2D ISING-HEISENBERG MODEL WITH QUARTIC INTERACTION

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(Received October 30, 2011)

We propose a two-dimensional spin-1/2 Ising-Heisenberg model with quartic interaction. In this sense the phase diagram for the model at zero temperature is studied, obtaining four different configurations for the unitary cell. It should be observed that using the rotation and spin inversion symmetry we have only three spin configurations that are relevant. In addition, the model under consideration will be straightforwardly mapped to the exactly solved eight-vertex model and the conditions for obtaining an exact solution will be investigated. The analysis was performed for two different conditions, namely, where the *free fermion condition* (FFC) and *the symmetrical eight-vertex condition* (SEVC) are satisfied.

PACS: 75.10.Jm

1. INTRODUCTION

It is well known that quartic interactions affect magnetic properties of several copper compounds and all efforts for obtaining exact results of models containing quartic interaction in the Hamiltonian can be useful to shed light on some important aspects of these magnetic properties [1]. Several results for 2*d* Ising Heisenberg model connected with its exact solutions were obtained in the works [2,3]. Actually, the main idea of similar transformation is to establish an equivalent form to write down the original partition function by means of a new set of interaction parameters. A detailed study of decoration-iteration transformation and exactly solved models can be found in [4,5]. On the other hand, as was showed by Kun-Fa [6] it is possible to obtain a region where the model can be solved approximately. This was considered by Fan and Wu [7,8], where the eight-vertex model was exactly solved as well as with good approximation.

With this motivation we study a two dimensional Ising-Heisenberg model where the quartic interaction is assigned to the outer spin-1/2 sites. The model is composed of a two-dimensional lattice of edge-sharing unitary cells, where each one is composed of two triangular prism converging in a basal plane with four Ising spins-1/2 (empty circles), the apical positions are also occupied by four Heisenberg spin-1/2 (filled circles). Interaction of the base plane containing the quartic Ising interaction has the parameter J_4, and the other two site interactions have parameter J. Then, we construct a phase diagram for the model at zero temperature. In order to solve the model, we perform the summation over the inner sites on each unitary cell of the whole lattice. The best way to achieve it, is by fixing the set of spin values of the outer sites. We obtain a complete set of sixteen eigenvalues for the unitary cell, where some of which are degenerated. It should be observed that using the rotation and spin inversion symmetry only three spin configurations are relevant.

The work is organized as follows. In Section 2 we present explicitly the two-dimensional Hamiltonian of XXZ-Ising model with quartic interaction. Section 3 is devoted to the study of the phase diagrams and the different ground states. In Section 4 we perform a straightforward mapping of our model to the zero field eight-vertex model, so a detailed analysis of the exact solution is realized. Finally, in Section 5 some concluding remarks are given.

2. THE MODEL

We give the Hamiltonian for the unitary cell $\mathcal{H}_u$ of the Ising-Heisenberg spin-1/2 model. This unitary cell is represented in Fig. 1, where the dashed and solid lines represent the Ising and Heisenberg interactions respectively. The total Hamiltonian can be written as $\mathcal{H} = \sum_u \mathcal{H}_u$, where

$$\mathcal{H}_u = \sum_{<i,j>} \left(\Delta J \big(\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y \big) + J \sigma_i^z \sigma_j^z \right) + J_4 \sum_{(i,j)} s_i s_j \sigma_i \sigma_j, \qquad (1)$$

the first sum runs over the nearest neighbor site while the second one runs over the next-nearest neighbor site. In the above relation we assume that $J_x = J_y = \Delta J$ and $J_z = J$, where Δ measure a relative strength of the exchange anisotropy in the XXZ interaction.

