Journal of Mathematical Physics, Analysis, Geometry
2014, vol. 10, No. 1, pp. 64–125

On High Moments and the Spectral Norm of Large Dilute Wigner Random Matrices

O. Khorunzhiy

Laboratoire de Mathématiques Université de Versailles-Saint-Quentin
45, Avenue des Etats-Unis 78035 Versailles, France

E-mail: oleksiy.khorunzhiy@uvsq.fr

Received August 8, 2011, revised July 17, 2013

We consider a dilute version of the Wigner ensemble of $n \times n$ random real symmetric matrices $H^{(n,\rho)}$, where ρ denotes an average number of non-zero elements per row. We study the asymptotic properties of the spectral norm $\|H^{(n,\rho_n)}\|$ in the limit of infinite n with $\rho_n = n^{2/3(1+\varepsilon)}$, $\varepsilon > 0$. Our main result is that the probability $\mathbf{P}\left\{\|H^{(n,\rho_n)}\| > 1 + xn^{-2/3}\right\}$, $x > 0$ is bounded for any $\varepsilon \in (\varepsilon_0, 1/2]$, $\varepsilon_0 > 0$ by an expression that does not depend on the particular values of the first several moments $V_{2l}, 2 \le l \le 6$ and $V_{12+2\phi_0}$, $\phi_0 = \phi(\varepsilon_0)$ of the matrix elements of $H^{(n,\rho)}$ provided they exist and the probability distribution of the matrix elements is symmetric. The proof is based on the study of the upper bound of the averaged moments of random matrices with truncated random variables $\mathbf{E}\{\operatorname{Tr}(\hat{H}^{(n,\rho_n)})^{2s_n}\}$, $s_n = \lfloor \chi n^{2/3} \rfloor$ with $\chi > 0$, in the limit $n \to \infty$.

We also consider the lower bound of $\mathbf{E}\{\operatorname{Tr}(H^{(n,\rho_n)})^{2s_n}\}$ and show that in the complementary asymptotic regime, when $\rho_n = n^{\epsilon}$ with $\epsilon \in (0, 2/3]$ and $n \to \infty$, the fourth moment V_4 enters the estimates from below and the scaling variable $n^{-2/3}$ at the border of the limiting spectrum is to be replaced by a variable related with ρ_n^{-1}.

Key words: random matrices, Wigner ensemble, dilute random matrices, spectral norm.

Mathematics Subject Classification 2010: 15B52.

1. Introduction

The spectral theory of random matrices of high dimensions was started by E. Wigner in the middle of the fifties, when the eigenvalue distribution of the

The financial support of the research grant ANR-08-BLAN-0311-11 "Grandes Matrices Aléatoires" (France) is gratefully acknowledged

ensemble $\{A^{(n)}\}$ of real symmetric matrices of the form

$$\left(A^{(n)}\right)_{ij} = \frac{1}{\sqrt{n}} a_{ij}, \quad 1 \le i \le j \le n, \tag{1.1}$$

where $\{a_{ij}, i \le j\}$ are jointly independent random variables with zero mean value and the variance v^2, was studied [27]. It was proved by E. Wigner that the normalized eigenvalue counting function of A_n converges in the limit $n \to \infty$ to a non-random limiting distribution that has the density of the semicircle form. It is common to refer to (1.1) as to the Wigner ensemble of random matrices. The limiting eigenvalue distribution of A_n is often referred to as the semicircle (or Wigner) law.

The semicircle law was then improved and generalized in various aspects, in particular, by relaxing the Wigner conditions on the probability distribution of the matrix elements a_{ij} [6, 18], by the studies of different random matrix ensembles generalizing the form of (1.1) [15], by the studies of the extremal eigenvalues of $A^{(n)}$ and related ensembles [2, 4, 5], and others.

Later on, being motivated by the universality conjecture of the level repulsion in the spectra of heavy atomic nuclei [19], a strong interest to the local properties of the eigenvalue distribution of $A^{(n)}$ at the bulk and at the border of the limiting spectrum has led to a number of powerful and deep results (see monographs [1] and [16] and references therein). These results were mostly related with the ensembles of the form (1.1) with the probability distribution of $A^{(n)}$ that belong to a certain class of laws.

In the general situation of the Wigner ensemble (1.1), the breakthrough results in the studies of the local properties of the eigenvalue distribution at the border of the limiting spectrum of random matrices were obtained in [22, 23] and [25], where the eigenvalue distribution of random matrices (1.1) was studied for the first time on the local scale, i.e., when the mean distance between the eigenvalues is of the order $n^{-2/3}$. These results were obtained with the help of a deep improvement of the moment method initially proposed by E. Wigner. The local asymptotic regime at the border of the spectrum is attained in the limit $n \to \infty$ when the order of the moments is proportional to $n^{2/3}$.

One of the generalizations of the Wigner ensemble (1.1) is given by an ensemble of $n \times n$ real random matrices such that each row contains a random number of non-zero elements and the mean value of this number ρ_n is a function of n. Following statistical mechanics terminology, where this kind of models was first considered, it is natural to refer to this class of random matrices as to the sparse or dilute random matrices [17, 20]. The limiting eigenvalue distribution of dilute random matrices or related ensembles is studied in a number of publications, where, in particular, the semicircle law is proved to be valid in the limit

$n, \rho_n \to \infty$ [10, 20]. The spectral properties at the edge of the limiting spectra was studied in papers [7, 11], however the local asymptotic regime was not reached there.

In the present paper, we consider the dilute version of Wigner random matrices and study its spectral properties on the local regime at the border of the limiting spectra. The paper is organized as follows. In Section 2, we describe the random matrix ensemble we study and formulate our main theorems. The main technical result on the upper bound of high moments of dilute Wigner random matrices is given and the general technique of the proof of this bound is described. In Section 3, we introduce the necessary definitions and formulate the basic principles of the estimates obtained. Section 4 is devoted to the proof of our main technical result and the proofs of the main theorems as well.

In Sections 5 and 6, we prove the auxiliary statements used in Sections 3 and 4. In Section 7, we obtain the estimates from below for the high moments of dilute Wigner random matrices; these estimates are related with certain generalizations of Catalan numbers. In Section 8, we discuss the results obtained.

2. Main Results

Let us consider a family of the real symmetric random matrices $\{H^{(n,\rho)}\}$ whose elements are determined by the equality

$$\left(H^{(n,\rho)}\right)_{ij} = a_{ij}\, b_{ij}^{(n,\rho)}, \quad 1 \le i \le j \le n, \tag{2.1}$$

where $\mathfrak{A} = \{a_{ij}, 1 \le i \le j\}$ is an infinite family of jointly independent identically distributed random variables, and $\mathfrak{B}_n = \{b_{ij}^{(n,\rho)}, 1 \le i \le j \le n\}$ is a family of jointly independent random variables that are also independent from $\mathfrak{A}$. We denote by $\mathbf{E} = \mathbf{E}_n$ the mathematical expectation with respect to the measure $\mathbf{P} = \mathbf{P}_n$ generated by the random variables $\{\mathfrak{A}, \mathfrak{B}_n\}$.

We assume that the probability distribution of the random variables a_{ij} is symmetric and denote their even moments by $V_{2l} = \mathbf{E}(a_{ij})^{2l}, l \ge 1$ with $V_2 = v^2 = 1/4$.

The random variables $b_{ij}^{(n,\rho)}$ are proportional to the Bernoulli ones,

$$b_{ij}^{(n,\rho)} = \frac{1}{\sqrt{\rho}} \begin{cases} 1 - \delta_{ij}, & \text{with probability } \rho/n, \\ 0, & \text{with probability } 1 - \rho/n, \end{cases} \tag{2.2}$$

where δ_{ij} is the Kronecker δ-symbol.

Our main result is related with the asymptotic behavior of the maximal in the absolute value eigenvalue of $H^{(n,\rho)}$,

$$\lambda_{\max}^{(n,\rho)} = \|H^{(n,\rho)}\| = \max_{1\le k\le n} |\lambda_k(H^{(n,\rho)})|,$$

in the limit when n and ρ tend to infinity.

Theorem 2.1. *Let the probability distribution of a_{ij} be such that the moment $V_{12+2\phi} = \mathbf{E}|a_{ij}|^{12+2\phi}$ exists with some $\phi > 0$. If $\rho_n = n^{2/3(1+\varepsilon)}$ with any given $\varepsilon > \frac{3}{6+\phi}$, then the limiting probability*

$$\limsup_{n\to\infty} \mathbf{P}\left\{\lambda_{\max}^{(n,\rho_n)} \geq \left(1+\frac{x}{n^{2/3}}\right)\right\} \leq \mathfrak{P}(x), \quad x>0, \tag{2.3}$$

admits the universal upper bound in the sense that $\mathfrak{P}(x)$ does not depend on the values of V_{2l} with $2 \leq l \leq 6$ and $V_{12+2\phi}$.

Assuming more about the probability distributions of a_{ij}, one can relax the restriction on ε of Theorem 2.1. The following statement is true.

Theorem 2.2. *Let $\tilde{a}_{ij}, 1 \leq i \leq j$ be independent identically distributed bounded random variables, $|\tilde{a}_{ij}| \leq U$, such that their probability distribution is symmetric. Then the maximal eigenvalue $\tilde{\lambda}_{\max}^{(n,\rho)} = \lambda_{\max}(\tilde{H}^{(n,\rho)})$ of the random matrix with elements $\tilde{H}_{ij}^{(n,\rho)} = \tilde{a}_{ij} b_{ij}^{(n,\rho)}$ admits the same asymptotic bound as* (2.3),

$$\limsup_{n\to\infty} \mathbf{P}\left\{\tilde{\lambda}_{\max}^{(n,\rho_n)} \geq \left(1+\frac{x}{n^{2/3}}\right)\right\} \leq \mathfrak{P}(x), \quad x>0, \tag{2.4}$$

in the limit $n \to \infty$, $\rho_n = n^{2/3(1+\tilde{\varepsilon})}$ with any given $\tilde{\varepsilon} \in (0, 1/2]$.

R e m a r k 1. Theorems 2.1 and 2.2 remain true in the case when the random matrices $H^{(n,\rho_n)}$ (2.1) are hermitian, where a_{ij} are complex jointly independent random variables and $b_{ij}^{(n,\rho_n)}$ are still determined by (2.2). In this case the upper bound $\mathfrak{P}(x)$ can be slightly diminished. This difference should disappear in the asymptotic regime of the infinite n and ρ_n such that $\rho_n \ll n^{2/3}$. We discuss this topic in more details in Section 8.

