# ON DISPERSION RELATION OF SLOW CIRCULARLY POLARIZED ELECTROMAGNETIC WAVES IN PLASMAS

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In the present communication, Hamilton equations for electrons interacting with slow circular polarized electromagnetic wave are solved in a self-consistent way. Basing on these solutions the interaction between the fast electrons and propagating circular wave is described kinetically, and the non-linear dispersion relation is obtained. As a result, specific conditions for the slow wave propagation in a two component plasma are analyzed.

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## INTRODUCTION

Finding the dispersion relation for electromagnetic waves in homogeneous media is basic fundamental problem of wave propagation. In cold isotropic plasma within a linear theory, the dispersion relation for transverse electromagnetic waves with frequency higher than electron plasma frequency results in the fact that only fast electromagnetic waves with $\upsilon_{ph} > c$ can propagate [1]. Some time ago the question was under discussion if there a possibility to arrange circularly polarized slow waves due to the trapping of some supra-thermal electron fraction into the wave field (see e.g. [2,3]). In such a situation slowing-down of waves is provided due to electron trapping by a finite amplitude electromagnetic wave. In the present communication we analyze specific conditions for the existence of circularly polarized slow waves in a plasma with two electron components. The nonlinear dispersion relation for such waves is obtained self-consistently with taking into account Maxwell equations and motion equations for electrons. The treatment is performed both in hydrodynamics approximation and kinetically basing on the solutions of Hamilton equations for electrons interacting with slow circular polarized electromagnetic wave. Problems of the formation of such waves and of electrons with two fractions are out of the scope of present communication.

## 1. SELF-CONSISTENT STATIC SHEARED MAGNETIC FIELD

In the investigation of slow waves it may be convenient to shift to the reference frame moving with the phase velocity, where the plane wave is presented as purely static magnetic configuration. In particular, circularly polarized in the laboratory frame wave corresponds to sheared magnetic field

$$\begin{aligned} B_x &= B\cos kz \\ B_y &= B\sin kz \end{aligned} . \tag{1}$$

To provide self-consistency of such magnetic configuration it is necessary to have corresponding electron current in which every electron perform the motion allowed by magnetic field (1). For the sake of simplicity we shall consider that the current is produced by electrons with constant longitudinal velocities $V_z = const$ (the ion motion is neglected).

For the electron with $V_z = const$ it is necessary to satisfy the following set of equations:

$$\begin{cases} V_x B_y = V_y B_x \\ \dot{V}_x = \omega_B V_z \sin kV_z t \\ \dot{V}_y = -\omega_B V_z \cos kV_z t \end{cases} , \tag{2}$$

where $\omega_B = eB/mc$. From Eqs. (2) for $V_z \neq 0$ it follows that $\vec{V}_\perp = -e\vec{B}/mck$, and this relation does not depend on the longitudinal velocity. Electrons with $V_z = 0$ ("trapped" electrons in the laboratory frame of reference), can have arbitrary $\vec{V}_{\perp 0}$ which is parallel to the magnetic field $\vec{B}$ in the corresponding $z$ =const plane. Transverse current of electrons with $V_z \neq 0$ ("untrapped" electrons) is

$$\vec{j}_\perp = -e(1-\alpha)N\vec{V}_\perp = (1-\alpha)e^2 N\vec{B}/mck , \tag{3}$$

where $N$ is the electron density, $\alpha$ is a fraction of "trapped" electrons. From the static Maxwell equation

$$rot\vec{B} = \frac{4\pi}{c}\vec{j} ,$$

we obtain following condition

$$k^2 + (1-\alpha)\frac{\omega_P^2}{c^2} = \alpha\frac{\omega_P^2}{c^2}\frac{kV_{\perp 0}}{\omega_B} , \tag{4}$$

with $\omega_P^2 = 4\pi Ne^2/m$. The condition that Eq.(4) possess real solutions for $k$ is $\frac{\alpha}{\sqrt{1-\alpha}}\frac{V_{\perp 0}}{c} > 2\frac{\omega_B}{\omega_P}$, which impose the following limitation to $\alpha$

