AHARONOV-BOHM EFFECT IN SCATTERING OF HIGH-ENERGY PARTICLES

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Quantum-mechanical scattering of coherent high-energy charged particles by a magnetic vortex is considered. The vortex core is assumed to be impermeable to scattered particles, and effects of its transverse size are taken into account. The limit of high energies of scattered particles corresponds to the quasiclassical limit, and we show that in the scattering the Aharonov-Bohm effect persists in this limit owing to the Fraunhofer diffraction in the forward direction. The issue of the experimental detection of the Fraunhofer diffraction peak and the Aharonov-Bohm effect in the quasiclassical limit is discussed.

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1. INTRODUCTION

Starting from its discovery in 1959, the Aharonov-Bohm effect [1] is considered in the framework of two somewhat different, although closely related, setups. The first one concerns the fringe shift in the interference pattern due to two coherent particle beams under the influence of the impermeable magnetic vortex placed between the beams. The second one deals with scattering of a particle beam directly on an impermeable magnetic vortex. All experiments are performed in the first setup, though the second setup is more elaborate from the theoretical point of view. The reason of this somewhat a paradoxical situation lies in the simple fact that the efforts in the elaboration of scattering theory for the Aharonov-Bohm effect were mostly concentrated on the case of long-wavelength scattered particles, when the transverse size of the impermeable magnetic vortex was neglected. Since a direct scattering experiment is hard to perform with long-wavelength (slow-moving) particles, thus elaborated theory remained actually unverified. The aim of the present paper is to extend the theory to the case of short-wavelength (fast-moving) particles and to reach the realm where the experimental verification is quite feasible (see also [2–4]).

2. DOUBLE-SLIT INTERFERENCE EXPERIMENT

But, first, let us recall briefly the setup which is conventionally used for the verification of the Aharonov-Bohm effect (see, e.g., [5, 6]). It involves the observation of the interference patterns resulting from the two coherent electron beams bypassing from different sides an impermeable magnetic vortex which is orthogonal to the plane defined by the beams. This is a so called double-slit interference experiment, although in reality an electrostatic biprism is used to bend the beams and to direct them on the detection screen. Let the detection screen be parallel to the screen with slits, L be the distance between the screens, and d be the distance between the slits; otherwise, in the biprism setting, the line connecting images of a source is parallel to the detection screen, L is the distance between the line and the screen, and d is the distance between the images. The interference pattern on the detection screen consists of equally spaced fringes which are in the same direction as the magnetic vortex,

$$I(y) = 4I_0(y)\cos^2\left[\left(\frac{yd}{\lambda L} + \frac{\Phi}{\Phi_0}\right)\pi\right], \qquad (1)$$

where y is the coordinate which is orthogonal to the fringes on the detection screen ($y = 0$ corresponds to the point which is symmetric with respect to the slits), $I_0(y)$ is the intensity in the case when either of the slits is closed, λ is the electron wavelength, Φ is the flux of the impermeable magnetic vortex placed just after the screen with slits (otherwise, after the biprism), $\Phi_0 = hce^{-1}$ is the London flux quantum. Intensity $I(y)$ (1) is oscillating with period

$$\Delta y = \lambda L d^{-1} \qquad (2)$$

and the enveloping function given by $4I_0(y)$ which is a Gaussian centred at $y = 0$. At the centre of the detection screen one gets

$$I(0) = 4I_0(0)\cos^2\left(\frac{\Phi}{\Phi_0}\pi\right). \qquad (3)$$

If $L \gg d$ and λ, then one can use dimensionless (angular) variable $\varphi = y/L$. The period of oscillations

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in this variable is

$$\Delta\varphi = \lambda d^{-1}. \quad (4)$$

Evidently, the period of oscillation decreases with the decrease of wavelength λ. The distance resolution of the detector should be at least as high as $\frac{1}{2}\Delta y$, that is why the observation of the interference pattern becomes more complicated in the short-wavelength limit. Since the enveloping function takes a form of a narrow peak in this limit, it is appropriate to define the visibility of the central point

$$V = \frac{\left|I(0) - I\left(\pm\frac{1}{2}\Delta y\right)\right|}{I(0) + I\left(\pm\frac{1}{2}\Delta y\right)}. \quad (5)$$