R e m a r k 2. Theorem 2.1 is in agreement with the statements of [8], where the existence of the moment $V_{12+2\phi}$ with any positive $\phi > 0$ is proved to be sufficient for the upper bound (2.3) to hold for $\rho_n = n$ when the dilute random matrices coincide with those of the Wigner ensemble (1.1). Moreover, the proofs of the theorems given in the present paper hold in the case of $\rho_n = n$ without any change. Thus, by using the technique developed here, we obtain once more the results of [8].

The proofs of Theorems 2.1 and 2.2 are related with the detailed study of the averaged moments $\mathrm{M}_{2s}^{(n,\rho)} = \mathbf{E}\mathrm{L}_{2s}^{(n,\rho)}$,

$$\mathrm{L}_{2s}^{(n,\rho)} = \mathrm{Tr}\left(H^{(n,\rho)}\right)^{2s} = \sum_{i_0,i_1,i_2,\dots,i_{2s-1}=1}^{n} H_{i_0i_1}^{(n,\rho)} H_{i_1i_2}^{(n,\rho)} \dots H_{i_{2s-1}i_0}^{(n,\rho)}, \tag{2.5}$$

in the limit of the infinite n, ρ and s. To study the case described by Theorem 2.1, we consider the random matrices with truncated random variables a_{ij}. This makes the proofs of Theorems 2.1 and 2.2 almost identical up to the final stages.

Given a sequence $U_n > 0, n \ge 1$, we introduce the truncated random variables

$$\hat{a}_{ij} = \hat{a}_{ij}^{(n)} = \begin{cases} a_{ij}, & \text{if } |a_{ij}| \le U_n, \\ 0, & \text{otherwise.} \end{cases}$$

Theorems 2.1 and 2.2 will follow from our main technical result related with moments (2.5) of the random matrices $\hat{H}_{ij}^{(n,\rho)} = \hat{a}_{ij}^{(n)}\, b_{ij}^{(n,\rho)}$.

Theorem 2.3. *Under conditions of Theorem* 2.1, *there holds the inequality*

$$\limsup_{n\to\infty} \mathbf{E}\,\mathrm{Tr}\left(\hat{H}^{(n,\rho_n)}\right)^{2s_n} \le \mathfrak{M}(\chi) < \infty, \tag{2.6}$$

where $s_n = \lfloor \chi n^{2/3} \rfloor$ *with* $\chi > 0$, $\rho_n = n^{2/3(1+\varepsilon)}$ *and* $U_n = n^{\delta}$ *with a proper choice of* $\delta > 0$. *The upper bound does not depend on the values of* V_{2l}, $2 \le l \le 6$ *and* $V_{12+2\phi}$.

Theorem 2.3 is proved in Section 4.

Following the original E. Wigner's idea, it is natural to consider the right-hand side of (2.5) as the weighted sum over paths of $2s$ steps. In particular, we can write that

$$\hat{\mathrm{M}}_{2s}^{(n,\rho)} = \mathbf{E}\,\mathrm{Tr}\left(\hat{H}^{(n,\rho)}\right)^{2s} = \sum_{\mathcal{I}_{2s}\in\mathbb{I}_{2s}(n)} \hat{\Pi}_a(\mathcal{I}_{2s})\,\Pi_b(\mathcal{I}_{2s}), \tag{2.7}$$

where the sequence $\mathcal{I}_{2s} = (i_0, i_1, \ldots, i_{2s-1}, i_0), i_k \in \{1, 2, \ldots, n\}$ is regarded as a closed path of $2s$ steps (i_{t-1}, i_t) with the discrete time $t \in [0, 2s]$. We will also say that $\mathcal{I}_{2s}$ is a *trajectory* of $2s$ steps. The set of all possible trajectories of $2s$ steps over $\{1, \ldots, n\}$ is denoted by $\mathbb{I}_{2s}(n)$.

The weights $\hat{\Pi}_a(\mathcal{I}_{2s})$ and $\Pi_b(\mathcal{I}_{2s})$ are naturally determined as the mathematical expectations of the products of the corresponding random variables,

$$\hat{\Pi}_a(\mathcal{I}_{2s}) = \mathbf{E}\left(\hat{a}_{i_0 i_1}\ldots\hat{a}_{i_{2s-1}i_0}\right), \quad \Pi_b(\mathcal{I}_{2s}) = \mathbf{E}\left(b_{i_0 i_1}\ldots b_{i_{2s-1}i_0}\right). \tag{2.8}$$

Here and below, we omit the superscripts in $b_{ij}^{(n,\rho)}$ when no confusion can arise.

In a series of papers by Ya. Sinai and A. Soshnikov [22, 23, 25], a powerful and deep approach was developed to study the moments $\mathrm{M}_{2s}^{(n)}$ of Wigner random matrices in the limit $n, s \to \infty$. It is based on the classification of the family of trajectories $\mathbb{I}_{2s}(n)$ according to the number of their self-intersections. Corresponding classes of equivalence depend also on the types of these self-intersections (simple self-intersections, simple open self-intersections, simple self-intersections with multiple edges). In [14], this approach was completed by the notions of the instants of broken tree structure and proper and imported cells that are important in the studies of the trajectories $\mathcal{I}_{2s}$ that have many steps with common starting point. The complete version of the Sinai–Soshnikov description was further developed in paper [8] where the results of [25] were generalized to the case of Wigner random matrices whose elements have a finite number of moments.

In the present paper, we propose an improvement of the approach described in [8] which enables us to study the high moments of dilute random matrices in the asymptotic regime that describes the local properties of their spectra at the border of the limiting eigenvalue distribution. The method proposed here, when applied to the ensemble of Wigner random matrices, essentially simplifies the technique used in [8].

3. Even Walks and Classes of Equivalence

Given a trajectory $\mathcal{I}_{2s}$, we write that $\mathcal{I}_{2s}(t) = i_t$, $t \in [0, \dots, 2s]$ and consider a subset $\mathbb{U}(\mathcal{I}_{2s}; t) = \{\mathcal{I}_{2s}(t'),\ 0 \le t' \le t\} \subseteq \{1, 2, \dots, n\}$. We denote by $|\mathbb{U}(\mathcal{I}_{2s}; t)|$ its cardinality. Each $\mathcal{I}_{2s}$ generates a *walk* $\mathcal{W}_{2s} = \mathcal{W}_{2s}^{(\mathcal{I}_{2s})} = \{\mathcal{W}(t), 0 \le t \le 2s\}$ that we determine as a sequence of $2s+1$ symbols (or, equivalently, letters) from an ordered alphabet, say, $\mathcal{A} = \{\alpha_1, \alpha_2, \dots\}$. The walk $\mathcal{W}_{2s}^{(\mathcal{I}_{2s})}$ is constructed with the help of the following recurrence rules [13]:

1) $\mathcal{W}_{2s}(0) = \alpha_1$;
2) if $\mathcal{I}_{2s}(t+1) \notin \mathbb{U}(\mathcal{I}_{2s}; t)$, then $\mathcal{W}_{2s}(t+1) = \alpha_{|\mathbb{U}(\mathcal{I}_{2s};t)|+1}$;
   if there exists $t' \le t$ such that $\mathcal{I}_{2s}(t+1) = \mathcal{I}_{2s}(t')$, then $\mathcal{W}_{2s}(t+1) = \mathcal{W}_{2s}(t')$.

For example, $\mathcal{I}_{16} = (5, 2, 7, 9, 7, 1, 2, 7, 9, 7, 2, 7, 2, 1, 7, 2, 5)$ produces the walk

$$\mathcal{W}_{16} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_3, \alpha_5, \alpha_2, \alpha_3, \alpha_4, \alpha_3, \alpha_2, \alpha_3, \alpha_2, \alpha_5, \alpha_3, \alpha_2, \alpha_1).$$

One can say that the pair $(\mathcal{W}_{2s}(t-1), \mathcal{W}_{2s}(t))$ represents the t-th step of the walk $\mathcal{W}_{2s}$ and that α_1 represents the *root* of the walk $\mathcal{W}_{2s}$.

Given two trajectories $\mathcal{I}'_{2s}$ and $\mathcal{I}''_{2s}$, we say that they are equivalent, $\mathcal{I}'_{2s} \sim \mathcal{I}''_{2s}$, if $\mathcal{W}_{2s}^{(\mathcal{I}'_{2s})} = \mathcal{W}_{2s}^{(\mathcal{I}''_{2s})}$, and denote by $\mathcal{C}_{\mathcal{W}} = \mathcal{C}_{\mathcal{W}_{2s}}$ the corresponding class of equivalence. It is clear that

$$|\mathcal{C}_{\mathcal{W}}| = n(n-1)\cdots(n - |\mathbb{U}(\mathcal{I}_{2s}; 2s)| + 1). \tag{3.1}$$

Given $\mathcal{W}_{2s}$, one can draw a *graphical representation* $g(\mathcal{W}_{2s}) = (\mathbb{V}_g, \mathbb{E}_g)$ that can be considered as a kind of multigraph with the set $\mathbb{V}_g$ of vertices labelled by $\alpha_1, \dots, \alpha_{|\mathbb{U}(\mathcal{I}_{2s};2s)|}$, and the set $\mathbb{E}_g$ of $2s$ oriented edges (or, equivalently, arcs) labelled by $t \in [1, \dots, 2s]$; the edge $e_t = (\alpha_i, \alpha_j)$ is in $\mathbb{E}_g$ in the case when $\mathcal{W}_{2s}(t-1) = \alpha_i$ and $\mathcal{W}_{2s}(t) = \alpha_j$. To describe the properties of $g(\mathcal{W}_{2s})$ in general situations, we will use Greek letters $\alpha, \beta, \gamma, \dots$ instead of the symbols from the ordered alphabet $\mathcal{A}$. In this case the root of the walk will be denoted by ϱ. Given vertex β such that $\mathcal{W}_{2s}(t) = \beta$, we will say that β is seen in $\mathcal{W}_{2s}$ at the instant of time t. By an abuse of terminology, we refer to $g(\mathcal{W}_{2s})$ as to the *graph* of the walk $\mathcal{W}_{2s}$.

Let us define the *current multiplicity* of the couple of vertices $\{\beta, \gamma\}$, $\beta, \gamma \in \mathbb{V}_g$ up to the instant t by the variable

$$\mathfrak{m}_{\mathcal{W}}^{(\{\beta,\gamma\})}(t) = \#\{t' \in [1,t] : (\mathcal{W}(t'-1), \mathcal{W}(t')) = (\beta, \gamma) \quad \text{or} \\ (\mathcal{W}(t'-1), \mathcal{W}(t')) = (\gamma, \beta)\}$$

and say that $\mathfrak{m}_{\mathcal{W}}^{(\{\beta,\gamma\})}(2s)$ represents the total multiplicity of the couple $\{\beta, \gamma\}$.