$$\alpha > 2\frac{c^2\omega_B^2}{V_{\perp 0}^2\omega_P^2}\left(\sqrt{1+\frac{V_{\perp 0}^2}{4c^2}\frac{\omega_P^2}{\omega_B^2}} - 1\right); \tag{5}$$

in the case $\alpha \ll 1$ this corresponds to the inequality

$$\alpha > 2\frac{\omega_B c}{\omega_P V_{\perp 0}} . \tag{6}$$

Taking into account the evident inequality $V_{\perp 0} < c$, one can obtain overestimated limitation to $\alpha$

$$\alpha > 2\frac{\omega_B^2}{\omega_P^2}\left(\sqrt{1+\frac{\omega_P^2}{4\omega_B^2}} - 1\right) , \tag{7}$$

which does not contain $V_{\perp 0}$.

Eq. (4) for a fixed plasma density impose the relation between field amplitude, fraction of "trapped" electrons and their transverse velocity, which may be considered as nonlinear dispersion relation. This relation can be modified with taking into account that longitudinal current formed by ions with the density $N$ moving with the velocity $-\upsilon_{ph}$ and by "untrapped" electrons with the density $\alpha N$ must be zero. This means that in the laboratory reference frame "untrapped" electrons must move also in $z$-direction relative to immovable ions to compensate z-component of "trapped" electron current. The mean velocity of "untrapped" electrons in $z$-direction $<V_z>$ is defined by the relation $1-\alpha=-\upsilon_{ph}/<V_z>$.

It should be noted that Eq. (4) has two real solutions for $k$, which means that in the laboratory reference frame there are two slow nonlinear circularly polarizes waves with the equal phase velocities.

## 2. ELECTRON MOTION IN CIRCULARLY POLARIZED ELECTROMAGNETIC WAVE

Above we present the simplest demonstration of the existing of nonlinear slow circularly polarized waves in a plasma with two cold electron fractions ("trapped" and "untrapped"), in which every "trapped" electron is moving with constant velocity along its own rectilinear trajectory, while "untrapped electrons are moving with constant velocity along spiral trajectories. In such a wave the "trapped" fraction can possess arbitrary spread over transverse velocities, and "untrapped" fraction allows spread over longitudinal velocities.

Now again having in mind investigation of slow nonlinear circularly polarized waves in a more general case, we consider arbitrary electron motion (non-relativistic) in the electromagnetic wave defined by vector-potential:

$$A_x=\frac{cE_0}{\omega}\cos(\phi);\ A_y=\frac{cE_0}{\omega}\sin(\phi),\qquad \phi=kz-\omega t \tag{8}$$

From the translation symmetry transverse momentum conservation follows:

$$m\vec{V}_\perp-\frac{e}{c}\vec{A}=\vec{P}_\perp=const\,. \tag{9}$$

The longitudinal motion is governed by the equation:

$$m\ddot{z}=\frac{eE_0}{m\omega}k(P_x\sin\phi-P_y\cos\phi)\,, \tag{10}$$

or

$$m\ddot{z}=-\frac{eE_0}{m\omega}kP_\perp\sin(\phi-\phi_0)\,, \tag{11}$$

where $P_\perp$ and $\phi_0$ are introduced by $P_x=-P_\perp\cos\phi_0;P_y=-P_\perp\sin\phi_0$. Note also the following condition

$$\vec{P}_\perp=-\frac{P_\perp\omega}{E_0c}\vec{A}\Big|_{\phi=\phi_0}\,. \tag{12}$$

By substitution $k\tilde{z}=kz-\omega t-\phi_0$ one can obtain the first integral of equation (11):

$$\frac{m\dot{\tilde{z}}^2}{2}-\frac{eE_0}{m\omega}P_\perp\cos k\tilde{z}=W_z=const\,, \tag{13}$$

from which it follows

$$z=\tilde{z}(t-\tau,P_\perp,E_z)-\frac{\omega}{k}t-\frac{\phi_0}{k}\,, \tag{14}$$

where $\tilde{z}$ – takes the form of inverse elliptical function

$$t-\tau=\sqrt{\frac{m}{2}}\int\frac{d\tilde{z}}{\sqrt{W_z-\frac{eE_0}{m\omega}P_\perp\cos k\tilde{z}}}\,, \tag{15}$$

with the constant $\tau$ characterizing the initial phase of electron oscillation.