Assuming $I_0\left(-\frac{1}{2}\Delta y\right) = I_0\left(\frac{1}{2}\Delta y\right)$, one finds immediately

$$V = \quad (6)$$
$$= \frac{\left|I_0(0) - I_0\left(\frac{1}{2}\Delta y\right) + \left[I_0(0) + I_0\left(\frac{1}{2}\Delta y\right)\right]\cos\left(2\frac{\Phi}{\Phi_0}\pi\right)\right|}{I_0(0) + I_0\left(\frac{1}{2}\Delta y\right) + \left[I_0(0) - I_0\left(\frac{1}{2}\Delta y\right)\right]\cos\left(2\frac{\Phi}{\Phi_0}\pi\right)}.$$

3. DIRECT SCATTERING EXPERIMENT

We start with the Schrödinger equation for the wave function describing the stationary scattering state

$$H\psi(r,\,\varphi) = \frac{\hbar^2 k^2}{2m}\psi(r,\,\varphi), \quad (7)$$

where m is the particle mass and k is the absolute value of the particle wave vector ($k = 2\pi/\lambda$); the impermeable magnetic vortex is assumed to be directed orthogonally to the plane with polar coordinates r and φ, and we confine ourselves to the particle motion in this plane, since the motion along the vortex is free. Out of the vortex core the Schrödinger hamiltinian takes form

$$H = -\frac{\hbar^2}{2m}\left[r^{-1}\partial_r r\partial_r + r^{-2}\left(\partial_\varphi - \mathrm{i}\Phi\Phi_0^{-1}\right)^2\right], \quad (8)$$

and we impose condition

$$\lim_{r\to\infty} \mathrm{e}^{\mathrm{i}kr}\psi(r,\,\pm\pi) = 1, \quad (9)$$

signifying that the incident wave comes from the far left; the forward direction is $\varphi = 0$, and the backward direction is $\varphi = \pm\pi$.

Without a loss of generality we assume that the vortex has a shape of cylinder of radius r_c and impose the Robin boundary condition on the wave function:

$$\left\{\left[\cos(\rho\pi) + r_c\sin(\rho\pi)\partial_r\right]\psi(r,\,\varphi)\right\}\big|_{r=r_c} = 0. \quad (10)$$

The solution to (7) with Hamiltonian (8), which satisfies (9) and (10), takes the following form

$$\psi(r,\,\varphi) = \sum_{n\in Z} \mathrm{e}^{\mathrm{i}n\varphi} e^{\mathrm{i}(|n| - \frac{1}{2}|n-\mu|\pi)}\left[J_{|n-\mu|}(kr)\right.$$
$$\left. - \Upsilon^{(\rho)}_{|n-\mu|}(kr_c) H^{(1)}_{|n-\mu|}(kr)\right], \quad (11)$$

where Z is the set of integer numbers, $\mu = \Phi\Phi_0^{-1}$, $J_\alpha(u)$ and $H^{(1)}_\alpha(u)$ are the Bessel and first-kind Hankel functions of order α, and

$$\Upsilon^{(\rho)}_\alpha(u) = \frac{J_\alpha(u) + \tan(\rho\pi)u\partial_u J_\alpha(u)}{H^{(1)}_\alpha + \tan(\rho\pi)u\partial_u H^{(1)}_\alpha(u)}. \quad (12)$$

Thus, wave function (11) consists of two parts: the one which will be denoted by $\psi_0(r,\,\varphi)$ is independent of r_c, and the other one which will be denoted by $\psi_c(r,\,\varphi)$ is dependent on r_c.

Taking the asymptotics of the first part at large distances from the vortex, one can get (see [7])

$$\psi_0(r,\,\varphi) = \mathrm{e}^{\mathrm{i}kr\cos\varphi + \mathrm{i}\mu\varphi}\Big\{\cos(\mu\pi) - \mathrm{i}\,\mathrm{sgn}(\varphi)\sin(\mu\pi)$$
$$\times\left[1 - \mathrm{e}^{\mathrm{i}\left(\frac{1}{2}+\nu-\mu\right)\varphi}\mathrm{erfc}\left(\mathrm{e}^{-\mathrm{i}\pi/4}\sqrt{2kr}\left|\sin\frac{\varphi}{2}\right|\right)\right]\Big\}, \quad (13)$$

where it is implied that $-\pi < \varphi < \pi$, $\mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}}\int_z^\infty du\, \mathrm{e}^{-u^2}$ is the complementary error function, $\nu = [\![\mu]\!]$ is the integer part of μ, and $\mathrm{sgn}(u) = \left\{\begin{array}{ll} 1, & u > 0 \\ -1, & u < 0\end{array}\right\}$. In the case $\sqrt{kr}\left|\sin\frac{\varphi}{2}\right| \gg 1$, one gets