The probability law of $\hat{a}_{ij}$ being symmetric, the weight of $\mathcal{I}_{2s}$ (2.8) is non-zero, $\hat{\Pi}_a(\mathcal{I}_{2s}) \neq 0$ only in the case when $\mathcal{I}_{2s}$ is such that in the corresponding graph of the walk $\mathcal{W}_{2s}^{(\mathcal{I}_{2s})}$ each couple $\{\alpha, \beta\}$ has an even multiplicity $\mathfrak{m}_{\mathcal{W}}^{(\{\alpha,\beta\})}(2s) = 0 (\mathrm{mod}\, 2)$. We refer to the walks of these trajectories as to the *even closed walks* [22] and denote by $\mathbb{W}_{2s}$ the set of all possible even closed walks of $2s$ steps. In what follows, we consider the even closed walks only and consider them simply as the walks.

3.1. Marked Steps and Self-Intersections

Regarding an instant of time t, we say that the couple $(t-1, t)$ represents the *step of time* number t. It is natural to say that the pair $(\mathcal{W}_{2s}(t-1), \mathcal{W}_{2s}(t)) = \mathfrak{s}_t$ represents the *step of the walk* number t. Given $\mathcal{W}_{2s} \in \mathbb{W}_{2s}$, we say that the instant of time t is *marked* [22] if the couple $\{\alpha, \beta\} = \{\mathcal{W}_{2s}(t-1), \mathcal{W}_{2s}(t)\}$ has an odd current multiplicity at the instant t, $\mathfrak{m}_{\mathcal{W}}^{(\{\alpha,\beta\})}(t) = 1 (\mathrm{mod}\, 2)$. We also say that the step of the walk $\mathfrak{s}_t$ and the corresponding edge e_t of $g(\mathcal{W}_{2s})$ are marked. All other steps and edges are called the *non-marked* ones. Regarding the collection of marked edges $\bar{\mathbb{E}}_s$ of $g(\mathcal{W}_{2s})$, we can consider the multigraph $\bar{g}_s = (\bar{\mathbb{V}}_s, \bar{\mathbb{E}}_s)$. Clearly, $\bar{\mathbb{V}}_s = \mathbb{V}_s$ and $|\bar{\mathbb{E}}_s| = s$. It is useful to keep the time labels of the edges $\bar{\mathbb{E}}_s$ as they are in $\mathbb{E}_s$.

Any even closed walk $\mathcal{W}_{2s} \in \mathbb{W}_{2s}$ generates a sequence θ_{2s} of s marked and s non-marked instants that can be regarded as a binary sequence of 0's and 1's. The sequence θ_{2s} is known to encode a Dyck path of $2s$ steps. We denote by $\theta_{2s} = \theta(\mathcal{W}_{2s})$ the Dyck path of $\mathcal{W}_{2s}$ and say that $\theta(\mathcal{W}_{2s})$ represents the *Dyck structure* of $\mathcal{W}_{2s}$.

Let us denote by Θ_{2s} the set of all Dyck paths of $2s$ steps. It is known that Θ_{2s} is in one-by-one correspondence with the set of all half-plane rooted trees $\mathcal{T}_s \in \mathbb{T}_s$ constructed with the help of s edges. Sometimes we will also use the denotation $\mathbf{T}_{2s} = \mathcal{T}_s$. The correspondence between Θ_{2s} and $\mathbb{T}_s$ can be established with the help of the chronological run $\mathfrak{R}$ over the edges of $\mathcal{T}_s$. The cardinality of $\mathbb{T}_s$ being given by the Catalan number that we denote by

$$\mathrm{t}_s = \frac{(2s)!}{s!\,(s+1)!}, \tag{3.2}$$

we refer to the elements of $\mathbb{T}_s$ as to the *Catalan trees*. We consider the edges of the tree $\mathcal{T}_s$ as the oriented ones in the direction away from its root.

Given a Catalan tree $\mathcal{T}_s \in \mathbb{T}_s$, one can label its vertices with the help of letters of $\mathcal{A}$ according to $\mathfrak{R}_{\mathcal{T}}$. The root vertex gets the label α_1 and each new vertex that has no label is labelled by the next in turn letter. We denote the walk obtained by $\overset{\circ}{\mathcal{W}}_{2s}[\mathcal{T}_s]$, and the corresponding Dyck path $\theta_{2s} = \theta(\overset{\circ}{\mathcal{W}}_{2s})$ will be denoted also as $\theta_{2s} = \theta(\mathcal{T}_s)$.

Given $\mathcal{W}_{2s}$, we denote by $\theta^*(\mathcal{W}_{2s})$ the height of the corresponding Dyck path,

$$\theta^*(\mathcal{W}_{2s}) = \max_{0\le t\le 2s} \theta_{2s}(t), \quad \theta_{2s} = \theta(\mathcal{W}_{2s}).$$

This is also the height of the tree $\mathcal{T}_s$, $\theta^*(\mathcal{T}_s) = \max_{0\le t\le 2s} \theta_{2s}(t)$, $\theta_{2s} = \theta(\mathcal{T}_s)$.

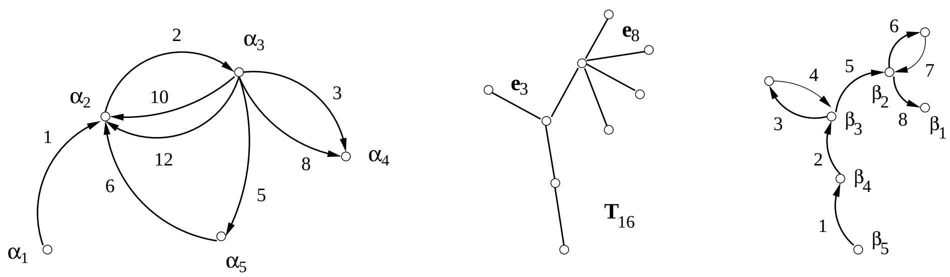

Fig. 1. Graph $\bar{g}(\mathcal{W}_{16})$, tree $\mathbf{T}_{16} = \mathcal{T}_8 = \mathcal{T}(\mathcal{W}_{16})$ and a part of the chronological run over $\mathcal{T}_8$

Any Dyck path θ_{2s} generates a sequence $(\xi_1, \xi_2, \dots, \xi_s)$, $\xi_i \in [1, 2s-1]$ such that each step of the walk $\mathfrak{s}_{\xi_i}$, $1 \le i \le s$, of $\overset{\circ}{\mathcal{W}}_{2s}[\theta_{2s}]$ is marked. We denote this sequence by $\Xi_s = \Xi(\theta_{2s})$. Given Ξ_s and $\tau \in [1, s]$, one can uniquely reconstruct θ_{2s} and find the corresponding instant of time $\xi_\tau \in [1, 2s-1]$. We will say that τ represents the *τ-marked instants* or *instants of marked time* that varies from 1 to s; sometimes we will simply say that τ is the *marked instant* when no confusion with the term "marked instant of time" can arise.

In Figure 1, we present the graph $\bar{g}(\mathcal{W}_{16}) = (\mathbb{V}_8, \bar{\mathbb{E}}_8)$ of the walk $\mathcal{W}_{16}$ as well as its Catalan tree $\mathbf{T_{16}} = \mathcal{T}_8 = \mathcal{T}(\mathcal{W}_{16})$ and a part of the chronological run over $\mathcal{T}_8$ on the time interval $[0, 8] = [0, \xi_6]$. We have $\theta^*(\mathcal{W}_{16}) = 4$. The set of five vertices $\{\beta_1, \dots, \beta_5\}$ represents the *descending part* of the tree $\mathcal{T}_6$, that is a part of $\mathcal{T}_8$.

Given a walk $\mathcal{W}_{2s}$ and a letter β such that $\beta \in \mathbb{V}_g(\mathcal{W}_{2s})$, we say that the instant of time t' such that $\mathcal{W}_{2s}(t') = \beta$ represents an *arrival* $\mathfrak{a}$ at β and that t' is the *arrival instant of time.* If t' is marked, we will say that $\mathfrak{a}$ is the marked arrival at β. In $\mathcal{W}_{2s}$, there can be several marked arrival instants of time at β that we denote by $1 \leq t_1^{(\beta)} < \cdots < t_N^{(\beta)}$. For any non-root vertex β, we have $N = N_\beta \geq 1$. The first arrival instant of time β is always the marked one. We can say that β is created at this instant of time. To unify the description, we assume that the root vertex ϱ is created at the zero instant of time $t_1^{(\rho)} = 0$ and add the corresponding zero marked instant to the list of the marked arrival instants at ϱ.

If $N_\beta \geq 2$, then we say that the N-plet $(t_1^{(\beta)}, \dots, t_N^{(\beta)})$ of marked arrival instants of time represents the *self-intersection* of $\mathcal{W}_{2s}$, β is the *vertex of self-intersection*, and this self-intersection is of the *degree* N [22]. We say that the self-intersection degree of β is equal to N and denote it by $\varkappa(\beta) = N_\beta$. Clearly, if $\beta \neq \varrho$, then the self-intersection degree $\varkappa(\beta)$ indicates the number of marked edges of $g(\mathcal{W}_{2s})$ that have β as their tails.

If $\varkappa(\beta) = 2$, then we say that β is the vertex of *simple self-intersection* [22]. If the walk is such that $\varkappa(\beta) = 2$ and at the second marked arrival instant $t_2^{(\beta)}$ there is at least one couple $\{\beta, \gamma\}$ with an odd current multiplicity, $\mathfrak{m}_{\mathcal{W}}^{(\{\beta,\gamma\})}(t_2^{(\beta)} - 1) = 1(\mathrm{mod}\, 2)$, then β is referred to as the vertex of *open simple self-intersection* and $t_2^{(\beta)}$ is the *instant of open simple self-intersection* [23]. We will also say that the vertex β is open at the instant of time $t' = t_2^{(\beta)} - 1$ or that β is a t'-open vertex.

