ADVANCED BOGOLYUBOV MODEL OF IMPERFECT BOSE GAS AND SUPERFLUIDITY

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The equilibrium properties of a system of interacting bosons are studied from a microscopic point of view. We calculate the superfluid density in the Bogolyubov model of imperfect Bose gas. The model superstable Hamiltonian is considered. We examine the case of some pair potential and find the estimate for temperature and density in the λ-point.

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1. INTRODUCTION

It took 30 years after the liquefaction of ^{4}He in 1908 to make the discovery that liquid helium was not just a "cold" liquid. Below 2.18 K, it is a "quantum" liquid which exhibits spectacular macroscopic quantum behavior that can be seen with our eyes. The first evidence for superfluidity was, in fact, a seminal experiment in 1935 by Misener. He found that the viscosity decreased sharply as one went just below 2.18 K, although it was still finite. This was the first evidence that He II was a new kind of liquid (this work was published under the name of E.F. Barton, the head of the Toronto lab, causing some later confusion [1]).

The superfluid behavior is commonly associated with the phenomenon of a Bose-Einstein condensation. However, the detailed theoretical investigation of superfluidity and condensation in liquid helium is a great challenge, as it is a strongly interacting quantum system and cannot be effectively described with a mean-field or perturbative approaches.

Experimental achievement of Bose-Einstein condensation in dilute trapped atomic gases gives a unique possibility to study their superfluid properties under condition of weak interactions. It has opened an opportunity for revisiting the concept of superfluidity. In this context the issue of the atomic Bose-Einstein condensation under rotation has attracted great interest. On the other hand, a little is known about superfluidity in a translational motion of the atomic condensate. There are still no experiments that would correspond to the demonstration of frictionless non-rotary flow, which is the most intuitive manifestation of this phenomenon.

In Section 2 we analyze the superfluid properties of Bogolyubov quasiparticles. We derive analytically superfluid density of the system.

In Section 3 we present some generalization of a standard Bogolyubov model. This model is exactly solvable in thermodynamic limit. The estimates for temperature and total density in the λ-point are found.

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2. BOGOLYUBOV MODEL. SUPERFLUIDITY

Let us consider a system of N spinless identical nonrelativistic bosons of mass m enclosed in a centered cubic box $\Lambda \subset \mathbb{R}^3$ of volume $V = |\Lambda| = L^3$ with periodic boundary conditions for the wave functions. The Hamiltonian of the system can be written in the second quantized form as:

$$\begin{aligned} \hat{H}_\Lambda(\mu) \quad &\equiv \hat{H}_\Lambda - \mu \hat{N}_\Lambda = \textstyle\sum_{k\in\Lambda^*}(\epsilon_k - \mu)\hat{a}_k^\dagger \hat{a}_k \\ &+ \frac{1}{2V}\sum_{p,q,k\in\Lambda^*} \nu(k)\hat{a}_p^\dagger \hat{a}_q^\dagger \hat{a}_{p+k}\hat{a}_{q-k}. \end{aligned} \qquad (1)$$

Here $\hat{a}_p^\# = \{\hat{a}_p^\dagger \text{ or } \hat{a}_p\}$ are the usual boson creation (annihilation) operators for the one-particle state $\psi_p(x) = V^{-1/2}\exp(ipx), p \in \Lambda^*, x \in \Lambda$, acting on the Fock space $F_\Lambda = \oplus_{n=0}^\infty \mathcal{H}_{\rm B}^{(n)}$, where $\mathcal{H}_{\rm B}^{(n)} \equiv [L^2(\Lambda^n)]_{\rm symm}$ is the symmetrized n-particle Hilbert space appropriate for bosons, and $\mathcal{H}_{\rm B}^{(0)} = \mathbb{C}$. The sums in (1) run over the dual set:

$$\Lambda^* = \{p \in \mathbb{R}^3 : p_\alpha = \frac{2\pi}{L}n_\alpha,$$
$$n_\alpha = 0, \pm1, \pm2, \ldots, \alpha = 1,2,3\},$$

$\epsilon_p = |p|^2/(2m)$ is the one-particle energy spectrum of free bosons in the modes $p \in \Lambda^*$ (we propose $\hbar = 1$), $\hat{N}_\Lambda = \sum_{k\in\Lambda^*}\hat{a}_k^\dagger \hat{a}_k$ is the total particle-number operator, μ is the chemical potential, $\nu(k)$ is the Fourier transform of the interaction pair potential $\Phi(x)$. We suppose that $\Phi(x) = \Phi(|x|) \in L^1(\mathbb{R}^3)$ and $\nu(k)$ is a real function with a compact support such that $0 \le \nu(k) = \nu(-k) \le \nu(0)$ for all $k \in \mathbb{R}^3$. Under these conditions the Hamiltonian (1) is superstable.