$$\psi_0(r,\,\varphi) = \mathrm{e}^{\mathrm{i}kr\cos\varphi}\mathrm{e}^{\mathrm{i}\mu[\varphi - \mathrm{sgn}(\varphi)\pi]} + f_0(k,\,\varphi)\frac{\mathrm{e}^{\mathrm{i}(kr+\pi/4)}}{\sqrt{r}}, \quad (14)$$

where

$$f_0(k,\,\varphi) = \mathrm{i}\frac{\mathrm{e}^{\mathrm{i}\left(\nu+\frac{1}{2}\right)\varphi}}{\sqrt{2\pi k}}\frac{\sin(\mu\pi)}{\sin(\varphi/2)} \quad (15)$$

is the scattering amplitude which was first obtained by Aharonov and Bohm [1]. In the case $kr \gg 1$ but $\sqrt{kr}\left|\sin\frac{\varphi}{2}\right| \ll 1$, one gets

$$\psi_0(r,\,\varphi) = \mathrm{e}^{\mathrm{i}kr}\cos(\mu\pi). \quad (16)$$

Taking the large-distance asymptotocs of the r_c-dependent part of the wave function, one gets

$$\psi_c(r,\,\varphi) = f_c(k,\,\varphi)\frac{\mathrm{e}^{\mathrm{i}(kr+\pi/4)}}{\sqrt{r}}, \quad (17)$$

where

$$f_c(k,\,\varphi) = \mathrm{i}\sqrt{\frac{2}{\pi k}}\sum_{n\in Z}\mathrm{e}^{\mathrm{i}n\varphi}\mathrm{e}^{\mathrm{i}(|\mathrm{n}| - |\mathrm{n}-\mu|\pi)}\Upsilon^{(\rho)}_{|n-\mu|}(kr_c). \quad (18)$$

In the low-energy (long-wavelength) limit amplitude f_c (18) is suppressed by powers of kr_c as compared to amplitude f_0 (15); however this limit is not feasible to experimental measurements. In the higt-energy (short-wavelength) limit, amplitude f_0 (15) is suppressed and wave function ψ_0 (13) is actually reduced to an incident wave, $\mathrm{e}^{ikr\cos\varphi}$, which is distorted by the flux-dependent factors, see first term in (14) and (16). Amplitude f_c (18) in this limit is prevailing, and we obtain the corresponding differential cross section, see [2],

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\varphi} = |f_c(k,\,\varphi)|^2$$
$$= 4r_c\Delta_{\frac{1}{2}kr_c}(\varphi)\cos^2\left(\frac{1}{2}kr_c\varphi + \mu\pi\right) + \frac{r_c}{2}\left|\sin\frac{\varphi}{2}\right|, \quad (19)$$

where

$$\Delta_x(\varphi) = \frac{1}{4\pi x}\frac{\sin^2(x\varphi)}{\sin^2(\varphi/2)} \qquad (-\pi < \varphi < \pi) \qquad (20)$$

is a strongly peaked at $\varphi = 0$ and $x \gg 1$ function which can be regarded as a regularization of the angular delta-function,

$$\lim_{x\to\infty} \Delta_x(\varphi) = \frac{1}{2\pi}\sum_{n\in Z} e^{in\varphi}, \ \Delta_x(0) = \frac{x}{\pi}. \qquad (21)$$

Thus, the first term on the right-hard side of (19) describes the forward peak of the Fraunhofer diffraction on the vortex, while the second term describes the classical reflection from the vortex. Using notations of the vortex diameter, $d = 2r_c$, and the particle wavelength, $\lambda = 2\pi/k$, we rewrite (19) in a form similar to (1):

$$\frac{d\sigma}{d\varphi} = 2d\Delta_{\frac{d\pi}{2\lambda}}(\varphi)\cos^2\left[\left(\frac{\varphi d}{2\lambda} + \frac{\Phi}{\Phi_0}\right)\pi\right] + \frac{d}{4}\left|\sin\frac{\varphi}{2}\right|; \qquad (22)$$

the differential cross section of the Fraunhofer diffraction is oscillating with period, cf. (4),

$$\Delta\varphi = 2\lambda d^{-1}. \qquad (23)$$

In the strictly forward direction one gets, cf. (3),

$$\left.\frac{d\sigma}{d\varphi}\right|_{\varphi=0} = \frac{d^2}{\lambda}\cos^2\left(\frac{\Phi}{\Phi_0}\pi\right). \qquad (24)$$