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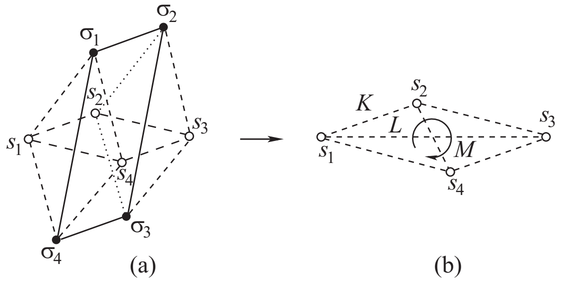

Fig. 1. *In (a) the unitary cell of Ising-Heisenberg spin-1/2 model with quartic interaction is given. The dashed lines indicate interactions of the Ising type while the solid lines are connected with the Heisenberg interaction. In (b) we depicted the mapping of the unitary cell to the effective two dimensional Ising square cell*

3. PHASE DIAGRAMS

In this section we construct the phase diagram for XXZ-Ising model with quartic interaction at zero temperature. At the beginning we perform the summation of Heisenberg σ_i sites on each unitary cell. The best way to achieve it, is by fixing the set of Ising spin values $\{s_1, s_2, s_3, s_4\}$, in this sense it is not difficult to obtain a complete set of sixteen eigenvalues for the unitary cell (depicted in fig.1b). Using the rotation and spin inversion symmetry, we conclude that only three spin configurations are relevant, therefore we analyze the spin configurations (i) $\{+,+,+,+\}$, (ii) $\{+,+,-,-\}$, and (iii) $\{+,+,+,-\}$.

3.1. Configuration $(+,+,+,+)$

For this case it is not tricky to obtain all sixteen energy eigenvalues. These eigenvalues are displayed in the first column of the table 1, while the second column indicates the degeneracy order of the corresponding eigenvalues. For the region with $J > 0$ and any value of the parameters J_4 and Δ, the energy $\varepsilon_{FI_1} = -2J - \sqrt{8\Delta^2 J^2 + (J - J_4)^2}$, is the lowest eigenvalue with the corresponding eigenvector

$$|FI_1\rangle = (1 + \mathsf{R}) \left| {}^{+}_{+}{\substack{-\,+\\+\,-}}{}^{+}_{+} \right\rangle + a^{(-)} \sum_{r=0}^{3} \mathsf{R}^r \left| {}^{+}_{+}{\substack{-\,+\\-\,+}}{}^{+}_{+} \right\rangle, \tag{2}$$

whith $a^{(-)}$ given by

$$a^{(\mp)} = \frac{1}{4\Delta J}\Big(J - J_4 - \sqrt{8\Delta^2 J^2 + (J \mp J_4)^2}\Big), \tag{3}$$

in (2) with the largest (+) signal (inner signs) we represent sites with spin σ. The magnetization of the unitary cell is neither null, nor saturated and corresponds to the ferrimagnetic state with magnetization 1/4, we represent this state as FI_1. By R we represent the rotation operator acting only on Heisenberg interaction particles with spin σ, each rotation is performed in $\frac{\pi}{2}$, around the axis perpendicular to the plane of lattice.

On the other hand in the region with $J < 0$, we observe a ground state dependence of the parameter Δ, in this case for large values of $\Delta \gg 1$ only the FI_1 state (2) is present, while for small values of $\Delta < 1$ we have additionally two other states, a ferromagnetic (FM_1) and antiferromagnetic (AF_1) states which we called type I, these states are degenerated and have the same eigenvalue, $\varepsilon_{FM_1} = \varepsilon_{AF_1} = 4J + 2J_4$. The corresponding eigenvectors are given by

$$|FM_1\rangle = \left| {}^{+}_{+}{\substack{+\,+\\+\,+}}{}^{+}_{+} \right\rangle, \tag{4}$$

$$|AF_1\rangle = \left| {}^{+}_{+}{\substack{-\,-\\-\,-}}{}^{+}_{+} \right\rangle, \tag{5}$$

these two states have magnetization equals to 1/2 for the FM_1 and 0 for the AF_1 state. In the Fig. 2 we depict the dependence of the different ground states on the parameter Δ.