3.2. Vertices and edges of $g(\mathcal{W}_{2s})$ and diagram $\mathcal{G}(\mathcal{W}_{2s})$

Given a walk $\mathcal{W}_{2s}$ and an integer $k_0 \geq 1$, we consider all vertices $\beta \in \mathbb{V}_g$ such that their self-intersection degree $\varkappa(\beta) \leq k_0$ and say that they are the *μ-vertices.* We will denote the collection of μ-vertices by $\mathbb{V}_g^{(k_0,\mu)} = \mathbb{V}_g^{(\mu)}$. The vertices γ with $\varkappa(\gamma) \geq k_0 + 1$ are referred to as the *ν-vertices*, $\gamma \in \mathbb{V}_g^{(\nu)}$.

Regarding a ν-vertex β, we say that all oriented marked edges of $\bar{\mathbb{E}}_g$ of the form (γ, β) are the *ν-edges.* We color the ν-edges in black and denote by ν_k the number of vertices β such that $\varkappa(\beta) = k$, $k \geq k_0 + 1$. We denote by $\bar{\mathbb{E}}_g^{(\nu)}$ the collection of the ν-edges and determine the subset $\bar{\mathbb{E}}_g^{(k_0)} = \bar{\mathbb{E}}_g \setminus \bar{\mathbb{E}}_g^{(\nu)}$.

The number of μ-vertices β such that $\varkappa(\beta) = 1$ will be denoted by μ_1. In this case, all marked edges of the form (γ, β) will be referred to as the μ_1-edges and colored in grey.

Let us choose a μ-vertex β such that $\varkappa(\beta) \geq 2$ and consider the marked arrivals $\bar{\mathfrak{a}}_i$ at β. Let $e_2 = (\gamma, \beta)$ be the edge of $\bar{\mathbb{E}}_g$ that corresponds to the second arrival $\bar{\mathfrak{a}}_2$ at β. In our considerations, we do not assume that γ is different from β. We denote by ξ_{τ_2} the marked instant of time of e_2 and consider the sub-walk $\mathcal{W}_{[0,t']}$ of $\mathcal{W}_{2s}$ with $t' = \xi_{\tau_2} - 1$. We classify the μ-vertices and corresponding edges according to the properties of the second and the third arrivals at them.

First, let us consider the edge $e_2 = (\gamma, \beta)$ of $\bar{\mathbb{E}}_g$ that corresponds to the second arrival $\bar{\mathfrak{a}}_2$ at β. We denote by ξ_{τ_2} the marked instant of time of e_2 and consider the the sub-walk $\mathcal{W}_{[0,t']}$ of $\mathcal{W}_{2s}$ with $t' = \xi_{\tau_2} - 1$. We distinguish the following properties with respect to $\bar{\mathfrak{a}}_2$:

(a) the edge $e_2 = (\gamma, \beta)$ is such that there exists a marked edge $e' = (\beta, \gamma)$ such that $e' \in \bar{\mathbb{E}}_g^{(k_0)}$ and $e' \in g(\mathcal{W}_{[0,t']})$; in this case we say that e_2 is the q-edge;

(b) if the edge e_2 does not verify condition (a) and there exists a marked edge $e'' = (\gamma, \beta)$ such that $e'' \in g(\mathcal{W}_{[0,t']})$, then we say that e_2 is the p-edge;

(c) if the edge e_2 does not verify (a) and does not verify (b) and the vertex β is t'-open, then we say that e_2 is the o-edge.

We denote by $\mathbb{M}_2'$ the collection of μ-vertices β such that the second arrival $\bar{\mathfrak{a}}_2$ at β verifies one of the three conditions listed above and denote its cardinality by $\mu_2' = |\mathbb{M}_2'|$. We say that the first arrival $\mathfrak{a}_1$ at β represents an f-edge of the graph $\bar{g}_s$ and color it in red. The o, p and q-edges are colored in blue color. All edges that correspond to the arrivals $\mathfrak{a}_i$, $i \geq 3$ at μ_2'-vertex β, if they exist, will be referred to as the u-edges and colored in green color. We denote by u_2 the total number of these edges of $\bar{g}_s$.

If β is such that $\varkappa(\beta) = 2$ and neither (a) nor (b) nor (c) is verified, we say that e_2 is the μ-edge at color it in blue color. The collection of vertices of this kind will be denoted by $\mathbb{M}_2''$ with the cardinality $|\mathbb{M}_2''| = \mu_2''$.

Let us take a μ-vertex β such that $\varkappa(\beta) \geq 3$ that does not belong to $\mathbb{M}_2'$. If the third arrival $\bar{\mathfrak{a}}_3$ verifies at least one of the two conditions, either (a) or (b), with τ_2 and e_2 replaced by τ_3 and $e_3 = e(\xi_{\tau_3}) = (\gamma, \beta)$, respectively, we say that $\beta \in \mathbb{M}_3'$. In this case, assuming the same ordering of conditions (a) and (b) as before, we attribute to the marked edge e_3 of the third arrival at β one of two labels, either q or p. All the edges that correspond to the arrivals $\bar{\mathfrak{a}}_i$, $i \geq 4$ at β, if they exist, will be referred to as the u-edges. We denote by u_3'' the total number of these edges of $\bar{g}_s$. The cardinality of $\mathbb{M}_3'$ will be denoted by $\mu_3' = |\mathbb{M}_3'|$.

If $\beta \in \mathbb{M}_3'$, then we say that e_3 is the μ-edge and color it in blue. We color the edges of the first and the second arrivals at β in red and refer to them as to the f-edges.

Finally, let us consider the vertices β such that $\varkappa(\beta) \geq 3$ and $\beta \notin \mathbb{M}_3' \cup \mathbb{M}_2'$. The family of these vertices will be denoted by $\mathbb{M}_3''$ with $|\mathbb{M}_3''| = \mu_3''$. In this case, we say that all three arrival edges at β are the μ-edges and color them in blue. The edges of all subsequent arrivals at β are referred to as the green u-edges.

The total number of green edges will be denoted by u_3'.

Summing up these considerations, we can see that a given walk $\mathcal{W}_{2s}$ generates a kind of *graphical diagram* $\mathcal{G}$ that describes the vertices of self-intersections of $\mathcal{W}_{2s}$ and the structure of the corresponding edges. More rigorously, we define a diagram $\mathcal{G}$ as a collection of vertices $v_i \in \mathcal{V}(\mathcal{G})$ $|\mathcal{V}(\mathcal{G})| = |\mathbb{V}_g|$ and *half-edges* $\mathfrak{e}_j \in \mathcal{E}(\mathcal{G})$ attached to v_i. The half-edges have heads but have no tails. Instead of the tail, we attach to the corresponding end of $\mathfrak{e}$ a circle that we refer to as the *window* $\mathfrak{o}$. The windows can contain the numerical data; in this case we will say that these numbers represent a *realization* of the windows. In general, we will say that the numerical data in the windows that come from $\mathcal{W}_{2s}$ represent a *realization of the diagram* $\mathcal{G}$ given by $\mathcal{W}$; we denote this realization by $\langle \mathcal{G} \rangle_{\mathcal{W}}$. In what follows, we will refer to the triplet $(v, \mathfrak{e}, \mathfrak{o})$ either as to the *edge-window* or simply as to the edge of $\mathcal{G}$. The edge-windows of $\mathcal{G}$ are colored according to the colors of the corresponding edges of $g(\mathcal{W}_{2s})$. Then we can determine the same classification of the elements of $\mathcal{V}(\mathcal{G})$ as it is done for those of $\mathbb{V}_g$.

Regarding the example walk $\mathcal{W}_{16}$ from Fig. 1, we can see that the graph $g(\mathcal{W}_{16})$ contains five vertices $\alpha_1, \ldots, \alpha_5$. The vertices of self-intersections are represented by two of them, $\alpha_2, \alpha_4 \in \mathbb{M}_2'$. Therefore the realization of the diagram $\mathcal{G}(\mathcal{W}_{16})$ contains two vertices, $\mathcal{V}(\mathcal{G}) = \{v_1, v_2\}$. There are four edge-windows at v_1 with $\langle(\mathfrak{o}_1, \mathfrak{o}_2, \mathfrak{o}_3, \mathfrak{o}_4)\rangle_{\mathcal{W}} = (1, 6, 10, 12)$ and two at v_2, $\langle(\mathfrak{o}_1, \mathfrak{o}_2)\rangle_{\mathcal{W}} = (3, 8)$. We order the vertices v_i according to the values in the blue windows of the second arrivals. If $k_0 \geq 4$, then v_1 has one red, one blue and two green edges attached, the vertex v_2 has one red edge and one blue edge.

The vertices of $\mathcal{V}(\mathcal{G})$ are not ordered, but those of $\langle \mathcal{G} \rangle$ are. In the general situation, we order the vertices of $\langle \mathcal{G} \rangle$ according either to the instants of the second arrivals, if the vertices are from $\mathbb{M}_2' \cup \mathbb{M}_2''$, or to the instants of the third arrivals, if they are from $\mathbb{M}_3' \cup \mathbb{M}_3''$, or to the instants of the last arrivals, if the corresponding vertices of $g(\mathcal{W}_{2s})$ are the ν-vertices.

3.3. Classes of walks and trajectories

Given θ_{2s}, an even closed walk $\mathcal{W}_{2s}$ is determined by its values at the marked and non-marked instants of time. The general estimation principle used in papers [22, 23] and [25] is based on the observation that the knowledge of the instants of self-intersections determines all values of $\mathcal{W}_{2s}$ at the marked instants of time; this knowledge being added by a rule Υ of the non-marked passages determines completely the walk $\mathcal{W}_{2s}$ (see Section 5 for the rigorous definition of Υ).

The vertices of self-intersections and the properties of the corresponding edges are described by diagrams $\mathcal{G}$. Any diagram is characterized by the following set of variables:

$$\mathcal{S} = (r, p, q, \mu_2'', u_2; \mu_3', \mu_3'', u_3, \bar{\nu}),$$

where $\bar{\nu} = \bar{\nu}^{(k_0+1)} = (\nu_{k_0+1}, \dots, \nu_s)$, and ν_k denotes the number of vertices v such that $\varkappa(v) = k$. Let us denote the set of the diagrams by $\mathbb{G}(\mathcal{S})$. The elements of $\mathbb{G}(\mathcal{S})$ differ by the positions of green edges attached to the μ-vertices.

