SCALING LAWS FOR THE HELICON EIGENMODES IN A NONUNIFORM PLASMA CYLINDER

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The wave eigenmodes of a radially nonuniform plasma cylinder are examined analytically in the helicon frequency range. Conditions for the existence of almost pure helicon eigenmodes with on-axis localization are pointed out. The scaling (i.e. the relationship between the plasma density, the magnetic field, and the mode frequency and wave number) for these eigenmodes is found and shown to depend on the azimuthal mode number.

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INTRODUCTION

The problem of helicon wave eigenmodes in a radially nonuniform plasma cylinder was examined in numerous papers in application to helicon discharges, fusion plasmas etc. (e.g., [1,2]). However, recently this problem was addressed again [3], for the following reason. As is known, the helicon wave dispersion relation for an infinite uniform plasma reads

$$\omega = \omega_{ce} \frac{k_z k c^2}{\omega_{pe}^2}, \qquad (1)$$

where ω_{pe} is the electron plasma frequency, and ω, k_z, $k_\perp$ and $k = (k_z^2 + k_\perp^2)^{1/2}$ are the mode frequency and wave numbers, the longitudinal, transverse and total ones, respectively. If the frequency is fixed (in a helicon plasma, it is normally determined by an rf generator), the dispersion relation (1) yields the scaling for plane waves propagating along the magnetic field in the form $n_0 / B_0 \propto k_z^2$ where n_0 is the plasma density and B_0 an ambient magnetic field. For a radially bounded nonuniform plasma, the dispersion is normally evaluated from Eq. (1) by assuming $k_\perp \sim a^{-1}$ where $a = \min\{R, L_n\}$ (R and L_n: the cylinder radius and the characteristic scale of density nonuniformity). For long waves, $k_z a << 1$, the total wave number $k \sim a^{-1}$, and then one obtains the scaling $n / B_0 \propto k_z$ (n: a radially averaged density).

In some experiments with the excitation of non-axisymmetric waves (with the aximuthal number $m \neq 0$; normally $m = +1$) [4,5], it was found that the long-wave scaling is $n / B_0 \propto k_z^2$, which is similar to the plane wave case. To explain this fact, it was assumed [3] that excited in these experiments can be radially localized helicon modes of special type which have off-axis localization near the surface of strongest radial density gradient [6]. In this paper we examine on-axis localized eigenmodes whose scaling is likely to explain the aforementioned experiments.

1. THE MODEL

We consider a radially nonuniform plasma cylinder of radius R immersed in a uniform axial magnetic field of strength B_0. The plasma density is assumed to decrease from the center to periphery. We describe the electromagnetic fields by Maxwell equation with a cold-plasma dielectric tensor. The eigenmode fields are represented as $\mathbf{F}(\mathbf{r}, t) = \mathbf{F}(r) \exp(-i\omega t + i k_z z + i m\theta)$ where k_z and $m = 0, \pm 1, \pm 2\ldots$ are the axial and azimuthal wave numbers. We shall examine the eigenmodes in the helicon approximation, i.e., assuming that the longitudinal electric field $E_z = 0$. This approximation is valid under two conditions. First, a plasma should be considerably nonuniform radially, with the density at the boundary with a confining vessel much lesser than the center density, so that the surface mode conversion is negligible. Second, the mode axial wave number should lie in the range

$$2(\omega / c)(\omega_{p0} / \omega_{ce}) < k_z < (\omega / c)\left(\omega_{p0} / \sqrt{\omega \omega_{ce}}\right), \qquad (2)$$

where ω_{p0} is the center plasma frequency. The right inequality in Eq. (2) signifies that the central part of the plasma column is transparent whereas the peripheral part is opaque for the helicon waves. These parts are separated by the helicon cut-off surface determined by equation

$$n(r) = (m_e / 4\pi e^2)(\omega_{ce} / \omega) k_z^2 c^2 . \qquad (3)$$

The left inequality in Eq. (2) signifies that the surface of the helicon wave coalescence with the quasi-electrostatic (Trivelpiece-Gould) wave is lacking in the plasma bulk, so that the process of bulk mode conversion of these waves is excluded [7]. If both aforementioned conditions are true, the helicon waves and the Trivelpiece-Gould waves are decoupled and the former can be described in the helicon approximation.

In this approximation one puts $E_z = 0$, and obtains from Maxwell equations the following equations for the E_θ and B_z fields

$$\frac{dE_\theta}{dr} + \left(1 + \frac{m \varepsilon_2}{N_z^2 - \varepsilon_1}\right)\frac{E_\theta}{r} = i k_v \left(1 + \frac{1}{k_v^2 r^2}\frac{m^2}{N_z^2 - \varepsilon_1}\right) B_z ;$$

$$\frac{dB_z}{dr} - \frac{m\varepsilon_2}{N_z^2 - \varepsilon_1}\frac{B_z}{r} = -i k_v \frac{V}{N_z^2 - \varepsilon_1} E_\theta , \qquad (4)$$

where

$$\varepsilon_1 = 1 - \frac{\omega_{pe}^2}{\omega^2 - \omega_{ce}^2} \text{ and } \varepsilon_2 = \frac{\omega_{pe}^2 \omega_{ce}}{\omega(\omega^2 - \omega_{ce}^2)}$$

are the dielectric tensor components, $N_z = k_z c / \omega$ is a longitudinal refractive index, $k_v = \omega / c$ is a vacuum wave number, and $V = (N_z^2 - \varepsilon_1)^2 - \varepsilon_2^2$. The rest of field components are determined from algebraic equations

