lambda_3$	$\partial_0, \partial_1, \mathcal{G}, M, Q_2 = e^{\lambda_8 x_0}(\lambda_6 I_1 + (\lambda_8 - \lambda_3) I_2)$
9.	$g^1 = u^1(\lambda_3 \ln u^1 + \lambda_6 \ln u^2)$, $g^2 = \lambda_3 u^2 \ln u^2$	$\partial_0, \partial_1, \mathcal{G}, M, Q_3 = e^{\lambda_3 x_0}(\lambda_6 x_0 I_1 + I_2)$

In Table 5, $\lambda_3, \ldots, \lambda_9$ are arbitrary constants, $\varphi^a = \varphi^a(u^2)$ are arbitrary smooth functions, and $I_2 = u^2\partial_{u^2}$.

Remark 7.2. Theorems 4.2, 5.2, 6.2 are proved similarly to Theorem 3.2. The proof of Theorem 7.2 is given in work [20].

Conclusions

The nonrelativistic movement of any macroobject is satisfied with transformations of shift and stretching and the Galilei law of the movement relativity. Therefore, it is obvious that the models of movement investigated in the given work, being invariant under the Galilei algebra and the algebras setting the transformation of shift and stretching, claim for the reliability of the description of the movement of objects within the Keller-Segel's model. In addition, the maximal algebras of invariance of systems established in the present work can considerably facilitate the work on the establishment of trajectories of movement of the objects whose movement is investigated within the mentioned model.

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