We say that $\mathcal{W}_{2s}$ belongs to the class $\mathbb{W}_{2s}(\mathcal{G})$, $\mathcal{G} = \mathcal{G}(\mathcal{S})$ if the graph $g_s = g(\mathcal{W}_{2s})$ has a collection of μ-vertices $\mathbb{M}_2', \mathbb{M}_2'', \mathbb{M}_3', \mathbb{M}_3''$ with corresponding cardinalities, where $r+p+q = \mu_2'$, and r is the number of self-intersections determined by o-edges, p is the number of self-intersections with p-edges, and q is the number of self-intersections with q-edges. Then obviously,

$$|\mathcal{V}(\mathcal{G})| = \mu_1 + \mu_2 + \mu_3 + \sum_{k=k_0+1}^{s} \nu_k,$$

where $\mu_2 = \mu_2' + \mu_2''$ and $\mu_3 = \mu_3' + \mu_3''$, and

$$|\mathcal{E}(\mathcal{G})| = \mu_1 + 2\mu_2 + 3\mu_3 + u_2 + u_3 + \sum_{k=k_0+1} k\nu_k = s. \tag{3.3}$$

Let us recall that $|\mathcal{V}(\mathcal{G})| = |\mathbb{V}_g|$ and $|\mathcal{E}(\mathcal{G})| = |\bar{\mathbb{E}}_g|$ and denote $\|\bar{\nu}\| = \sum_{k=k_0+1}^{s} k\nu_k$.

Lemma 3.1. *If $\mathcal{I}_{2s}$ is such that $\mathcal{W}(\mathcal{I}_{2s}) \in \mathbb{W}_{2s}(\mathcal{G})$ with $\mathcal{G} \in \mathbb{G}(\mathcal{S})$, then*

$$\hat{\Pi}_a(\mathcal{I}_{2s})\,\Pi_b(\mathcal{I}_{2s}) \le \left(\frac{V_2}{n}\right)^{\mu_1+\mu_2'+r+2\mu_2''+2\mu_3'+3\mu_3''} \left(\frac{U_n^2}{\rho}\right)^{p+q+\mu_3'+u_2+u_3+\|\bar{\nu}\|}. \tag{3.4}$$

P r o o f of Lemma 3.1. Regarding the factor $\hat{\Pi}_a(\mathcal{I}_{2s})$ of (2.8) determined by a given diagram $\mathcal{G}$, we replace by U_n all random variables $\hat{a}_{ij}$ that correspond to the ν-edges, u-edges, p-edges and q-edges together with all their non-marked counterparts. When doing this, only one case needs a special attention, namely the case when the vertices γ and β are joined by two q-edges of the form (β, γ) and (γ, β). However, it is easy to consider all possible configurations of the marked edges with heads γ and β and to show that (3.4) is valid for this class of diagrams. ■

Given a walk $\mathcal{W}_{2s}$, we determine the *enter cluster* of β of its graph $g(\mathcal{W}_{2s})$ as the set of all marked edges of the form (α_i, β), $\Lambda(\beta) = \Lambda(\beta; \mathcal{W}_{2s}) = \{(\alpha_j, \beta) \in \bar{\mathbb{E}}_g\}$. Similarly, we determine the *exit cluster* of a vertex β as the set of all edges (β, γ_i), $\Delta(\beta) = \Delta(\beta; \mathcal{W}_{2s}) = \{(\beta, \gamma_i) \in \bar{\mathbb{E}}_g\}$. Sometimes, when no confusion can arise, we will use the same denotations, Λ and Δ, for the collections of vertices that are the heads (or tails, respectively) of the corresponding edges.

Regarding $\mathcal{W}_{2s}$, we find the exit cluster of maximal cardinality and say that it determines the *maximal exit degree* of the walk,

$$\mathcal{D}(\mathcal{W}_{2s}) = \max_{\beta \in \mathbb{V}_g} |\Delta(\beta; \mathcal{W}_{2s})|.$$

Given $\theta = \theta_{2s}$, let us consider a sub-class $\mathbb{W}_{2s}^{[\theta]}(\mathrm{D};\mathcal{G}) \subset \mathbb{W}_{2s}(\mathcal{G})$ of walks $\mathcal{W}_{2s}$ of Dyck structure θ such that the maximal exit degree of $\mathcal{W}_{2s}$ is equal to D, $\mathcal{D}(\mathcal{W}_{2s}) = \mathrm{D}$. Given $\mathcal{G}$, let us denote by $\mathcal{G}_\Diamond$ the collection of its blue, green and black edge-windows and by $\mathcal{G}_\circ$ the collection of its red edge-windows.

Let $\langle \mathcal{G}_\Diamond \rangle_s$ be a realization of the corresponding edge-windows filled with the values from $(1,\ldots,s)$. Given such a realization, we perform a run of the walk $\mathcal{W}_{2s}$ with the self-intersections prescribed. If such a walk exists, we denote by $\langle \mathcal{G}_\circ \rangle = \langle \mathcal{G}_\circ \rangle_{\mathcal{W}}$ the realization of values in the red edge-windows of $\mathcal{G}$ recorded during the run of $\mathcal{W}_{2s}$.

Lemma 3.2. *Given a rule Υ, denote by $\mathbb{W}_{2s}^{[\theta]}(\mathrm{D};\mathcal{G},\Upsilon)$ the family of walks $\mathcal{W}_{2s}$ such that $\mathcal{W}_{2s} \in \mathbb{W}_{2s}^{[\theta]}(\mathrm{D};\mathcal{G},\Upsilon)$, $\mathcal{G} \in \mathbb{G}(\mathcal{S})$. Then*

$$|\mathbb{W}_{2s}^{[\theta]}(\mathrm{D};\mathcal{G},\Upsilon)| \; = \sum_{\langle \mathcal{G}_\circ \rangle} \; \sum_{\langle \mathcal{G}_\Diamond \rangle_s} 1,$$

where

$$\sum_{\langle \mathcal{G}_\Diamond \rangle_s} 1 \le \frac{s^{r+p+q+u_2}}{r!\,p!\,q!} \frac{1}{\mu_2''!} \left(\frac{s^2}{2}\right)^{\mu_2''} \frac{s^{\mu_3'+u_3}}{\mu_3'!} \frac{1}{\mu_3''!} \left(\frac{s^3}{6}\right)^{\mu_3''} \prod_{k=k_0+1}^{s} \frac{1}{\nu_k!} \left(\frac{s^k}{k!}\right)^{\nu_k} \tag{3.5a}$$

and

$$\sum_{\langle \mathcal{G}_\circ \rangle} 1 \le (2\theta_{2s}^*)^r \, \mathrm{D}^p \, k_0^q \, (s(\mathrm{D}+k_0))^{\mu_3'}, \tag{3.5b}$$

and therefore

$$|\mathbb{W}_{2s}^{[\theta]}(\mathrm{D};\mathcal{G},\Upsilon)| \;\le\; \frac{(2s\theta_{2s}^*)^r}{r!} \frac{(s\mathrm{D})^p}{p!} \frac{(sk_0)^q}{q!} \frac{1}{\mu_2''!} \left(\frac{s^2}{2}\right)^{\mu_2''} s^{u_2+u_3}$$
$$\times \; \frac{(s^2(\mathrm{D}+k_0))^{\mu_3'}}{\mu_3'!} \frac{1}{\mu_3''!} \left(\frac{s^3}{6}\right)^{\mu_3''} \prod_{k=k_0+1}^{s} \frac{1}{\nu_k!} \left(\frac{s^k}{k!}\right)^{\nu_k},$$

where θ_{2s}^* is the height of the Dyck path θ_{2s}.

We will prove Lemma 3.2 in Section 5.

Corollary of Lemma 3.2. Let us consider the family of walks

$$\mathbb{W}_{2s}^{[\theta]}(\mathrm{D};\mathcal{S}) \;=\; \bigsqcup_{\mathcal{G}\in\mathbb{G}(\mathcal{S})} \; \bigsqcup_{\Upsilon\in\mathbb{Y}} \mathbb{W}_{2s}^{[\theta]}(\mathrm{D};\mathcal{G},\Upsilon).$$

Then

$$|\mathbb{W}_{2s}^{[\theta]}(\mathrm{D};\mathcal{S})| \le \; \frac{(6s\theta_{2s}^*)^r}{r!} \frac{(3s\mathrm{D})^p}{p!} \frac{(3sk_0)^q}{q!} \frac{1}{\mu_2''!} \left(\frac{s^2}{2}\right)^{\mu_2''} \frac{(8k_0^4 s\mu_2')^{u_2}}{u_2!}$$

$$\times \frac{(s^2(\mathrm{D}+k_0))^{\mu_3'}}{\mu_3'!} \frac{1}{\mu_3''!} \left(\frac{s^3}{6}\right)^{\mu_3''} \frac{(16k_0^5 s\mu_3)^{u_3}}{u_3!} \prod_{k=k_0+1}^{s} \frac{1}{\nu_k!} \left(\frac{(2ks)^k}{k!}\right)^{\nu_k}. \tag{3.6}$$

Let us introduce a class of walks $\mathbb{W}_{2s}^{(u)}(\mathrm{D};\mathcal{S})$ such that the height of their Dyck paths is equal to u. Combining the results of Lemma 3.1 and Corollary of Lemma 3.2, it is not difficult to prove the following statement.

Lemma 3.3. *Let us denote by $\mathcal{C}(\mathbb{W}_{2s}^{(u)}(\mathrm{D};\mathcal{S}))$ the family of trajectories $\mathcal{I}_{2s}$ such that their walks belong to $\mathbb{W}_{2s}^{(u)}(\mathrm{D};\mathcal{S})$. Then*

$$\sum_{\mathcal{I}_{2s}\in\mathcal{C}(\mathbb{W}_{2s}^{(u)}(\mathrm{D};\mathcal{S}))} \hat{\Pi}_a(\mathcal{I}_{2s})\ \Pi_b(\mathcal{I}_{2s}) \le V_2^s\ |\Theta_{2s}^{(u)}|\ \exp\left\{-\frac{(s-\sigma)^2}{2n}\right\}$$

$$\times \frac{1}{\mu_2''!}\left(\frac{s^2}{2n}\right)^{\mu_2''} \mathrm{H}_{(\mathcal{S};2)}^{(u,\mathrm{D},k_0)}(1)\mathrm{H}_{(\mathcal{S};3)}^{(\mathrm{D},k_0)}(1)\mathrm{H}_{(\mathcal{S};\bar{\nu})}^{(k_0+1)}(1)\,, \tag{3.7}$$

where $\Theta_{2s}^{(u)} = \{\theta_{2s}\in\Theta_s,\ \theta_{2s}^* = u\}$, $\sigma = \mu_2+\mu_3+u_2+u_3+|\bar{\nu}|_1$ *with*