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$$E_r = \frac{1}{N_z^2 - \varepsilon_1}\left(i\varepsilon_2 E_\theta + \frac{m}{k_v r} B_z \right);$$

$$B_r = -N_z E_\theta, \quad B_\theta = N_z E_r . \tag{5}$$

One can eliminate the B_z field from Eqs. (4) to obtain a second-order equation for the E_θ field

$$\frac{d}{dr}\left[\frac{1}{qr}\frac{d(rE_\theta)}{dr}\right] + \frac{d}{dr}\left(\frac{\varepsilon_2}{N_z^2 - \varepsilon_1}\frac{m}{qr^2}\right)E_\theta +$$

$$+\frac{k_z^2}{N_z^2 - \varepsilon_1}\left[\frac{V}{N_z^2} + \frac{\varepsilon_2^2}{q\left(N_z^2 - \varepsilon_1\right)}\frac{m^2}{k_z^2 r^2}\right]E_\theta = 0, \tag{6}$$

where

$$q = 1 + \frac{N_z^2}{N_z^2 - \varepsilon_1}\frac{m^2}{k_z^2 r^2}. \tag{7}$$

2. AXISYMMETRIC MODES

Examine first the helicon eigenmodes with the azimuthal wave number $m = 0$. In this case, the parameter $q = 1$, and Eq. (6) simplifies to the form

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d\left(rE_\theta\right)}{dr}\right] + k_z^2\frac{\varepsilon_2^2 - \left(N_z^2 - \varepsilon_1\right)^2}{N_z^2\left(N_z^2 - \varepsilon_1\right)}E_\theta = 0. \tag{8}$$

As long as the left inequality in Eq. (2) implies that $N_z^2 > 4\varepsilon_{10}$ where $\varepsilon_{10} = \varepsilon_1(r=0)$, with a reasonable accuracy one can neglect ε_1 in comparison to N_z^2 in Eq. (8).

Assuming that the density profile is parabolic

$$n(r) = n_0\left(1 - r^2/a^2\right), \tag{9}$$

where $a \ge R$, and that $\omega_{ce} >> \omega$, one can write

$$\varepsilon_2^2 \approx \varepsilon_{20}^2\left(1 - 2r^2/a^2\right), \tag{10}$$

where $\varepsilon_{20}^2 = \omega_{p0}^4/\omega^2\omega_{ce}^2$. Then Eq. (8) takes the form

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d\left(rE_\theta\right)}{dr}\right] + \left(b_0 - b_2 r^2\right)E_\theta = 0, \tag{11}$$

where

$$b_0 = k_z^2\left(\frac{\varepsilon_{20}^2}{N_z^4} - 1\right), \quad b_2 = k_z^2\frac{2\varepsilon_{20}^2}{N_z^4 a^2}. \tag{12}$$

Introducing a dimensionless variable

$$\varsigma = r/d, \quad d = b_2^{-1/4} \tag{13}$$

reduces Eq. (11) to the following

$$\frac{d}{d\varsigma}\left[\frac{1}{\varsigma}\frac{d\left(\varsigma E_\theta\right)}{d\varsigma}\right] + \left(\lambda - \varsigma^2\right)E_\theta = 0, \tag{14}$$

where

$$\lambda = d^2 b_0 = \frac{N_z^2 k_z a}{\sqrt{2}\varepsilon_{20}}\left(\frac{\varepsilon_{20}^2}{N_z^4} - 1\right) \tag{15}$$

is an eigenvalue to be determined.

Introducing a new function g

$$E_\theta = \frac{\exp\left(-\varsigma^2/2\right)}{\varsigma}g \tag{16}$$

and changing the independent variable, $\xi = \varsigma^2$, converts Eq. (14) to the following

$$\xi\frac{d^2 g}{d\xi^2} - \xi\frac{dg}{d\xi} + \frac{\lambda}{4}g = 0. \tag{17}$$

With the eigenvalue specified as

$$\frac{\lambda}{4} = p, \tag{18}$$

where $p = 1, 2, 3, ...$ is a natural number, solutions of Eq. (17) are the orthogonal Laguerre polynomials $L_p^{-1}(\xi)$. Taking the first polynomial, $L_1^{-1}(\xi) = \xi^2$, one obtains finally the E_θ field of first radial helicon eigenmode in the form

$$E_\theta^{(1)} = C_0 r\exp\left(-r^2/2d^2\right), \tag{19}$$

where C_0 is a constant. The value of d, Eq. (13), determines the width of the mode localization.