So long as the rigorous analysis of the Hamiltonian (1) is very knotty problem, Bogolyubov introduced the model Hamiltonian of the superfluidity theory [2, 3]. He proposed to disregard the terms of the third and fourth order in operators $\hat{a}_p^{\#}, p \neq 0$ in the Hamiltonian (1):

$$\hat{H}_\Lambda^{\mathrm{B}}(\mu) = \sum_{k\in\Lambda^*}(\epsilon_k - \mu)\hat{a}_k^\dagger \hat{a}_k + \frac{1}{V}\hat{a}_0^\dagger \hat{a}_0 \sum_{k\neq 0}\nu(k)\hat{a}_k^\dagger \hat{a}_k + \frac{1}{2V}\sum_{k\neq 0}\nu(k)(\hat{a}_k^\dagger \hat{a}_{-k}^\dagger \hat{a}_0\hat{a}_0 + \hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_{-k}\hat{a}_k) + \frac{\nu(0)}{V}\hat{a}_0^\dagger \hat{a}_0 \sum_{k\neq 0}\hat{a}_k^\dagger \hat{a}_k + \frac{\nu(0)}{2V}\hat{a}_0^\dagger \hat{a}_0^\dagger \hat{a}_0 \hat{a}_0. \quad (2)$$

Then Bogolyubov takes advantage of the macroscopic occupation of the zero momentum one-particle state to replace the corresponding creation and annihilation operators $\hat{a}_0^{\#}$ by c-numbers:

$$\frac{\hat{a}_0^\dagger}{\sqrt{V}} \to \bar{c}, \quad \frac{\hat{a}_0}{\sqrt{V}} \to c, \quad (3)$$

where $c \in \mathbb{C}$ and the bar means complex conjugation.

The validity of substituting a c-number for the $k = 0$ mode operators $a_0^{\#}$ was established rigorously for the Bogolyubov Hamiltonian (2) in papers [4, 5]. So one can prove that the model Hamiltonian $\hat{H}_\Lambda^{\mathrm{B}}(\mu)$ is thermodynamically equivalent to the approximate Hamiltonian:

$$\hat{H}_\Lambda^{\mathrm{B}}(\mu, c) = \sum_{k\neq 0}[\epsilon_k - \mu + |c|^2(\nu(0) + \nu(k))]\hat{a}_k^\dagger \hat{a}_k + \frac{1}{2}\sum_{k\neq 0}\nu(k)(c^2 \hat{a}_k^\dagger \hat{a}_{-k}^\dagger + \bar{c}^2 \hat{a}_k \hat{a}_{-k}) + \frac{1}{2}\nu(0)|c|^4 V - \mu|c|^2 V. \quad (4)$$

The self-consistency parameter c in the method is determined by the condition that the approximate pressure $p[\hat{H}_\Lambda^{\mathrm{B}}(\mu, c)]$ be maximal. In the same time, the stability condition $\mu \leq \nu(0)|c|^2$ must be fulfilled. A necessary condition for $p[\hat{H}_\Lambda^{\mathrm{B}}(\mu, c)]$ to be maximum (self-consistency equation) in the case of the Bogolyubov model is

$$\left\langle \frac{\partial \hat{H}_\Lambda^{\mathrm{B}}(\mu, c)}{\partial c} \right\rangle_{\hat{H}_\Lambda^{\mathrm{B}}(\mu,c)} = 0. \quad (5)$$

