TO THE LANDAU THEORY OF RELAXATION PHENOMENA IN PLASMA

A.I. Sokolovsky*, V.N. Gorev, Z.Yu. Chelbaevsky

Dnipropetrovs'k National University, Dnipropetrovs'k, Ukraine

(Received November 7, 2011)

The distribution function of quasi-equilibrium plasma in linear approximation in differences of temperatures and velocities of its components has been obtained. It was done on the basis of our generalization of the Chapman-Enskog method. It is found that the Maxwell distribution with different velocities and temperatures of components is not a true nonequilibrium distribution function even in the linear approximation. The Landau theory of relaxation phenomena in plasma gives a leading approximation of the developed theory. It was established that relaxation takes place in the case of small difference between temperatures and velocities of components regardless of the mass ratio.

PACS: 05.20.Dd, 51.10.+y

1. INTRODUCTION

Landau in his well known work [1] has derived a kinetic equation for completely ionized plasma and solved a problem about temperature relaxation of electron and ion components. According to Landau equilibrium is established in the electron and ion subsystems at first (over periods of time τ_m and τ_M respectively), then temperature relaxation of the components is observed over a period of time τ_T. The last process is the slowest one because of a small electron and ion masses ratio $\sigma \equiv (m/M)^{1/2}$. Analogously a problem about velocity relaxation of the components over a period of time τ_u has been solved (see, for example, [2]). The Landau theory is based on an assumption that quasi-equilibrium state of the plasma can be described by the Maxwell distribution:

$$w_a(p) = \frac{n_a}{(2\pi m_a T_a)^{3/2}} \exp\left(-\frac{(p - m_a \upsilon_a)^2}{2m_a T_a}\right) \qquad (1)$$

$(a = e, i)$. The Landau kinetic equation was in the basis of the study which in considered here spatially uniform case in standard notations has the form

$$\frac{\partial f_a(p,t)}{\partial t} = I_a(p, f(t)),$$

$$I_a(p_a, f) \equiv \frac{\partial}{\partial p_{an}} \sum_b 2\pi(e_a e_b)^2 \times$$
$$\times L \int d^3p_b \frac{u_{ab}^2 \delta_{nl} - u_{ab,n} u_{ab,l}}{u_{ab}^3} \times$$
$$\times \left(\frac{\partial f_a(p_a)}{\partial p_{al}} f_b(p_b) - f_a(p_a) \frac{\partial f_b(p_b)}{\partial p_{bl}}\right) \qquad (2)$$

$(u_{ab,n} \equiv u_{an} - u_{bn},\ u_{an} \equiv p_{an}/m_a)$. Temperatures T_a, velocities υ_{an} and densities of the components n_a are defined by the usual relations:

$$\int d^3p\, f_a(p) = n_a, \qquad \int d^3p\, f_a(p) p_n = m_a n_a \upsilon_{an},$$
$$\int d^3p\, f_a(p)\varepsilon_a(p) = \frac{3}{2} n_a T_a + \frac{1}{2} m_a n_a \upsilon_a^2 \qquad (3)$$

$(\varepsilon_a(p) \equiv p^2/2m_a)$ which express them through the distribution function $f_a(p)$. One can immediately obtain time equations for T_a, υ_{an} which can be written in the form

$$\frac{\partial \upsilon_{an}}{\partial t} = \frac{1}{m_a n_a} R_{an}(f),$$
$$\frac{\partial T_a}{\partial t} = \frac{2}{3n_a}\{Q_a(f) - \upsilon_{an} R_{an}(f)\};$$
$$R_{an}(f) \equiv \int d^3p\, I_a(p,f) p_n,$$
$$Q_a(f) \equiv \int d^3p\, I_a(p,f)\varepsilon_a(p) \qquad (4)$$

(n_a does not depend on time). In the Landau theory these equations formed a closed set of equations after using of the mentioned assumption $f_a(p) = w_a(p)$ where $w_a(p)$ is the Maxwell distribution (1). This set of equations was studied by Landau in a leading approximation in small parameter σ.