$$|\bar{\nu}|_1 = \sum_{k=k_0+1}^{s}(k-1)\nu_k,$$

and

$$\mathrm{H}_{(\mathcal{S};2)}^{(u,\mathrm{D},k_0)}(h) = \frac{1}{r!}\left(\frac{6hsu}{n}\right)^r \frac{1}{p!}\left(\frac{3hsD\hat{U}_n^2}{\rho}\right)^p$$

$$\times \frac{1}{q!}\left(\frac{3hsk_0\hat{U}_n^2}{\rho}\right)^q \frac{1}{u_2!}\left(\frac{8hk_0^4 s\mu_2'\hat{U}_n^2}{\rho}\right)^{u_2}, \tag{3.8a}$$

$$\mathrm{H}_{(\mathcal{S};3)}^{(\mathrm{D},k_0)}(h) = \frac{1}{\mu_3'!}\left(\frac{9h(D+k_0)s^2\hat{U}_n^2}{n\rho}\right)^{\mu_3'} \frac{1}{\mu_3''!}\left(\frac{3hs^3}{2n^2}\right)^{\mu_3''} \frac{1}{u_3!}\left(\frac{16hk_0^5 s\mu_3\hat{U}_n^2}{\rho}\right)^{u_3}, \tag{3.8b}$$

and

$$\mathrm{H}_{(\mathcal{S};\bar{\nu})}^{(k_0+1)}(h) = \prod_{k=k_0+1}^{s}\frac{1}{\nu_k!}\left(\frac{n(2hks)^k\,\hat{U}_n^{2k}}{k!\,\rho^k}\right)^{\nu_k} \tag{3.8c}$$

with $h\ge 1$ and $\hat{U}_n^2 = U_n^2/V_2$.

The upper bound (3.7) represents a natural generalization of the estimates obtained for the first time in [22, 23, 25] and [21]. Further modifications of these estimates were presented in [8]. The form of (3.6) together with expressions (3.7), (3.8), (3.9) and (3.10) is based on a new description related with the form and the structure of diagrams $\mathcal{G}$. It gives a considerable simplification and powerful improvement of the general approach used in [8]. The rigorous proof of Lemma 3.3 answers a number of questions that arise when reading papers [21, 25]. Notice, we will prove Lemma 3.3 in Section 5.

The results of Lemma 3.3 are sufficient to get the upper bound of the leading term of (2.5) and to show that the contribution of the walks that have multiple edges and that have bounded maximal exit degree $\mathcal{D}(\mathcal{W}_{2s}) \le n^{\epsilon}$ with certain ϵ vanishes in the limit $n \to \infty$ (see Section 4). To study the family of walks such that $\mathcal{D}(\mathcal{W}_{2s}) > n^{\epsilon}$, we need to consider the properties of the corresponding graphs in more details.

3.4. Vertex of maximal exit degree

Let us consider a walk $\mathcal{W}_{2s}$ and find the first letter $\alpha_i = \breve{\beta}$ such that $|\Delta(\breve{\beta})| = \max_{\beta \in \mathbb{V}(\mathcal{W}_{2s})} |\Delta(\beta)|$. We will refer to $\breve{\beta}$ as to the *vertex of maximal exit degree* and denote $\mathcal{D}(\breve{\beta}) = |\Delta(\breve{\beta})|$. In this section we study the properties of even closed walks related with the vertex of maximal exit degree $\breve{\beta}$. To classify the arrival edges at $\breve{\beta}$, we need to determine the reduction procedures related with $\breve{\beta}$. These procedures are similar to those considered in [14].

3.4.1. Reduction procedures and reduced sub-walks. Any walk $\mathcal{W}_{2s}$ can be considered as an ordered set of its steps $\mathfrak{s}_t$, $1 \le t \le 2s$, where $\mathfrak{s}_t = (\mathcal{W}_{2s}(t-1), \mathcal{W}_{2s}(t))$. Inversely, each pair of letters α, β such that $\mathcal{W}_{2s}(t-1) = \alpha$ and $\mathcal{W}_{2s}(t) = \beta$ with some t represents an element of the ordered set of the steps of $\mathcal{W}_{2s}$ that we denote by $\mathfrak{S} = \mathfrak{S}(\mathcal{W}_{2s})$. To each element $\mathfrak{s}_i \in \mathfrak{S}$ we attribute the label i which is simply the number of the step in $\mathcal{W}_{2s}$. These labels order in natural way the elements of $\mathfrak{S}$. We do not change these labels during the reduction procedures considered below.

Given $\mathcal{W}_{2s}$, let t' be the minimal instant of time such that

i) the step $\mathfrak{s}_{t'}$ is the marked step of $\mathcal{W}_{2s}$;
ii) the consecutive to $\mathfrak{s}_{t'}$ step $\mathfrak{s}_{t'+1}$ is non-marked;
iii) $\mathcal{W}_{2s}(t'-1) = \mathcal{W}_{2s}(t'+1)$.

If this t' exists, we can apply to $\mathfrak{S}$ a reduction procedure $\hat{\mathcal{R}}$ that removes from $\mathfrak{S}$ two consecutive elements $\mathfrak{s}_{t'}$ and $\mathfrak{s}_{t'+1}$; we denote $\hat{\mathcal{R}}(\mathfrak{S}) = \mathfrak{S}'$. The ordering labels of the elements of $\mathfrak{S}'$ are inherited from those of $\mathfrak{S}$.

It is clear that the ordered set $\mathfrak{S}'$ can be considered as a new walk $\mathcal{W}'_{2s-2}$ that is again an even closed walk. We denote $\mathcal{W}'_{2s-2} = \hat{\mathcal{R}}(\mathcal{W}_{2s})$ and say that $\hat{\mathcal{R}}$ is the *strong reduction* procedure of the walk $\mathcal{W}_{2s}$. Then we can apply $\hat{\mathcal{R}}$ to $\mathcal{W}'_{2s-2}$ and get an even closed walk $\mathcal{W}''_{2s-4} = \hat{\mathcal{R}}(\mathcal{W}_{2s-2})$. Repeating this action maximally possible number of times m, we get the walk

$$\hat{\mathcal{W}}_{2\hat{s}} = (\hat{\mathcal{R}})^m(\mathcal{W}_{2s}), \quad \hat{s} = s - m,$$

which we refer to as the *strongly reduced walk*. We denote $\hat{\mathfrak{S}} = (\hat{\mathcal{R}})^m(\mathfrak{S})$.

We introduce a *weak reduction* procedure $\check{R}$ of $\mathfrak{S}$ that removes from $\mathcal{W}_{2s}$ the pair of consecutive steps, $\mathfrak{s}_{t'}, \mathfrak{s}_{t'+1}$, such that the conditions (i)-(iii) are verified and

iv) $\mathcal{W}_{2s}(t') \neq \check{\beta}$.

We denote by

$$\check{\mathcal{W}}_{2\check{s}} = (\check{\mathcal{R}})^l(\mathcal{W}_{2s}), \quad \check{s} = s - l, \tag{3.9}$$

the result of the action of maximally possible number of consecutive weak reductions $\check{\mathcal{R}}$ and denote $\check{\mathfrak{S}} = (\check{\mathcal{R}})^l(\mathfrak{S})$. In what follows, we sometimes omit the subscripts $2\hat{s}$ and $2\check{s}$.

Let us consider the example walk $\mathcal{W}_{16}$ (3.1). It is seen that $\mathcal{D}(\mathcal{W}_{2s}) = 5$ and the vertex of maximal exit degree is α_3. The strongly reduced walk $(\hat{\mathcal{R}})^3(\mathcal{W}_{16}) = \hat{\mathcal{W}}_{10}$ is as follows:

$$\hat{\mathcal{W}}_{10} = (\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_2, \alpha_3, \alpha_2, \alpha_5, \alpha_3, \alpha_2, \alpha_1),$$

and the corresponding reduced set of the steps $\hat{\mathfrak{S}} = \hat{\mathfrak{S}}(\hat{\mathcal{W}}_{10})$ is

$$\hat{\mathfrak{S}} = \{\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_5, \mathfrak{s}_6, \mathfrak{s}_7, \mathfrak{s}_{12}, \mathfrak{s}_{13}, \mathfrak{s}_{14}, \mathfrak{s}_{15}, \mathfrak{s}_{16}\}.$$

For this example walk, we have $\check{\mathcal{W}}_{10} = \hat{\mathcal{W}}_{10}$.

Regarding the difference $\check{\mathfrak{S}} \setminus \hat{\mathfrak{S}} = \breve{\mathfrak{S}}$, one can see that it represents a collection of sub-walks, $\breve{W} = \cup_j \breve{\mathcal{W}}^{(j)}$. Each sub-walk $\breve{\mathcal{W}}^{(j)}$ can be reduced by a sequence of the strong reduction procedures $\hat{\mathcal{R}}$ to an empty walk. We say that $\breve{\mathcal{W}}^{(j)}$ is of the *tree-type* structure, or that $\breve{\mathcal{W}}^{(j)}$ is a tree-type sub-walk. It is easy to see that any $\breve{\mathcal{W}}^{(j)}$ starts by a marked step and ends by a non-marked steps and there is no steps of $\hat{W}$ between these two steps of $\breve{\mathcal{W}}^{(j)}$. We say that $\breve{\mathcal{W}}^{(j)}$ is *non-split*.

It is not hard to see that the remaining part $\tilde{\mathfrak{S}} = \mathfrak{S} \setminus \check{\mathfrak{S}}$ is given by a collection of the subsets $\tilde{\mathfrak{S}} = \cup_k \tilde{\mathfrak{S}}^{(k)}$, each of $\mathfrak{S}^{(k)}$ represents a non-split tree-type sub-walk $\tilde{\mathcal{W}}^{(k)}$,

$$\tilde{W} = \cup_k \tilde{\mathcal{W}}^{(k)}. \tag{3.10}$$

In this definition we assume that each sub-walk $\tilde{\mathcal{W}}^{(k)}$ is maximal by its length.

3.4.2. Tree-type sub-walks attached to $\check{\beta}$. Given $\mathcal{W}_{2s}$, let us consider the instants of time $0 \leq t_1 < t_2 < \dots t_L \leq 2s$ such that for all $i = 1, \dots, L$ the walk arrives at $\check{\beta}$ by the steps of $\check{\mathcal{W}}_{2\check{s}}$,

$$\mathcal{W}_{2s}(t_i) = \check{\beta} \quad \text{and} \quad \mathfrak{s}_{t_i} \in \check{\mathcal{W}}_{2\check{s}}. \tag{3.11}$$

We say that t_i are the $\breve{t}$-*arrival* instants of time of $\mathcal{W}_{2s}$. Let us consider a sub-walk that corresponds to the subset $\mathfrak{S}_{[t_i+1,t_{i+1}]} = \{\mathfrak{s}_t, t_i + 1 \le t \le t_{i+1}\} \subseteq \mathfrak{S}$; we denote this sub-walk by $\mathcal{W}_{[t_i,t_{i+1}]}$. In general situation, we also denote by $\mathcal{W}_{[t',t'']}$ a sub-walk which is not necessarily even and/or closed.