Defining the visibility of the central point in the differential cross section as

$$V = \frac{|d\sigma|_{\varphi=0} - d\sigma|_{\varphi=\pm\frac{1}{2}\Delta\varphi}|}{d\sigma|_{\varphi=0} + d\sigma|_{\varphi=\pm\frac{1}{2}\Delta\varphi}}, \qquad (25)$$

we get

$$V = \frac{\left|1 - \frac{4}{\pi^2} + \left(1 + \frac{4}{\pi^2}\right)\cos\left(2\frac{\Phi}{\Phi_0}\pi\right)\right|}{1 + \frac{4}{\pi^2} + \left(1 - \frac{4}{\pi^2}\right)\cos\left(2\frac{\Phi}{\Phi_0}\pi\right)}. \qquad (26)$$

The maximal visibility ($V = 1$) is attained for the flux which is quantized in the units of the Abrikosov vortex flux $\left(\Phi = \frac{1}{2}n\Phi_0\right)$; the minimal visibility ($V = 0$) is attained at

$$\Phi = \left\{n \pm \frac{1}{4} \pm \frac{1}{2\pi}\arcsin\left[\left(1 - \frac{4}{\pi^2}\right)\left(1 + \frac{4}{\pi^2}\right)^{-1}\right]\right\}\Phi_0.$$

4. SUMMARY AND DISCUSSION

We have shown that the fringe shift emerging under the influence of a magnetic vortex in the diffraction pattern in the forward direction in a direct scattering experiment with high-energy particles is completely analogous to that in the interference pattern in a double-slit experiment. It should be emphasized that permeability of the magnetic vortex does not affect the diffraction pattern (first term in (22)), only the classical reflection (second term in (22)) is affected. The latter circumstance facilitates the observation of the scattering Aharonov-Bohm effect in the quasiclassical limit.

Certainly, the Fraunhofer diffraction (i.e. the diffraction in almost parallel rays) is a well-known phenomenon of wave optics. Poisson was the first who predicted theoretically in the early nineteenth century the emergence of a spot of brightness in the centre of a shadow of an opaque disc; the prediction was in a contradiction with the laws of geometric (ray) optics. It is curious that Poisson used his prediction as an argument to disprove wave optics: this demonstrates how unexpected and unbelievable was Poisson's result at that time. Nevertheless, the brightness spot in the centre of the disc shadow was soon observed; the decisive experiments were performed by Arago and Fresnel. The diffraction on the opaque disc bears the name of Poisson, and the brightness spot in the shadow centre bears the name of Arago, see [8]. The same effect persists for scattering of light on an opaque ball and other obstacles. However, in the case of obstacles in the form of a long strip or cylinder, the streak of brightness in the centre of a shadow of such obstacles is elusive to experimental measurements: as is noted in the well-known treatise [9], it seems more likely that the measurable quantity is the classical cross section, although the details of this phenomenon depend on the method of measurement. Almost six decades have passed from the time when this assertion was made by Morse and Feshbach, and experimental facilities have improved enormously since then. It is a challenge for experimentalists to reconsider the Fraunhofer diffraction on the cylindrical obstacles. In the present paper we draw attention to this long-standing problem by pointing at the circumstances when the detection of the forward diffraction peak will be the detection of the Aharonov-Bohm effect persisting in the quasiclassical limit.

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ЭФФЕКТ ААРОНОВА-БОМА В РАССЕЯНИИ ВЫСОКОЭНЕРГЕТИЧЕСКИХ ЧАСТИЦ

Ю.А. Ситенко, Н.Д. Власий

Рассматривается квантово-механическое рассеяние высокоэнергетических заряженных частиц магнитным вихрем. Ядро вихря предполагается непроницаемым для рассеиваемых частиц, и учитываются эффекты его поперечных размеров. Предел высоких энергий рассеиваемых частиц соответствует квазиклассическому пределу, и мы показываем, что эффект Ааронова-Бома в рассеянии в этом пределе выживает благодаря дифракции Фраунгофера в направлении вперед. Обсуждаются вопросы экспериментального детектирования пика Фраунгоферовой дифракции и эффекта Ааронова-Бома в квазиклассическом пределе.

ЕФЕКТ ААРОНОВА-БОМА В РОЗСІЯННІ ВИСОКОЕНЕРГЕТИЧНИХ ЧАСТИНОК

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Розглядається квантово-механічне розсіяння високоенергетичних заряджених частинок магнітним вихором. Припускається, що ядро вихора є непроникним для розсіюваних частинок, та враховуються ефекти його поперечних розмірів. Границя високих енергій розсіюваних частинок відповідає квазікласичній границі, і ми показуємо, що ефект Ааронова-Бома в розсіянні в цій границі виживає завдяки дифракції Фраунгофера в напрямку вперед. Обговорюються питання експериментального детектування піка Фраунгоферової дифракції та ефекта Ааронова-Бома в квазікласичній границі.