Table 1. *The energy levels for configuration* $\{+,+,+,+\}$

Energy $\{+,+,+,+\}$	Degeneracy
$-2J \pm \sqrt{8\Delta^2 J^2 + (J - J_4)^2}$	1
$\pm 4\Delta J$	2
$4J + 2J_4$	2
$-4J + 2J_4$	1
$-2J_4$	3
0	4

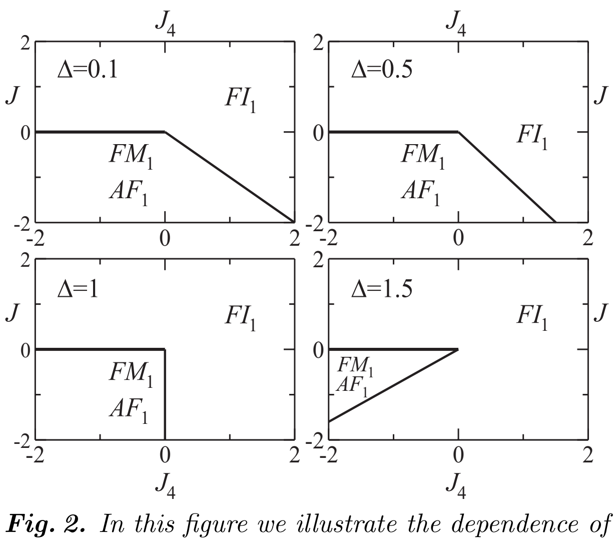

Fig. 2. *In this figure we illustrate the dependence of the ground state on the parameter* Δ*. It is clear that for large values of relative strength of the exchange anisotropy (*$\Delta \gg 1$*), only the ferrimagnetic state of type I is present*

3.2. Configuration $(+,+,-,-)$

As before we found sixteen eigenvalues listed in Table 2. Firstly, we have that for positive values of $J > 0$ and any value of the parameters J_4 and Δ, the antiferromagnetic state of type II (AF_2) with energy, $\varepsilon_{AF_2} = -2J - \sqrt{8\Delta^2 J^2 + (J + J_4)^2}$, becomes the ground state energy

$$|AF_2\rangle = (1+\mathsf{R})\left|\begin{smallmatrix} - & - & + & - \\ + & + & - & + \end{smallmatrix}\right\rangle + a^{(+)}\sum_{r=0}^{3}\mathsf{R}^r\left|\begin{smallmatrix} - & - & - & - \\ + & + & + & + \end{smallmatrix}\right\rangle, \quad (6)$$

with $a^{(+)}$ given by eq.(3). In the other region for negative values of $J < 0$ and large values of the parameter $\Delta \gg 1$ only the AF_2 given by the eq.(6) is present, while for small values of the parameter Δ, we have a new ferrimagnetic state of type II (FI_2). This state has the energy value, $\varepsilon_{FI_2} = 4J - 2J_4$,

$$|FI_2\rangle = \left|\begin{smallmatrix} - & + & + & - \\ + & + & + & + \end{smallmatrix}\right\rangle, \quad (7)$$

this is a two degenerated state with magnetization equal to 1/4. The other eigenvector state, with the same energy, is equivalent to (7) and it is obtained by applying the spin inversion operator to whole unitary cell, magnetization of this state is equal to $-1/4$. All ground states are represented in Fig. 3.

Table 2. *The energy levels for configuration* $\{+,+,-,-\}$

Energy $\{+,+,-,-\}$	Degeneracy
$-2J \pm \sqrt{8\Delta^2 J^2 + (J+J_4)^2}$	1
$\pm 4\Delta J$	2
$4J - 2J_4$	2
$-4J - 2J_4$	1
$2J_4$	3
0	4

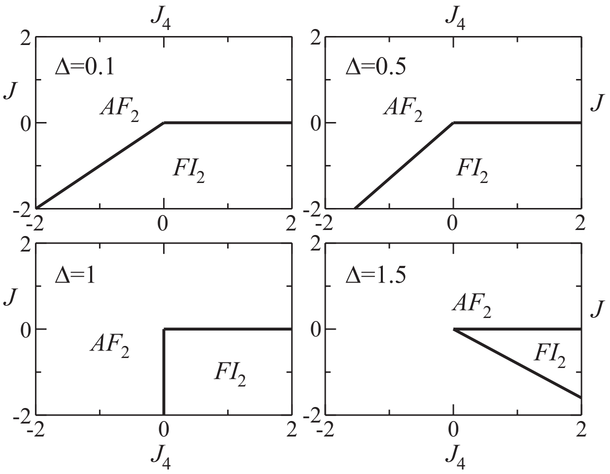

Fig. 3. *The ground states are depicted as a function of the parameter Δ. For large values of $\Delta \gg 1$ only the ground state AF_2 is maintained and for small values of Δ, two degerated state FI_2 with magnetization equal to 1/4 appears*