Let us consider the interval of time $[t_i + 1, t_{i+1} - 1]$ between two consecutive $\breve{t}$-arrivals at $\breve{\beta}$. It can happen that $\mathcal{W}_{2s}$ arrives at $\breve{\beta}$ at some instants of time $t' \in [t_i + 1, t_{i+1} - 1]$, $\mathcal{W}_{2s}(t') = \breve{\beta}$. We denote by $\tilde{t}_{(i)}$ the maximal value of t'.

Lemma 3.4. *The sub-walk $\mathcal{W}_{[t_i,\tilde{t}_{(i)}]}$ is of the tree structure and coincides with one of the maximal tree-type sub-walks $\tilde{\mathcal{W}}^{(k')}$ of (3.10).*

This statement means that the walk $\mathcal{W}_{2s}$ is such that after an arrival at $\breve{\beta}$ by a step of $\breve{\mathcal{W}}$ it creates a tree-type sub-walk $\mathcal{W}^{(k')}$, which is not interrupted by the steps of $\breve{\mathcal{W}}$, and that all steps performed after $\mathcal{W}^{(k')}$ belong again to $\breve{\mathcal{W}}$,

$$\{\mathfrak{s}_t,\ t_i + 1 \le t \le \tilde{t}_{(i)}\} \subseteq \tilde{\mathfrak{S}}, \quad \{\mathfrak{s}_t,\ \tilde{t}_{(i)} + 1 \le t \le t_{i+1}\} \subseteq \breve{\mathfrak{S}}.$$

P r o o f of Lemma 3.4. It is clear that the step $\mathfrak{s}_{\tilde{t}_{(i)}}$ does not belong to $\breve{\mathfrak{S}}$ and it is non-marked. Then this step makes a part of a non-split tree-type sub-walk $\tilde{\mathcal{W}}^{(k')} = \mathcal{W}_{[t'',\tilde{t}_{(i)}]}$ such that $\tilde{\mathcal{W}}^{(k')}(t'') = \breve{\beta}$. Then the previous step $\mathfrak{s}_{t''} \in \breve{\mathfrak{S}}$ is such that $\mathcal{W}_{2s}(t'') = \breve{\beta}$. Then $t'' = t_i$. The lemma is proved.

As a consequence of Lemma 3.4, the sub-walk $\mathcal{W}_{[t_i,\tilde{t}_{(i)}]} = \tilde{\mathcal{W}}^{(k')}$ is of the tree-type structure that ends at $\breve{\beta}$ by a non-marked step and therefore starts at $\breve{\beta}$ by the marked step. Let us consider the family of all marked exit edges from $\breve{\beta}$ performed by the marked steps on the interval of time $[t_i, \tilde{t}_{(i)}]$ and denote these edges of $\bar{\mathbb{E}} = \bar{\mathbb{E}}(\mathcal{W}_{2s})$ by $\tilde{\Delta}_i$. Regarding the non-empty subsets $\tilde{\Delta}_j$, we say that $\tilde{\Delta}_j$ represents the *exit sub-clusters of tree type* attached to $\breve{\beta}$ and denote by $d_j = |\tilde{\Delta}_j|$ its cardinality, $d_j \ge 1$, $j = 1, \dots, L'$. These tree-type sub-clusters are numerated in natural chronological order. Clearly, any non-empty tree-type sub-cluster is attributed to a uniquely determined $\breve{t}$-arrival instant at $\breve{\beta}$.

Regarding the even walk $\breve{\mathcal{W}}_{2\breve{s}}$, we can determine the corresponding Dyck path $\breve{\theta}_{2\breve{s}} = \theta(\breve{\mathcal{W}}_{2\breve{s}})$ and the tree $\breve{\mathcal{T}}_{\breve{s}} = \mathcal{T}(\breve{\theta})$. It is clear that $\breve{\mathcal{T}}_{\breve{s}}$ can be considered as a subtree of $\mathcal{T}_s = \mathcal{T}(\theta(\mathcal{W}_{2s}))$. One can determine the difference $\tilde{\mathcal{T}} = \mathcal{T}_s \diagdown \breve{\mathcal{T}}_{\breve{s}}$ such that it is represented by a collection of sub-trees $\tilde{\mathcal{T}}^{(j)}$. Not to overload the paper, we do not present rigorous definitions here.

Regarding the Catalan tree $\mathcal{T}(\theta_{2s})$, we say that the chronological run $\mathfrak{R}_{\mathcal{T}}$ is represented by $2s$ directed arcs ϖ_i, $1 \le i \le 2s$ drawn over $\mathcal{T}_s$ (see Fig. 1). This chronological run uniquely determines L arcs $\breve{\varpi}_l$, $1 \le l \le L$ that correspond to the arrival instants of time $\breve{t}_l$ at $\breve{\beta}$. The corresponding vertices υ_l of the tree $\mathcal{T}_s$ are also determined. It is clear that υ_l are not necessarily different for different l.

The sub-trees $\tilde{\mathcal{T}}^{(l)}$ are attached at υ_l and the chronological run over $\tilde{\mathcal{T}}^{(l)}$ starts immediately after the arc $\breve{\varpi}_l$ is drawn. We will say that these arcs $\breve{\varpi}_l$ represent the *nest cells* from where the sub-trees $\tilde{\mathcal{T}}^{(l)}$, $1 \le l \le L$ grow. It is clear that the sub-tree $\mathcal{T}_l$ has $d_l \ge 0$ edges attached to its root ρ_l represented by the vertex υ_l. Returning to $\mathcal{W}_{2s}$, we will say that the arrival instants of time $\breve{t}_l$ represent the *arrival cells* at $\breve{\beta}$. In the next sub-section, we will give a classification of the arrival cells at $\breve{\beta}$ which is a natural improvement of the approach proposed in [14].

3.4.3. Classification of arrival cells at $\breve{\beta}$ and BTS-instants. Let us consider a $\breve{t}$-arrival cell t_i (3.11). If the step $\mathfrak{s}_{t_i}$ of $\mathcal{W}_{2s}$ is marked, then we say that t_i is a *proper cell* at $\breve{\beta}$. If the step $\mathfrak{s}_{t_i}$ is non-marked and $\mathfrak{s}_{t_i} \in \check{\mathcal{W}} = \breve{\mathcal{W}} \setminus \hat{\mathcal{W}}$, then we say that t_i is a *mirror cell* at $\breve{\beta}$. If the step $\mathfrak{s}_{t_i} \in \hat{\mathcal{W}}$ is non-marked, then we say that t_i is an *imported cell* at $\breve{\beta}$.

Let us consider I proper cells $\breve{t}_i$ such that the corresponding step $\mathfrak{s}_{\breve{t}_i}$ belongs to $\breve{\mathfrak{S}}$. We denote by x_i the corresponding to $\breve{t}_i$ marked instants, $x_i = \xi_{\breve{t}_i}$, $1 \le i \le I$, and write that $\bar{x}_I = (x_1, \dots, x_I)$. Each proper cell x_i is associated with a number of corresponding mirror cells. We denote this number by $m_i \ge 0$ and write that $M = \sum_{i=1}^{I} m_i$ and $\bar{m}_I = (m_1, \dots, m_I)$.

Regarding the strongly reduced walk $\hat{\mathcal{W}}_{2\hat{s}}$, we denote by $\hat{t}_k$ the proper cells such that the steps $\mathfrak{s}_{\hat{t}_k} \in \hat{\mathfrak{S}}$. The corresponding to $\hat{t}_k$ marked instants will be denoted by z_k, $1 \le k \le K$. Then $\bar{z}_K = (z_1, \dots, z_K)$ and, clearly, $\varkappa(\breve{\beta}) = I + K$.

Regarding any walk $\mathcal{W}_{2s}$, we observe that if the set $\hat{\mathfrak{S}}$ is non-empty, then there exists at least one pair of the elements of $\hat{\mathfrak{S}}$, $(\mathfrak{s}', \mathfrak{s}'')$ such that $\mathfrak{s}'$ is a marked step of $\hat{\mathcal{W}}_{2\hat{s}}$, $\mathfrak{s}''$ is a non-marked one and $\mathfrak{s}''$ follows immediately after $\mathfrak{s}'$ in $\hat{\mathfrak{S}}$. We refer to each pair of this kind as to the pair of *broken tree structure* of $\mathcal{W}_{2s}$ or in abbreviated form, the BTS-pair of $\mathcal{W}_{2s}$. If τ' is the marked instant that corresponds to $\mathfrak{s}'$, we will simply say that τ' is the *BTS-instant* of $\mathcal{W}_{2s}$ [14]. We will refer to the vertex $\gamma = \mathcal{W}_{2s}(\xi_{\tau'})$ as to the *BTS-vertex* of the walk.

Regarding the strongly reduced walk $\hat{\mathcal{W}}$, let us consider a non-marked arrival step at $\breve{\beta}$ that we denote by $\bar{\mathfrak{s}} = \mathfrak{s}_{\bar{t}}$. Then one can uniquely determine the marked instant τ' such that all steps $\mathfrak{s}_t \in \hat{\mathfrak{S}}$ with $\xi_{\tau'} + 1 \le t \le \bar{t}$ are the non-marked ones. Let us denote by t'' the instant of time of the first non-marked step $\mathfrak{s}_{\bar{t}''} \in \hat{\mathfrak{S}}$ of this series of non-marked steps. Then $(\mathfrak{s}_{t'}, \mathfrak{s}_{t''})$ with $t' = \xi_{\tau'}$ is the BTS-pair of $\mathcal{W}_{2s}$ which corresponds to $\bar{t}$. We will say that $\bar{t}$ is attributed to the corresponding BTS-instant τ'. Several arrival instants $\bar{t}_i$ can be attributed to the same BTS-instant τ'. We will also say that the BTS-instant τ' *generates* the imported cells that are attributed to it.