## 3. DISPERSION RELATION

The plane wave (8) can propagate in the plasma without support from external sources if the following equations are satisfied:

$$\begin{cases}\left(k^2-k_0^2\right)\vec{A}=\dfrac{4\pi}{c}\vec{j}_\perp\\ j_z=0\end{cases}. \tag{16}$$

The transverse current can be calculate as

$$\vec{j}_\perp(z,t)=-e\int\vec{V}_\perp f\delta(z-z(t))d...\,, \tag{17}$$

where $f$ is the – electron distribution function, $z(t)$ is the solution of motion equation; integration is performed over the set of constants characterizing electron motion. In our case such these constants are $E_z,P_\perp,\phi_0,\tau$. Actually the electron motion must be characterized by 6 constant, but two constants arising from initial transverse coordinates can be omitted. When the distribution function depends on $E_z$ and $P_\perp$, Eq. (17) takes the form:

$$\vec{j}_\perp(z,t)=-e\int\vec{V}_\perp(z,t)\times\\ \times\delta\left(z-\tilde{z}-\frac{\omega}{k}t-\frac{\varphi_0}{k}\right)\frac{1}{2\pi}f(W_z,P_\perp)\,d\varphi_0d\tau dE_zdP_\perp, \tag{18}$$

where transverse velocity $\vec{V}_\perp$ is also dependent on $E_z,P_\perp$. From Eqs. (9) and (12) it follows:

$$\vec{V}(z,t)=\frac{e}{cm}\vec{A}+\frac{1}{m}\vec{P}=\frac{e}{cm}\vec{A}(\phi)-\frac{P_\perp\omega}{mcE_0}\vec{A}(\phi_0)\,. \tag{19}$$

Integrating over variable $\phi_0$ and using normalization condition

$$\int\delta(z-z(t))\frac{1}{2\pi}f(W_z,P_\perp)\,d\varphi_0d...=\frac{k}{2\pi}\int fd...=const=N,$$

one can obtain

$$\vec{j}_\perp(z,t)=-\frac{e^2}{mc}N\vec{A}(\varphi)+\\ +\frac{e\omega}{mcE_0}\int\vec{A}\big(\varphi-k\tilde{z}(t-\tau,W_z,P_\perp)\big)\frac{k}{2\pi}P_\perp f(W_z,P_\perp)\,d\tau dE_zdP_\perp. \tag{20}$$

Substituting (20) into (16) results in the nonlinear dispersion relation:

$$k^2-k_0^2+\frac{\omega_p^2}{c^2}=\frac{4\pi}{c}\frac{e\omega}{mcE_0}\iint\Phi(W_z,P_\perp)P_\perp F(W_z,P_\perp)dE_zdP_\perp\,, \tag{21}$$

where $F$ is the distribution function normalized as:

$$\int FdW_zdP_\perp=N\,, \tag{22}$$

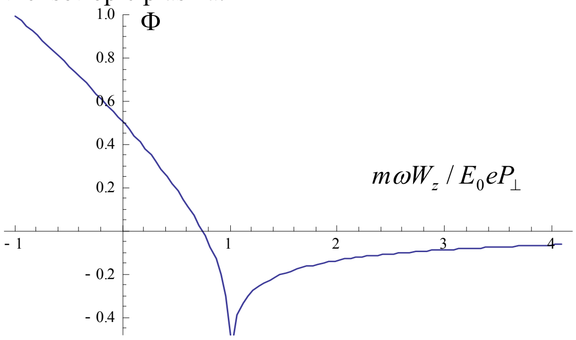
and $\Phi = \dfrac{\int \cos\zeta \left( W_z + \dfrac{eE_0}{m\omega} P_\perp \cos\zeta \right)^{-1/2} d\zeta}{\int \left( W_z + \dfrac{eE_0}{m\omega} P_\perp \cos\zeta \right)^{-1/2} d\zeta}$ (23)

integration limits in (23) are determined by zero points of expression $W_z + eE_0 P_\perp \cos\zeta / m\omega$.