3.3. Configuration $(+,+,+,-)$

For the last configuration and after some manipulations we found all sixteen eigenvalues listed in Table 3. In this case we have different situations depending on values of the anisotropy parameter Δ. For example, for value of the parameter $\Delta = 1$, we see that the ground state energies are given by an antiferromagnetic state ($AF^{(+)}$), with energy $\varepsilon_{AF^{(+)}} = -2J(1+\sqrt{1+8\Delta^2})$, a ferrimagnetic state of type III (FI_3) and a antiferromagnetic state of type III (AF_3), these states are two-degenerated and have energies $\varepsilon_{FI_3} = \varepsilon_{AF_3} = -2\sqrt{J_4^2 + 4\Delta^2 J^2}$. This situation is illustrated in Fig 4b. The corresponding eigenvectors in this case are given by

$$|AF^{(\pm)}\rangle = (1+\mathsf{R})\left|\begin{smallmatrix} - & - & + & + \\ + & + & - & + \end{smallmatrix}\right\rangle + c^{(\pm)}\sum_{r=0}^{3}\mathsf{R}^r\left|\begin{smallmatrix} - & - & - & + \\ + & + & + & + \end{smallmatrix}\right\rangle, \quad (8)$$

$$|FI_3\rangle = (1+c\mathsf{R})(1+\mathsf{R}^2)\left|\begin{smallmatrix} - & - & + & + \\ + & + & + & + \end{smallmatrix}\right\rangle, \quad (9)$$

$$|AF_3\rangle = (1+c\mathsf{R})(1+\mathsf{R}^2)\left|\begin{smallmatrix} - & + & - & + \\ + & - & - & + \end{smallmatrix}\right\rangle, \quad (10)$$

where $c^{(\pm)}$ and c are equal to

$$c^{(\pm)} = -\frac{1}{4\Delta}(1 \pm \sqrt{1+8\Delta^2}), \quad (11)$$

$$c = -\frac{1}{2\Delta J}(J_4 + \sqrt{J_4^2 + 4\Delta^2 J^2}). \quad (12)$$

For small values of the anisotropy parameter, as for example, $\Delta = 0.5$, a new ferrimagnetic state ($FI^{\pm}$) appears as depicted in Fig. 4a. The energy of this state is two degenerated and equal to $\varepsilon_{FI^{(\pm)}} = 4J$ with the corresponding eigenvector

$$|FI^{(\pm)}\rangle = \left|\begin{smallmatrix} - & \pm & \pm & + \\ + & \pm & \pm & + \end{smallmatrix}\right\rangle. \quad (13)$$

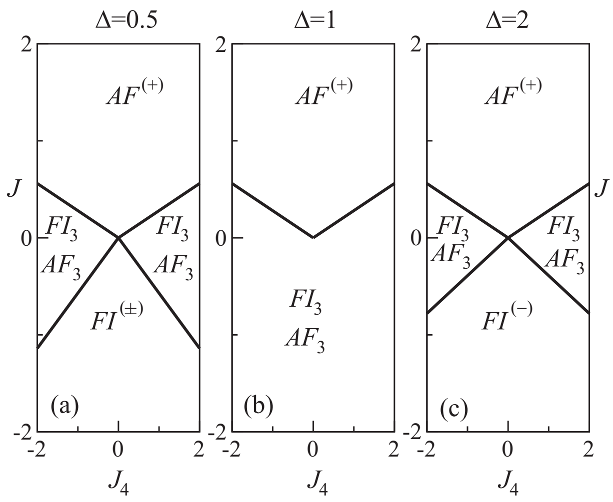

Fig. 4. *The ground state energies for different values of the parameter Δ. In (b) for $\Delta = 1$ we have three ground states $AF^{(+)}$, FI_3 and AF_3. For values of the parameter $\Delta < 1$ or $\Delta > 1$, three new ground states appear, the $FI^{(\pm)}$ and the $AF^{(-)}$ respectively. This is illustrated in figure (a) and (c)*