This equation always has the trivial solution $c = 0$ (no Bose condensation). By explicit calculations we get the following equation to obtain a nontrivial solution:

$$\mu - x\nu(0) = \frac{1}{2V}\sum_{k\neq 0}\left[-\nu(k)\frac{h_k}{E_k}\coth\frac{\beta E_k}{2} + (\nu(0) + \nu(k))\left(\frac{f_k}{E_k}\coth\frac{\beta E_k}{2} - 1\right)\right], \quad (6)$$

where

$$u_k = \sqrt{\tfrac{1}{2}\left(\tfrac{f_k}{E_k} + 1\right)}, \quad v_k = -\sqrt{\tfrac{1}{2}\left(\tfrac{f_k}{E_k} - 1\right)},$$
$$f_k = \epsilon_k - \mu + x(\nu(0) + \nu(k)),$$
$$h_k = x\nu(k), \quad E_k = \sqrt{f_k^2 - h_k^2}$$

and we denote $x \equiv |c|^2$.

We shall use the function

$$\nu(k) = \begin{cases} \nu(0) & \text{for } |k| \leq k_0, \\ 0 & \text{for } |k| > k_0 \end{cases} \quad (7)$$

as the Fourier transform of the pair potential. Here $\nu(0) = 4\pi r_0/m,\ k_0 r_0 = 1,\ r_0 = 2.56\,\text{Å},\ \ m = m_{\mathrm{He}^4}$.

In this case it was shown [4–6], that if the potential $\nu(k)$ in the Bogolyubov model of superfluidity (2) satisfies the condition

$$\nu(0) \geq \frac{1}{2V}\sum_{k\neq 0}\frac{\nu^2(k)}{\epsilon_k}, \quad (8)$$

then there exists the domain of stability on the phase diagram $\{0 < \mu \leq \mu^*, 0 \leq \theta \leq \theta_0(\mu)\}$, where the nontrivial solution of the self-consistency equation takes place. In this domain there is the non-zero Bose condensate. At the boundary $\theta = \theta_0(\mu)$ of this domain the Bose condensate density equals $\rho_0 = \mu/\nu(0)$. In this case the quasi-particles spectrum of the Bogolyubov Hamiltonian (2)

$$E_k = \sqrt{\epsilon_k(\epsilon_k + 2\rho_0\nu(k))}$$

has a gapless type and the famous Landau's criterion of superfluidity

$$\min_k \frac{E_k}{|k|} > 0$$

holds.

We shall now study the fluid motion of our system at a uniform, constant velocity. This will provide a means for defining normal and superfluid components. The normal density may be defined by the effective mass for drift, as was originally asserted by Landau [7]. The mass density of the normal fluid is found from

$$\rho_{\mathrm{n}} = -\frac{1}{3V}\sum_k k^2\frac{\partial n_k}{\partial \epsilon_k} = \frac{\beta}{3V}\sum_k k^2 n_k(1 + n_k). \quad (9)$$

The total mass density $\rho = mn = \rho_{\mathrm{n}} + \rho_{\mathrm{s}}$, where ρ_{s} is the density of superfluid. Of course, ρ_{s} is not to be interpreted as the density of the particles in the zero-momentum state $\rho_0 = mn_0$.

On evaluation of equation (9), we find $(\rho - \rho_{\mathrm{s}})/\rho \approx 0.1$, $(\rho - \rho_0)/\rho \approx 0.1, \rho_{\mathrm{s}} > \rho_0$ for $\mu = \mu^*$ and $\theta = \theta_0(\mu^*)$. For the model pair potential (7) one can find $\rho \approx 0.02\,\mathrm{g/cm^3}$ and $\theta_0 \approx 0.18\,\mathrm{K}$. (For real He^4 $\theta_0 \approx 2.18\,\mathrm{K}$ and $\rho \approx 0.13\,\mathrm{g/cm^3}$.The Bogolyubov Hamiltonian H^{Bog} does not take into account interactions between the excitations. So the physical results based on the Hamiltonian H^{Bog} are far from "real superfluidity".