In the present paper we consider a problem of obtaining of nonequilibrium distribution function which should be substituted into right side of the time equations (4). It is only possible when in the considered time interval the distribution function $f_a(p,t)$ depends on time only through variables $T_a(t)$, $\upsilon_{an}(t)$:

$$f_a(p,t) \xrightarrow[t \gg \tau_0]{} f_a(p, T_e(t), T_i(t), \upsilon_e(t), \upsilon_i(t)) \qquad (5)$$

$(\tau_m, \tau_M \ll \tau_0 \ll \tau_T, \tau_u)$. This assumption actually gives a generalization of the Chapman-Enskog

*Corresponding author E-mail address: alexsokolovsky@mail.ru

method of solving of kinetic equations and is called the Bogolyubov functional hypothesis (see, for example, [3]).

Note that the Landau problem was studied in many papers (see, for example, [4] and references within). Important advantage of the generalized Chapman-Enskog method is possibility do not choose time scale in the terms of small parameter of the theory. Moreover, this method gives description of non-equilibrium state of the system till equilibrium.

2. BASIC EQUATIONS

Total energy and momentum of the system are conserved, therefore equations (4) have motion integrals T and υ_n which are defined by formulas

$$\sum_a m_a n_a \upsilon_{an} = \upsilon_n \sum_a m_a n_a,$$

$$\sum_a \left(\frac{3}{2} n_a T_a + \frac{1}{2} m_a n_a \upsilon_a^2\right) =$$

$$= \frac{3}{2} T \sum_a n_a + \frac{1}{2}\upsilon^2 \sum_a m_a n_a. \qquad (6)$$

Parameters T and υ_n are equilibrium temperature and velocity of the system. Let study relaxation phenomena close to the equilibrium and introduce small deviation of the electron parameters from the mentioned equilibrium values and the corresponding values for ions:

$$\upsilon_{en} \equiv \upsilon_n + u_n, \quad \upsilon_{in} = \upsilon_n - z\sigma^2 u_n,$$

$$T_e \equiv T + \tau, \quad T_i = T - z\tau - \frac{1}{3} z(1+\sigma^2) m u^2. \quad (7)$$

We assumed here that the system is electrically-neutral and therefore $n_i \equiv n$, $n_e = zn$ where z is charge of an ion.

It is convenient to consider the problem in the reference system where velocity $\upsilon_n = 0$. The functional hypothesis can be written in the form

$$f_a(p,t) \xrightarrow[t \gg \tau_0]{} f_a(p, \tau(t), u(t)). \qquad (8)$$

Evolution equation (4) give equations for parameters $\tau(t)$, $u_n(t)$ in the form

$$\frac{\partial \tau}{\partial t} = L_e(f(\tau,u)), \quad \frac{\partial u_n}{\partial t} = L_{en}(f(\tau,u)). \quad (9)$$

According to equations (8), (9) distribution function $f_a(p,\tau,u)$ satisfies equation

$$\frac{\partial f_a(p)}{\partial \tau} L_e(f) + \frac{\partial f_a(p)}{\partial u_n} L_{en}(f) = I_a(p,f). \qquad (10)$$

Definitions (3) lead to additional conditions for function $f_a(p,\tau,u)$:

$$\int d^3p\, f_a \varepsilon_a(p) = \frac{3}{2} n_a (T + \delta T_a) + \frac{1}{2} m_a n_a \upsilon_a^2,$$

$$\int d^3p\, f_a p_n = m_a n_a \upsilon_{an}, \quad \int d^3p\, f_a = n_a$$

$$(\delta T_a \equiv T_a - T). \qquad (11)$$

We will solve equation (10) with conditions (11) in a perturbation theory in small values τ, u_n and consider them as values of the same order $\tau, u_n \sim \mu$, $\mu \ll 1$. Calculations in the perturbation theory in μ give

$$f_a(p,\tau,u) = f_a^{(0)} + f_a^{(1)} + O(\mu^2),$$

$$f_a^{(0)} = w_a^o(p), \quad w_a^o(p) \equiv \frac{n_a}{(2\pi m_a T)^{3/2}} e^{-\frac{\varepsilon_a(p)}{T}} \quad (12)$$