Table 3. *The energy levels for configuration* $\{+,+,+,-\}$

Energy $\{+,+,+,-\}$	Degeneracy
$\pm 2\sqrt{J_4^2 + 4\Delta^2 J^2}$	2
$-2J(1 \pm \sqrt{1+8\Delta^2})$	1
$\pm 2J_4$	2
$-4J$	1
$4J$	2
0	3

For values of $\Delta > 1$, as for example $\Delta = 2$, another antiferromagnetic state $AF^{(-)}$ appears, this ground state has the eigenvalue, $\varepsilon_{AF^{(-)}} = -2J(1 + \sqrt{1-8\Delta^2})$, with the eigenvector given by the relation (8). This is depicted in fig.4c

4. EQUIVALENCE TO THE ZERO-FIELD EIGHT-VERTEX MODEL

Now we proceed to study the conditions necessary to obtain an exact solution. The best way to achieve it, is by performing a straightforward mapping to the exactly solved eight-vertex model. This procedure was already discussed in several works [2–4] where a decoration transformation was applied to different Ising-Heisenberg models. The main goal of similar transformation is to establish an equivalent form to write down the original partition function by means of a new interaction parameters set.

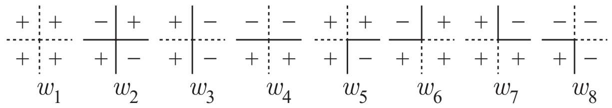

Fig.5. *We Illustrate the eight different spin arrangements corresponding to different Boltzmann weights. Inversion of all spins coresponds to the same vertex.*

We begin writing the partition function as,

$$\mathcal{Z} = \sum_{\{s\}} \prod_{u=1}^{N} w\left(\{s\}\right), \tag{14}$$

where N is the number of unitary square cells in the whole lattice and $w(\{s\})$ are the Boltzmann weights assigned to the u-th unitary cell as a function of the spin s, they are given by

$$w(\{s\}) \equiv \mathrm{Tr}_{\{\sigma\}}\left(e^{-\beta\mathcal{H}_u}\right), \tag{15}$$

here $\mathcal{H}_u$ is the Hamiltonian of the unitary cell and is given by the relation (1). By $\beta = 1/kT$, we denote the inverse of temperature, k is the Boltzmann constant and the $\mathrm{Tr}_{\{\sigma\}}$ indicates the trace on the spin-1/2 sites inside the unitary square cell. As showed in the works [2,4] it can be established a complete equivalence between the partition function of the original Ising-Heisenberg model and the partition function of the eight-vertex Ising model on square lattice. This transformation is illustrated in fig.1b. In this way we introduce the effective Boltzmann weight $\widetilde{w}$

$$\widetilde{w}\left(\{s\}\right) = f e^{-\beta\widetilde{\mathcal{H}}_u}, \tag{16}$$

here f is a new constant and $\widetilde{\mathcal{H}}_u$ is the new effective Hamiltonian. The eight different spin arrangements of these Boltzmann weights are schematically depicted in the fig.5. Next, we give the effective Hamiltonian $\widetilde{\mathcal{H}}_u$ of the unitary cell

$$\widetilde{\mathcal{H}}_u = K \sum_{(k,k')} s_k s_{k'} + L \sum_{<k,k'>} s_k s_{k'} + M s_1 s_2 s_3 s_4,$$
$$\widetilde{\mathcal{H}} = \sum_{\text{all square}} \widetilde{\mathcal{H}}_u, \tag{17}$$

here K, L and M represent a new set of interaction parameters and $\widetilde{\mathcal{H}}$ is the total effective Hamiltonian. The first sum in (17) with $k, k' = 1..4$, runs over the nearest neighbor spin-1/2 Ising site of the effective unitary cell, while the second one runs over the next-nearest neighbor spin-1/2 Ising site. The main idea to establishing a complete equivalence between these both models, is the fact that the Boltzmann weights contained in the expression (15) and the effective Boltzmann weight given by (16) are equivalent. We write the effective partition function as