Electrons may be specified as belonging to three groups. The first group includes trapped electrons with $-\frac{cE_0}{m\omega} P_\perp < W_z < W_C$ ($W_C \approx 0.65 \frac{cE_0}{m\omega} P_\perp$), for these electrons $0 < \Phi < 1$ and $< V_z >= \omega / k$; the second group includes trapped electrons with $W_C < W_z < \frac{cE_0}{m\omega} P_\perp$, for these electrons $-1 < \Phi < 0$ and $< V_z >= \omega / k$; the third group includes untrapped electrons with $\frac{cE_0}{m\omega} P_\perp < W_z$ for this electrons $-1 < \Phi < 0$ and $< V_z > \neq \omega / k$.

In Figure the dependence of $\Phi$ on $W_z m\omega / cE_0 P_\perp$ is shown.

For fast waves (with $\omega / k > c$) the expression (23) after some cumbersome calculations can be simplified and for waves of low enough amplitude the nonlinear dispersion relation (21) can be transformed into well known linear dispersion relation for transverse waves in the isotropic plasma.




*Weight function Φ dependence on normalized $W_z$*

Condition for existence of self-consistent non-linear slow waves in plasma may be derived from Eqs. (21), (23) (see also Figure). Such waves can exist, if there is sufficiently large fraction of trapped electrons with high enough $P_\perp$. This condition may be written in the form:

$$\iint \left( \Phi(W_z, P_\perp) P_\perp - \frac{eE_0}{\omega} \right) F(W_z, P_\perp) dW_z dP_\perp > 0 . \quad (24)$$

So for existence of such a wave two necessary conditions must be satisfied: for the group of trapped particles the condition $P_\perp > eE_0 / \omega$ must be fulfilled; more strictly the trapped particle density should satisfy the condition

$$N_{tr} P_{\perp tr} > NeE_0 / \omega . \quad (25)$$

If one take into account the conservation of transverse momentum in the process of wave switch-on, and assume that before switching-on the wave field electrons had Maxwellian distribution, the condition of existence of nonlinear slow waves takes the form

$$\frac{m\upsilon_T \omega}{3e} > E_0 . \quad (26)$$

This is a rather strong limitation, and more simple realization of slow waves may be achieved in the plasma with two electron fractions, e.g., in the presence of electron beam in the direction of wave propagation with large enough transverse energy.

In conclusion, we demonstrate the simple way of constructing nonlinear slow circularly polarized waves in the plasma, which allows evident modification to the relativistic case.

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## О ДИСПЕРСИОННОМ СООТНОШЕНИИ ДЛЯ ЦИРКУЛЯРНО ПОЛЯРИЗОВАННЫХ ЗАМЕДЛЕННЫХ ВОЛН В ПЛАЗМЕ

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Решаются уравнения Гамильтона для электронов, взаимодействующих с замедленной циркулярно поляризованной электромагнитной волной. На основе этих решений кинематически строится нелинейное дисперсионное соотношение. Обсуждаются специфические условия, при выполнении которых замедленная волна может распространяться в двухкомпонентной плазме.

## ПРО ДИСПЕРСІЙНІ СПІВВІДНОШЕННЯ ДЛЯ ЦИРКУЛЯРНО ПОЛЯРИЗОВАНОГО УПОВІЛЬНЕННЯ ХВИЛЬ У ПЛАЗМІ

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Розв'язується рівняння Гамільтона для електронів, що взаємодіють з уповільненою циркулярно поляризованою електромагнітною хвилею. На основі цих рішень кінематично будується нелінійне дисперсійне співвідношення. Обговорюються специфічні умови, при виконанні яких уповільнена хвиля може поширюватися в двокомпонентній плазмі.