($A^{(m)}$ is contribution to A of the order μ^m). According to (10), (11) the first order contribution $f_a^{(1)}$ satisfies the equation and additional conditions

$$\frac{\partial f_a^{(1)}(p)}{\partial \tau} L_e^{(1)} + \frac{\partial f_a^{(1)}(p)}{\partial u_n} L_{en}^{(1)} =$$

$$= \sum_b \int d^3p\, M_{ab}(p,p') f_b^{(1)}(p');$$

$$\int d^3p\, f_a^{(1)} \varepsilon_a(p) = \frac{3}{2} n_a \delta T_a^{(1)},$$

$$\int d^3p\, f_a^{(1)} p_n = m_a n_a \upsilon_{an}^{(1)}, \quad \int d^3p\, f_a^{(1)} = 0, \quad (13)$$

where

$$M_{ab}(p,p') \equiv \left.\frac{\delta I_a(p,f)}{\delta f_b(p')}\right|_{f=w^o}. \qquad (14)$$

Contributions of the first order to the right sides of equations (9) taking into account considerations of rotational invariance have the structure

$$L_e^{(1)} = -\lambda_T \tau, \quad L_{en}^{(1)} = -\lambda_u u_n. \qquad (15)$$

The constants λ_T, λ_u define the corresponding temperature and velocity relaxation times in the system $\tau_T \equiv \lambda_T^{-1}$, $\tau_u \equiv \lambda_u^{-1}$ (it is shown below that λ_T, λ_u are positive values). Solution of equations (13) has the structure

$$f_a^{(1)} = w_a^o(p)\{A_a(p)\tau + B_a(p) p_n u_n\}, \qquad (16)$$

where $A_a(p)$, $B_a(p)$ are scalar function of momentum module. Function $A_a(p)$ according to (13) satisfies integral equation and additional conditions

$$\sum_b \int d^3p'\, K_{ab}(p,p') A_b(p') = \lambda_T A_a(p),$$

$$\langle A_a(p) \rangle = 0,$$

$$\langle A_a(p) \varepsilon_a(p) \rangle = \frac{3}{2} nz(\delta_{a,e} - \delta_{a,i}), \qquad (17)$$

where the notation

$$\langle g(p) \rangle = \int d^3p\, w_p^o g(p)$$

is introduced. Kernel of this integral equation is defined by the formula

$$M_{ab}(p,p') w_b^o(p') = -w_a^o(p) K_{ab}(p,p'). \qquad (18)$$

Like to the previous the function $B_a(p)$ is solution of the integral equation with additional conditions

$$\sum_b \int d^3p'\, K_{ab}(p,p')B_b(p')p'_n = \lambda_u B_a(p)p_n,$$

$$\langle B_a(p)\varepsilon_a(p)\rangle = \frac{3}{2}nz(\delta_{a,e}-\delta_{a,i}). \qquad (19)$$

So, the functions $A_a(p)$, $B_a(p)p_n$ are eigenfunctions of the linearized collision integral $\hat{K}$. This operator is defined in space of pair functions of the type $g_a(p)$, $h_a(p)$ $(a=i,e)$ which we denote as H. According to kinetic equation (2) and definitions (14), (18) this operator act on arbitrary function $h_a(p)\in$ H as it follows

$$\begin{aligned}(\hat{K}h)_a(p) &\equiv \sum_b \int d^3p' K_{ab}(p,p')h_b(p')\\ &= w_a^{o\,-1}\sum_b 2\pi(e_ae_b)^2L\frac{\partial}{\partial p_{an}}\int d^3p_b w_a^o w_b^o \qquad (20)\\ &\times \frac{u_{ab}^2\delta_{nl}-u_{ab,n}u_{ab,l}}{u_{ab}^3}\left(\frac{\partial h_a}{\partial p_{al}}-\frac{\partial h_b}{\partial p_{bl}}\right).\end{aligned}$$