$$\widetilde{\mathcal{Z}} = f^N \mathcal{Z}_0, \tag{18}$$

with N being the number of square cells in the whole lattice. In the above relation $\mathcal{Z}_0$ is the partition function for spin-1/2 of the eight-vertex model. After some manipulations we find the following values for the interaction parameters

$$f = \left(w_1 w_3 w_5^2\right)^{1/4}, \tag{19}$$

$$\beta L = \ln\left(\frac{w_3}{w_1}\right)^{1/4}, \tag{20}$$

$$\beta M = \ln\left(\frac{w_5^2}{w_1 w_3}\right)^{1/4}, \tag{21}$$

$$K = 0, \tag{22}$$

the Boltzman weigths defined by (15) take the form

$$w_1 = 2e^{2\beta J}\mathrm{ch}\left(2\beta\sqrt{8\Delta^2 J^2 + (J - J_4)^2}\right) + e^{-2\beta J_4}\left(e^{4\beta J} + 2e^{-4\beta J}\right) + 3e^{2\beta J_4} + 4\mathrm{ch}(4\beta\Delta J) + 4, \tag{23}$$

$$w_3 = 2e^{2\beta J}\mathrm{ch}\left(2\beta\sqrt{8\Delta^2 J^2 + (J + J_4)^2}\right) + e^{2\beta J_4}\left(e^{4\beta J} + 2e^{-4\beta J}\right) + 3e^{-2\beta J_4} + 4\mathrm{ch}(4\beta\Delta J) + 4, \tag{24}$$

$$w_5 = 4\mathrm{ch}\left(2\beta\sqrt{J_4^2 + 4\Delta^2 J^2}\right) + 2e^{2\beta J}\mathrm{ch}\left(2\beta J\sqrt{1 + 8\Delta^2}\right) + 4\mathrm{ch}(2\beta J_4) + e^{4\beta J} + 2e^{-4\beta J} + 3, \tag{25}$$

the other Boltzmann weights are obtained taking into account the following identities

$$w_1 = w_2, \quad w_3 = w_4, \quad w_5 = w_6 = w_7 = w_8. \tag{26}$$

It directly follows from the relations (23)-(25) that the greatest Boltzmann weight is given by w_1 or w_3. It is also possible to observe the symmetry of the Boltzmann weights w_1 and w_3 in relation to the parameter J_4, namely, $w_1(\pm J_4) = w_3(\mp J_4)$, further, because w_5 is an even function in J_4 we can assume in the next lines a possitive value of $J_4 > 0$ without lossing generality.

4.1. Free Fermion Condition (FFC)

An extensive study of the exactly solvable model is given in [5]. On the other hand, recently, two dimensional Ising-Heisenberg model with quartic interaction was solved mapping into the zero field eight-vertex model [1]. In our case, the model defined by the Hamiltonian (1) is mapped into the eight-vertex model, verifying the conditions where this mapping is successfully done. In the following lines we discuss in details the conditions for obtaining an exact solvable model.

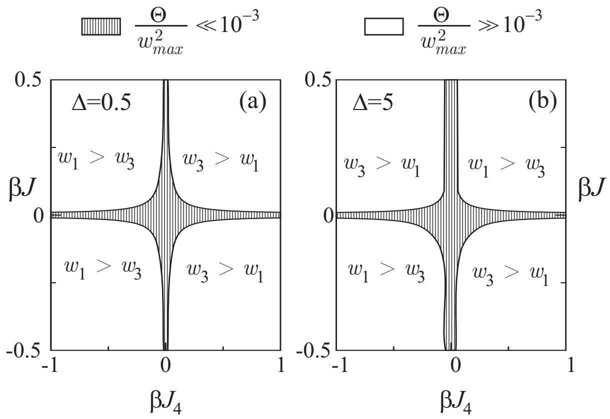

Fig.6. *In (a) and (b) we depict for* $\Delta = 0.5$ *and* $\Delta = 5$*, respectively, the region where the FFC is satisfied approximately, the shadow region satisfies the condition* $\Theta/w^2_{max} \ll 0.001$ *whereas the white region is for the case* $\Theta/w^2_{max} \gg 0.001$.