3. BOGOLYUBOV MODEL. λ-POINT

We attempt to take into account the terms of fourth order in operators $a_k^{\#}, k \neq 0$ in the full Hamiltonian (1). Consider the model superstable Hamiltonian

$$\hat{H}_\Lambda^{\mathrm{adv}}(\mu) = \hat{H}_\Lambda^{\mathrm{B}}(\mu) + \frac{\nu(0)}{2V}\hat{\tilde{N}}^2, \quad (10)$$

where $\hat{\tilde{N}} = \sum_{k\neq 0} \hat{a}_k^\dagger \hat{a}_k$. It is possible to prove that this Hamiltonian is equivalent to the approximate Hamiltonian

$$\hat{H}_\Lambda^{\rm adv}(\mu, x) = \hat{H}_\Lambda^{\rm B}(\mu) + \nu(0)x\hat{\tilde{N}} + \frac{\nu(0)}{2}x^2 V, \quad (11)$$

where the self-consistency equations are

$$x = \frac{1}{2V}\sum_{k\neq 0}\left(\frac{f_k}{E_k}\coth\frac{\beta E_k}{2} - 1\right), \quad (12)$$

$$\mu - \nu(0)(n_0 + x) = \frac{1}{2V}\sum_{k\neq 0}\nu(k)\left(\frac{f_k - h_k}{E_k}\coth\frac{\beta E_k}{2} - 1\right). \quad (13)$$

Here

$$f_k = \epsilon_k - \mu + \nu(0)(n_0 + x) + \nu(k)n_0, \quad (14)$$

$$h_k = \nu(k)n_0,\ E_k^2 = f_k^2 - h_k^2. \quad (15)$$

When $n_0 = 0$ the equation (12) reads

$$x = \frac{1}{V}\sum_{k\neq 0}\frac{1}{\exp\left[\beta(\epsilon_k - \mu + \nu(0)x)\right] - 1}. \quad (16)$$

This equation has the unique positive solution for $\mu \leq \mu_{\rm c} = \nu(0)\rho_{\rm c}^{(0)}$, where

$$\rho_{\rm c}^{(0)} = \frac{1}{V}\sum_{k\neq 0}\frac{1}{\exp(\beta\epsilon_k) - 1}. \quad (17)$$

Then, the condensate density $n_0 \neq 0$ for $\mu > \mu_{\rm c}$. So, for $n_0 \neq 0$ we must solve the pair of self-consistency equations (12)-(13). For $\mu = \nu(0)n$ the Landau's criterion of superfluidity holds. In this case the equation (9) and the condition $\rho = \rho_n$ give an estimate for the λ-point $\theta_{\rm s} \approx 1.9\,{\rm K}$ for $\rho \approx 0.08\,{\rm g/cm^3}$. This estimate is in sufficiently good-enough agreement with experimental data. As is easy to see this method does not describe a case of small density. Yet it permits to investigate the neighborhood of the λ-point.

4. CONCLUSIONS

We have investigated the superfluid properties of a Bogolyubov's weakly interacting Bose gas at thermal equilibrium. Using the conventional definition of the normal fraction, we find that the gas has a significant superfluid fraction only in the Bose condensed regime. However, it is impossible to describe the λ-point region by a standard Bogolyubov model. To investigate more carefully the λ-point regime where the superfluid density tends to zero, we examined the advanced Bogolyubov model. In this model we take into account the terms of fourth order in operators $\hat{a}_k^\#, k \neq 0$ in the full Hamiltonian of a non-ideal Bose-gas. Quantitatively, we have found that the superfluid phase transition temperature is in good agreement with experimental data.

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РАСШИРЕННАЯ МОДЕЛЬ БОГОЛЮБОВА НЕИДЕАЛЬНОГО БОЗЕ-ГАЗА И СВЕРХТЕКУЧЕСТЬ

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С микроскопической точки зрения изучены равновесные свойства системы взаимодействующих бозонов. Мы вычисляем плотность сверхтекучей компоненты в модели Боголюбова неидеального бозе-газа. Рассмотрен модельный суперстабильный гамильтониан. Мы изучаем случай некоторого парного потенциала и получаем оценку для температуры и плотности в λ-точке.

РОЗШИРЕНА МОДЕЛЬ БОГОЛЮБОВА НЕІДЕАЛЬНОГО БОЗЕ-ГАЗУ І НАДПЛИННІСТЬ

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З мікроскопічної точки зору вивчені рівноважні властивості системи взаємодіючих бозонів. Ми обчислюємо густину надплинної компоненти в моделі Боголюбова неідеального бозе-газу. Розглянуто модельний суперстабільний гамільтоніан. Ми вивчаємо випадок деякого парного потенціалу і одержуємо оцінку для температури і густини в λ-точці.