On the basis of this operator the bilinear form in the space H can be introduced:

$$\begin{aligned}\{g,h\} &= \sum_{a,b}\int d^3p d^3p' w_a^o(p)g_a(p)K_{ab}(p,p')h_b(p')\\ &= \sum_{a,b}\pi(e_ae_b)^2L\int d^3p_a d^3p_b w_a^o w_b^o\\ &\times \frac{u_{ab}^2\delta_{nl}-u_{ab,n}u_{ab,l}}{u_{ab}^3} \qquad (21)\\ &\times\left(\frac{\partial g_a}{\partial p_{an}}-\frac{\partial g_b}{\partial p_{bn}}\right)\left(\frac{\partial h_a}{\partial p_{al}}-\frac{\partial h_b}{\partial p_{bl}}\right),\end{aligned}$$

which is symmetric and positively definite

$$\{g,h\}=\{h,g\}, \quad \{g,g\}\geq 0,$$

$$\{g,g\}=0 \Rightarrow g_a = 1,\, p_{an},\, \varepsilon_a(p_a). \qquad (22)$$

This show that eigenvalues λ_T, λ_u are positive because equations (17), (19) lead to

$$\{A,A\}=\lambda_T(A,A),$$

$$\{p_nB,p_nB\}=\lambda_u(p_nB,p_nB), \qquad (23)$$

where scalar product in the space H

$$(g,h)=\sum_a\int d^3p w_a^o(p)g_a(p)h_a(p).$$

is introduced. Therefore, following from (9), (15) time equations for τ and u_n

$$\frac{\partial\tau}{\partial t}=-\lambda_T\tau+O(\mu^2), \quad \frac{\partial u_n}{\partial t}=-\lambda_u u_n+O(\mu^2) \quad (24)$$

really describe temperature and velocity of the plasma component equalization. Note that from (17), (19) formulas for attenuation coefficients follow:

$$\lambda_T=\frac{2}{3nz}\sum_a\{\varepsilon_e,A_a\}_{ea},$$

$$\lambda_u=\frac{1}{3mnz}\sum_a\{p_n,p_nB_a\}_{ea}, \qquad (25)$$

where one more bilinear form

$$\begin{aligned}&\{u_1,u_2\}_{ab}=\\ &=\int d^3p\, d^3p'\, w_a^o(p)u_1(p)K_{ab}(p,p')u_2(p') \qquad (26)\end{aligned}$$

is introduced ($u_1(p)$, $u_2(p)$ are some functions).

3. ANALYSIS OF OBTAINED RESULTS

For the purpose of further analysis of the equations (17), (19) it is convenient to represent them in a dimensionless form. Introducing the notations

$$\tilde{A}_a(q)=A_a(q(m_aT)^{1/2})T/3(2\pi)^{3/2};$$

$$\tilde{\lambda}_T=\lambda_T/\lambda_0, \quad \lambda_0\equiv\frac{nLe^4}{(2\pi)^{1/2}T^{3/2}m^{1/2}}, \qquad (27)$$

gives us the following system of integral equations for the functions $\tilde{A}_a(q)$ $(a=i,e)$:

$$\begin{aligned}&\left(q_{in}-\frac{\partial}{\partial q_{in}}\right)\sigma z^3\int d^3q e^{-q^2/2}\Delta_{nl}(q-q_i\sigma)\left(\frac{\partial\tilde{A}_i(q_i)}{\partial q_{il}}\sigma-\frac{\partial\tilde{A}_e(q)}{\partial q_l}\right)+\\ &+\left(q_{in}-\frac{\partial}{\partial q_{in}}\right)\sigma z^2\int d^3q e^{-q^2/2}\Delta_{nl}(q-q_i)\left(\frac{\partial\tilde{A}_i(q_i)}{\partial q_{il}}-\frac{\partial\tilde{A}_i(q)}{\partial q_l}\right)=\tilde{\lambda}_T\tilde{A}_i(q_i), \quad (28)\end{aligned}$$

$$\begin{aligned}&\left(q_{en}-\frac{\partial}{\partial q_{en}}\right)z^2\int d^3q e^{-q^2/2}\Delta_{nl}(\sigma q-q_e)\left(\frac{\partial\tilde{A}_e(q_e)}{\partial q_{el}}-\frac{\partial\tilde{A}_i(q)}{\partial q_l}\sigma\right)+\\ &+\left(q_{en}-\frac{\partial}{\partial q_{en}}\right)z\int d^3q e^{-q^2/2}\Delta_{nl}(\sigma q-q_e)\left(\frac{\partial\tilde{A}_e(q_e)}{\partial q_{el}}-\frac{\partial\tilde{A}_e(q)}{\partial q_l}\right)=\tilde{\lambda}_T\tilde{A}_e(q_e), \quad (29)\end{aligned}$$

$(\Delta_{nl}(q)\equiv(q^2\delta_{nl}-q_nq_l)/q^3)$.