We start discussing which conditions are necessary to obtain an exactly solved model. The first one is the FFC

$$\Theta = w_1 w_2 + w_3 w_4 - w_5 w_6 - w_7 w_8, \qquad (27)$$

this condition implies $\Theta = 0$. Unfortunately, when imposing the FFC we cannot find an exact solution. However, as it was pointed out by Fun and Tang, it is possible to obtain a region where the model can be solved approximately, this happens when the condition $\Theta/w^2_{max} \ll 1$ takes place, this procedure is detailed in the works [6, 7]. Actually, for any values of Δ in the range $\Delta \leq 1$, we have that for $J_4 < 0$ the maximum values of the Boltzmann weight is w_1 and for $J_4 > 0$ the maximum value becomes w_3. To illustrate this fact we fixed the value $\Delta = 0.5$ and depict the region where $\Theta/w^2_{max} \ll 0.001$, this is showed by the shadow region in the fig.6a. The solution is more exactly if the parameters J, J_4 converge to the point $(0, 0)$ and becomes exact for the trivial case $J = 0$, $J_4 = 0$. For large values of Δ, in the range $\Delta > 1$, the maximum value, as depicted in the fig.6b, are rearranged. For the sake of comparison we point out that only few changes of the shadown region are observed.

4.2. Symmetric Eight-Vertex Condition (SEVC)

The second branch where the model can be solved exactly is the SEVC given by

$$w_1 = w_2, \quad w_3 = w_4, \quad w_5 = w_6 \qquad w_7 = w_8, \quad (28)$$

it is not difficult to see from the relations (26) that this condition is fully satisfied for any values of the interaction parameters J, J_4 and any values of the anysotropy parameter Δ.

5. CONCLUSIONS

In this work we studied a two dimensional XXZ-Ising model with quartic interaction. We discussed the ground state energy and plotted the phase diagrams at zero temperature as a function of the strenght parameter Δ. We observed that only three spin configurations are relevant, in this way we analyzed the configurations $(+,+,+,+)$,$(+,+,-,-)$ and $(+,+,+,-)$ obtaining different ground states. Then, we performed a straightforward mapping to the eigth-vertex model and explored the conditions under which the model is exactly solved. The Boltzmann weights were calculated and the FFC and the SEVC were discussed. First, we verified that the FFC is not satisfied exactly, however it was possible to give the region where this condition is satisfied approximately. Secondly, we found that the SEVC is satisfied in unrestricted manner.

Acknowledgments

J.S.V. thanks FAPERGS for full financial support.

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ДВУМЕРНАЯ МОДЕЛЬ ISING-HEISENBERG СО СПИНОМ 1/2 И С ЧЕТВЕРНЫМ ВЗАИМОДЕЙСТВИЕМ

Х.С. Вальверде

Предлагается двумерная модель Ising-Heisenberg со спином 1/2 и с четверным взаимодействием. Анализируется фазовая диаграмма для модели при нулевой температуре, и получены четыре различных конфигурации для унитарной ячейки. Отмечается, что при учете симметрии вращения и инверсии спина имеются только три конфигурации спина, имеющие смысл. Кроме того, рассматриваемая модель прямо отображается на точно решаемую модель с восемью вершинами, и условия получения точного решения исследованы. Анализ был сделан для двух различных условий, а именно, для *случая свободных фермионов* (FFC) и *случая восьми симметричных вершин* (SEVC).

ДВОВИМІРНА МОДЕЛЬ ISING-HEISENBERG ЗІ СПІНОМ 1/2 І ІЗ ЧЕТВЕРНОЮ ВЗАЄМОДІЄЮ

Х.С. Вальверде

Пропонується двовимірна модель Ising-Heisenberg зі спіном 1/2 і із четверною взаємодією. Аналізується фазова діаграма для моделі при нульовій температурі, й отримані чотири різних конфігурації для унітарної комірки. Відзначається, що при обліку симетрії обертання й інверсії спіна є тільки три конфігурації спіна, які мають сенс. Крім того, розглянута модель прямо відображається на точно розв'язувану модель із вісьма вершинами. Умови одержання точного рішення досліджені. Аналіз був зроблений для двох різних умов, а саме, для *случаю вільних ферміонів* (FFC) і *случаю восьми симетричних вершин* (SEVC).