The eigenvalue equation for the first radial mode, $\lambda = 4$, can be resolved to give the following relationship

$$\omega_{p0}^2 = 2\sqrt{2}\frac{\omega_{ce}c^2}{\omega a^2}k_z a\left(1 + \sqrt{1 + (1/16)k_z^2 a^2}\right). \tag{20}$$

The consequence of this equation is the scaling law for the $m = 0$ modes:

for long waves ($k_z a << 1$)

$$n_0/B_0 \propto k_z a \tag{21}$$

and for short waves ($k_z a >> 1$)

$$n_0/B_0 \propto k_z^2 a^2. \tag{22}$$

3. NON-AXISYMMETRIC MODES

For the non-axisymmetric modes with azimuthal wave numbers $m \ne 0$, an analytic analysis can be performed for the axially long waves, $k_z a << 1$. In this case, the parameter q, Eq. (7), can be approximated as

$$q \approx m^2/k_z^2 r^2. \tag{23}$$

Substituting this into Eq. (6) and neglecting ε_1 in comparison with N_z^2, one obtains

$$rE_\theta'' + 3rE_\theta' + \left(\frac{m}{N_z^2}r\varepsilon_2' + 1 - m^2\right)E_\theta = 0, \tag{24}$$

where the prime denotes the derivative with respect to *r*.

Assuming again a parabolic density profile, as in Eq. (9), and changing the variable, $E_\theta = u/r$, one obtains the following equation

$$u'' + \frac{1}{r}u' + \left(\mu - \frac{m^2}{r^2}\right)u = 0, \tag{25}$$

where the eigenvalue to be determined

$$\mu = \frac{2m\omega_{p0}^2}{a^2 N_z^2\omega\omega_{ce}}. \tag{26}$$

Equation (25) is the *m*th-order Bessel equation which has physically reasonable solution only for the positive azimuthal numbers ($m > 0$), $u = J_m(\sqrt{\mu}\,r)$.

Finally, for the long waves $k_z a << 1$ with $m > 0$, the E_θ field takes the form

$$E_\theta^{(m)} = \frac{C_m}{r} J_m(\sqrt{\mu}\, r), \quad (27)$$

where C_m is a constant. Note that the right hand side of Eq. (27) is finite at $r \to 0$, as long as $E_\theta^{(m)} \propto r^m$ there.

The eigenvalue (26) is specified by the boundary condition. For instance, if the plasma confining vessel is conducting, $E_\theta^{(m)}(r = R) = 0$, one has to put $\sqrt{\mu} R = \beta_{mp}$ where β_{mp} is the *p*th root of the *m*th Bessel function. This gives the following relation for the *p*th radial mode

$$\frac{2m\omega_{p0}^2 R^2}{N_z^2 \omega\omega_{ce} a^2} = \beta_{mp}. \quad (28)$$

The relationship (28) determines the scaling for the long ($k_z a << 1$) helicon modes with the azimuthal wave numbers $m > 0$ in the form

$$n_0 / B_0 \propto k_z^2 a^2. \quad (29)$$

It is parabolic on k_z, contrary to the linear scaling for the long $m = 0$ modes, Eq. (21).

CONCLUSIONS

It was found that the scaling of cylindrical helicon eigenmodes in a radially nonuniform plasma depends on the azimuthal mode number *m*. For the axisymmetric modes ($m = 0$) the scaling is parabolic on k_z for the short modes, $n_0 / B_0 \propto k_z^2 a^2$ ($k_z a >> 1$), whereas it is linear for the long modes, $n_0 / B_0 \propto k_z a$ ($k_z a << 1$).

On the contrary, the scaling for the non-axisymmetric ($m > 0$) long modes is parabolic on k_z. The latter finding can explain the appropriate experimental results [4,5].

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ЗАКОНЫ ПОДОБИЯ ДЛЯ ГЕЛИКОННЫХ СОБСТВЕННЫХ МОД В НЕОДНОРОДНОМ ПЛАЗМЕННОМ ЦИЛИНДРЕ

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Аналитически исследованы волновые собственные моды радиально-неоднородного плазменного цилиндра в геликонном диапазоне частот. Указаны условия существования почти чистых геликонных мод, локализованных вблизи оси системы. Найден скейлинг (т.е. взаимоотношение между плотностью плазмы, магнитным полем и частотой, и волновым числом моды) для таких мод, и показано, что он зависит от азимутального волнового числа.

ЗАКОНИ ПОДІБНОСТІ ДЛЯ ГЕЛІКОННИХ ВЛАСНИХ МОД У НЕОДНОРІДНОМУ ПЛАЗМОВОМУ ЦИЛІНДРІ

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Аналітично досліджено хвильові власні моди радіально-неоднорідного плазмового циліндра в геліконному діапазоні частот. Виявлено умови існування майже чистих геліконних мод, локалізованих поблизу осі системи. Знайдено скейлінг (тобто співвідношення між густиною плазми, магнітним полем та частотою, і хвильовим числом моди) для таких мод, і показано, що він залежить від азимутального хвильового числа.