In (27) dimension of functions $A_a(p)$ is taken into account. The dimension is determined by the additional conditions (17) which can be written in the form

$$\int d^3q e^{-q^2/2}\tilde{A}_a(q) = 0,$$

$$\int d^3q q^2 e^{-q^2/2}\tilde{A}_a(q) = \delta_{ae} - z\delta_{ai}. \quad (30)$$

Introducing the notations

$$\tilde{B}_a(q) = B_a(q(m_aT)^{1/2})T/3(2\pi)^{3/2},$$

$$\tilde{\lambda}_u = \lambda_u/\lambda_0, \quad (31)$$

gives us the following system of integral equations for the function $\tilde{B}_a(q)$ $(a = i, e)$ from (19):

$$\left(q_{in} - \frac{\partial}{\partial q_{in}}\right)\sigma^2 z^3 \int d^3q e^{-q^2/2}\Delta_{nl}(q - q_i\sigma)\left(\frac{\partial q_{is}\tilde{B}_i(q_i)}{\partial q_{il}} - \frac{\partial q_s\tilde{B}_e(q)}{\partial q_l}\right)$$
$$+ \left(q_{in} - \frac{\partial}{\partial q_{in}}\right)\sigma z^2 \int d^3q e^{-q^2/2}\Delta_{nl}(q - q_i)\left(\frac{\partial q_{is}\tilde{B}_i(q_i)}{\partial q_{il}} - \frac{\partial q_s\tilde{B}_i(q)}{\partial q_l}\right) = \tilde{\lambda}_u q_{is}\tilde{B}_i(q_i), \quad (32)$$

$$\left(q_{en} - \frac{\partial}{\partial q_{en}}\right) z^2 \int d^3q e^{-q^2/2}\Delta_{nl}(\sigma q - q_e)\left(\frac{\partial q_{es}\tilde{B}_e(q_e)}{\partial q_{el}} - \frac{\partial q_{is}\tilde{B}_i(q)}{\partial q_l}\right)$$
$$+ \left(q_{en} - \frac{\partial}{\partial q_{en}}\right) z \int d^3q e^{-q^2/2}\Delta_{nl}(q - q_e)\left(\frac{\partial q_{es}\tilde{B}_e(q_e)}{\partial q_{el}} - \frac{\partial q_s\tilde{B}_e(q)}{\partial q_l}\right) = \tilde{\lambda}_u\tilde{B}_e(q_e) \quad (33)$$

In (31) dimension of functions $B_a(p)$ is taken into account. The dimension is determined by the additional conditions (19) which can be written in the form

$$\int d^3q q^2 e^{-q^2/2}\tilde{B}_a(q) = \delta_{ae} - \sigma^2 z\delta_{ai}. \quad (34)$$

We find solution of the equations (28)–(30), (32)–(34) in a series in small parameter $\sigma = (m/M)^{1/2}$:

$$\tilde{A}_a = \tilde{A}_a^{(0)} + \tilde{A}_a^{(1)} + O(\sigma^2),$$
$$\tilde{B}_e = \tilde{B}_e^{(0)} + \tilde{B}_e^{(1)} + O(\sigma^2),$$
$$\tilde{B}_i = \tilde{B}_i^{(2)} + \tilde{B}_i^{(3)} + O(\sigma^4),$$
$$\tilde{\lambda}_T = \tilde{\lambda}_T^{(2)} + \tilde{\lambda}_T^{(3)} + O(\sigma^4),$$
$$\tilde{\lambda}_u = \tilde{\lambda}_u^{(0)} + \tilde{\lambda}_u^{(1)} + O(\sigma^2). \quad (35)$$

Unfortunately, these equations cannot be solved by an iteration procedure but simple substitution shows that in the initial notations the leading contribution is given by relations

$$A_e^{(0)}(p) = \frac{1}{T^2}\{\varepsilon_a(p) - \frac{3}{2}T\},$$

$$A_i^{(0)}(p) = -\frac{z}{T^2}\{\varepsilon_a(p) - \frac{3}{2}T\}, \quad (36)$$

$$\lambda_T^{(2)} = \frac{2^{7/2}\pi^{1/2}ne^2z^2(z+1)L\sigma^2}{3m^{1/2}T^{3/2}}, \quad (37)$$

$$B_e^{(0)}(p) = \frac{1}{T}, \quad B_i^{(2)}(p) = -\frac{z}{T}\sigma^2, \quad (38)$$

$$\lambda_u^{(0)} = \frac{2^{5/2}\pi^{1/2}ne^2nz^2L}{3m^{1/2}T^{3/2}}. \quad (39)$$

Comparison of the formulas (36) with the expressions (1) and (16) shows that exactly such functions as $A_a^{(0)}(p)$ lie in the basis of the Landau theory [1]. Moreover, the expression (37) coincides with the temperature relaxation coefficient obtained by Landau [1]. In the same way the functions $B_e^{(0)}(p)$, $B_i^{(2)}(p)$ from (38) lie in the basis of the velocities relaxation theory [2] which is analogous to the Landau theory. At the same time the expression (39) coincides with the velocity relaxation coefficient [2]. So, the Landau relaxation theory is given by the leading approximation of the developed here theory.

4. CONCLUSIONS

The distribution function of quasi-equilibrium plasma in linear approximation in differences of temperatures and velocities of plasma components has been obtained. In was done on the basis of our generalization of the Chapman-Enskog method. It is found that the Maxwell distribution with different velocities and temperatures of components is not a true nonequilibrium distribution function even in the linear approximation. This means that the traditional idea about a universal role of the Maxwell distribution in description of quasi-equilibrium states is not confirmed. It was shown that the Landau theory of relaxation phenomena in plasma (i.e. the theory of temperatures and velocities relaxation of components) gives a leading approximation of the developed here theory in small electron and ion masses ratio. It was shown as well that relaxation takes place in the case of small difference between temperatures and velocities of components regardless of the mass ratio.

The proposed theory can be generalized for description of nonlinear relaxation phenomena.

This work supported in part by the State Foundation for Fundamental Research of Ukraine under project No. 25.2/102.

References

1. L.D. Landau. Kinetic equation in case of Coulomb interaction // *ZhETF.* 1937, v. 7, p. 203-209 (in Russian).
2. A.F. Alexandrov, L.S. Bogdankevich, A.A. Rukhadze. *Fundamentals of plasma electrodynamics.* Moscow: Vysshaya Shkola, 1988, 424 p. (in Russian).
3. A.I. Akhiezer, S.V. Peletminsky. *Methods of Statistical Physics.* Oxford: Pergamon Press, 1981, 368 p.
4. A.V. Bobylev, I.F. Potapenko, and P.H. Sakanaka. Relaxation of two-temperature plasma // *Phys. Rev. E.* 1997, v. 56 (2), p. 2081-2093.

К ТЕОРИИ ЛАНДАУ РЕЛАКСАЦИОННЫХ ЯВЛЕНИЙ В ПЛАЗМЕ

А.И. Соколовский, В.Н. Горев, З.Ю. Челбаевский

Получена функция распределения квазиравновесной плазмы в линейном приближении по разностям температур и скоростей её компонент. Это сделано на основе нашего обобщения метода Чепмена-Энскога. Найдено, что распределение Максвелла с различными скоростями и температурами компонент не является истинной неравновесной функцией распределения даже в линейном приближении. Теория Ландау релаксационных явлений в плазме дает главное приближение развитой теории. Установлено, что релаксация имеет место в случае малой разности между температурами и скоростями компонент независимо от отношения масс частиц.

ДО ТЕОРІЇ ЛАНДАУ РЕЛАКСАЦІЙНИХ ЯВИЩ У ПЛАЗМІ

О.Й. Соколовський, В.М. Горев, З.Ю. Челбаєвський

Одержано функцію розподілу квазірівноважної плазми в лінійному наближенні за різницями температур і швидкостей її компонент. Це зроблено на основі нашого узагальнення методу Чепмена-Енскога. Знайдено, що розподіл Максвелла з різними швидкостями і температурами компонент не є істинна нерівноважна функція розподілу навіть у лінійному наближенні. Теорія Ландау релаксаційних явищ у плазмі дає головне наближення розвинутої теорії. Встановлено, що релаксація має місце у випадку малої різниці між температурами і швидкостями компонент незалежно від відношення мас частинок.