Let us consider a BTS-instant τ such that $\mathcal{W}_{2s}(\xi_\tau) = \breve{\beta}$. As it is said above, we denote these marked instants by z_k, $1 \le k \le K$. Assuming that a marked BTS-instant z_k generates $f'_k \ge 0$ imported cells, we denote by $\varphi_1^{(k)}, \dots, \varphi_{f'_k}^{(k)}$ the positive numbers such that

$$\mathcal{W}_{2s}(\xi_{z_k} + \sum_{j=1}^{l} \varphi_j^{(k)}) = \breve\beta \quad \text{for all} \quad 1 \le l \le f_k'. \tag{3.12}$$

If for some $\tilde k$ we have $f'_{\tilde k} = 0$, then we will say that $z_{\tilde k}$ does not generate any imported cell at $\breve\beta$. We denote $\bar\varphi^{(k)} = (\varphi_1^{(k)}, \dots, \varphi_{f_k'}^{(k)})$.

Let us consider a BTS-instant τ that generates imported cells at $\breve\beta$ and such that $\mathcal{W}_{2s}(\xi_\tau) \ne \breve\beta$. We denote these BTS-instants by y_j, $1 \le j \le J$. Assuming that a marked BTS-instant y_j generates $f_j''+1$ imported cells, $f_j'' \ge 0$, we denote by $\ell_j, \psi_1^{(j)}, \dots, \psi_{f_j''}^{(j)}$ the positive numbers such that $\mathcal{W}_{2s}(\xi_{y_j} + \ell_j) = \breve\beta$ and

$$\mathcal{W}_{2s}\left(\xi_{y_j} + \ell_j + \sum_{l=1}^{k} \psi_l^{(j)}\right) = \breve\beta \quad \text{for all} \quad 1 \le k \le f_j''. \tag{3.13}$$

In this case we will say that the first arrival at $\breve\beta$ given by the instant of time $\xi_{y_j}+\ell_j$ represents the *principal* imported cell at $\breve\beta$. All subsequent arrivals at $\breve\beta$ given by (3.13) represent the *secondary* imported cells at $\breve\beta$. We will say that y_j is the *remote* BTS-instant with respect to $\breve\beta$ and will use denotations $\bar y_J = (y_1, \dots, y_J)$ and $\bar\ell_J = (\ell_1, \dots, \ell_J)$. We also denote $\bar\psi^{(j)} = (\psi_1^{(j)}, \dots, \psi_{f_j''}^{(j)})$.

All arrivals determined by (3.12) will also be referred to as the secondary imported cells at $\breve\beta$. We will say that the corresponding BTS-instant z_k is the *local* one with respect to $\breve\beta$.

For a given walk $\mathcal{W}_{2s}$, the proper, mirror and imported cells at its vertex of maximal exit degree are characterized by a set of parameters, $(\bar x, \bar m)_I$, $(\bar z, \Phi, \bar f')_K$, where $\Phi_K = (\bar\varphi^{(1)}, \dots \bar\varphi^{(K)})$, $\bar f_K' = (f_1', \dots, f_K')$ and $(\bar y, \bar\ell, \Psi, \bar f'')_J$, where $\Psi_J = (\bar\psi^{(1)}, \dots, \bar\psi^{(J)})$, $\bar f_J'' = (f_1'', \dots, f_J'')$. We also denote $F' = \sum_{k=1}^K f_k'$ and $F'' = \sum_{j=1}^J f_j''$.

Summing up, we observe that the vertex $\breve\beta$ with $\varkappa(\breve\beta) = I + K$ has the total number of cells given by $R = I + M + K + 2J + F$, where $F = F' + F''$. In what follows, we will use the following denotation for the set of parameters described above: $\left\{(\bar x, \bar m)_I, (\bar z, \Phi, \bar f')_K, (\bar y, \bar\ell, \Psi, \bar f'')_J\right\} = \langle \mathcal{P}_R \rangle$. It would be an instructive exercise to consider the example walk $\mathcal{W}_{16}$ from Fig. 1 and to determine its $\langle \mathcal{P}_R \rangle$.

4. Proof of Main Results

Remembering that $\varepsilon > \dfrac{3}{6+\phi} = \varepsilon_0$, let us choose

$$k_0 = \lfloor \frac{3}{\varepsilon - \varepsilon_0} \rfloor + 2 \ , \ \delta = \frac{\varepsilon_0 + \varepsilon}{6} \ , \ \text{and} \ \ U_n = n^{\delta}. \tag{4.1}$$

We assume k_0 to be an even number.

Our aim is to show that the averaged trace $\hat{\mathrm{L}}_{2s_n}^{(n,\rho)}$ (2.7) admits an upper bound in the limit $n, s_n \to \infty$, $s_n = \lfloor \chi n^{2/3} \rfloor$, $\chi > 0$ that we denote by $(n,s)_\chi \to \infty$. We also prove that the trajectories $\mathcal{I}_{2s}$ such that the graphs of their walks have multiple edges vanish in this limit. Then Theorem 2.3 will follow.

In the spirit of [21, 25], we consider the following partition of the sum (2.7):

$$\hat{\mathrm{M}}_{2s}^{(n,\rho)} = \mathbf{E}\,\mathrm{Tr}\left(\hat{H}^{(n,\rho_n)}\right)^{2s_n} = Z_{2s}^{(1)} + Z_{2s}^{(2)} + Z_{2s}^{(3)}, \tag{4.2}$$

where

- $Z_{2s}^{(1)}$ is the sum over the trajectories $\mathcal{I}_{2s} \in \mathcal{C}(\mathcal{W}_{2s})$ such that the graph $g(\mathcal{W}_{2s})$ has neither p-edges nor μ_3'-vertices;
- $Z_{2s}^{(2)}$ is the sum over the trajectories $\mathcal{I}_{2s} \in \mathcal{C}(\mathcal{W}_{2s})$ such that the graph $g(\mathcal{W}_{2s})$ has at least one p-edge or one μ_3'-vertex and the maximal exit degree $g(\mathcal{W}_{2s})$ is bounded, $\mathcal{D}(\mathcal{W}_{2s}) \le n^{\epsilon}$, and
- $Z_{2s}^{(3)}$ is the sum over the trajectories $\mathcal{I}_{2s}$ such that the graph $g(\mathcal{W}_{2s})$ has the maximal exit degree $\mathcal{D}(\mathcal{W}_{2s}) > n^{\epsilon}$.

As we will see further, the choice of $\epsilon = (\varepsilon - \varepsilon_0)/12$ is sufficient for our purposes. The sub-sum $Z_{2s}^{(1)}$ will also be represented as a sum of two parts in dependence of the presence of q-edges, or u-edges, or ν-vertices.

4.1. Estimate of $Z_{2s}^{(1)}$

Following the definitions of Section 3, we can write that

$$Z_{2s}^{(1)} = \sum_{u=1}^{s} \sum_{\langle \mathcal{S}^{(1)} \rangle} \sum_{\mathcal{G} \in \mathbb{G}(\langle \mathcal{S}^{(1)} \rangle)} \sum_{\langle \mathcal{G}_\Diamond \rangle_s} \sum_{\mathcal{W}_{2s} \in \mathbb{W}_{2s}^{(u)}(\langle \mathcal{G}_\Diamond \rangle_s)} \sum_{\mathcal{I}_{2s} \in \mathcal{C}(\mathcal{W}_{2s})} \hat{\Pi}_a(\mathcal{I}_{2s}) \Pi_b(\mathcal{I}_{2s}), \tag{4.3}$$

where $\mathcal{S}^{(1)} = (r, 0, q, \mu_2'', u_2; 0, \mu_3'', u_3; \bar{\nu})$ represents a particular case of the set of variables $\mathcal{S}$ (see sub-section 3.3) and the sum over $\langle \mathcal{S}^{(1)} \rangle$ runs over all non-negative integer values of its parameters such that condition (3.3) is verified. Let us represent $Z_{2s}^{(1)}$ as a sum of two terms,

$$Z_{2s}^{(1)} = Z_{2s}^{(1,1)} + Z_{2s}^{(1,2)},$$

where $Z_{2s}^{(1,1)}$ is the sum over the realizations of $\mathcal{S}^{(1,1)} = (r, 0, 0, \mu_2'', 0; 0, \mu_3'', 0; 0)$. Then the corresponding walks have neither q-edges, nor u-edges, nor ν-vertices.

To estimate the right-hand side of (4.3) from above, we use Lemma 3.3 of Section 3 with $p = q = u_2 = \mu_3' = u_3 = |\bar{\nu}|_1 = 0$. Then $\sigma = r + \mu_2'' + \mu_3''$, and we can write that

$$\exp\left\{-\frac{(s-\sigma)^2}{2n}\right\} \le \exp\left\{-\frac{s^2}{2n}\right\} \left(e^{\frac{s}{n}}\right)^{\mu_2''+r+\mu_3''}. \tag{4.4}$$

Applying the upper bound (3.7) to the right-hand side of (4.3), replacing the sum over $\langle \mathcal{S}^{(1)} \rangle$ by the sum over all possible values of the variables r, μ_2'', and μ_3'' and using (4.4), we can write the inequality

$$Z_{2s}^{(1;1)} \le nV_2^s \exp\left\{-\frac{s^2}{2n}\right\} \sum_{u=1}^{s} |\Theta_{2s}^{(u)}| \sum_{r=0}^{s} \frac{1}{r!} \left(\frac{6su}{n} e^{s/n}\right)^r$$
$$\times \sum_{\mu_2''=0}^{s} \frac{1}{\mu_2''!} \left(\frac{s^2}{2n} e^{s/n}\right)^{\mu_2''} \sum_{\mu_3''=0}^{s} \frac{1}{\mu_3''!} \left(\frac{3s^3}{2n^2} e^{s/n}\right)^{\mu_3''}.$$

Taking into account that $e^{s/n} \le 2$ for large values of n, we get the bound

$$Z_{2s}^{(1,1)} \le nV_2^s \, \mathrm{t}_s \, \mathrm{B}_s(12\chi^{3/2}) \exp\left\{\frac{s^2}{2n}\left(e^{s/n} - 1\right) + 3\chi^3\right\},$$

where we denoted

$$\mathrm{B}_s(x) = \frac{1}{\mathrm{t}_s} \sum_{u=1}^{s} |\Theta_{2s}^{(u)}| \exp\left\{\frac{xu}{\sqrt{s}}\right\} = \mathbf{E}_s\left(\exp\{x\theta^*/\sqrt{s}\}\right). \tag{4.5}$$

In (4.5), $\mathbf{E}_s(\cdot)$ denotes the mathematical expectation with respect to the uniform measure on the set of Catalan trees $\mathbb{T}_s$. It is proved in [12] that $\mathrm{B}_s(x)$ in the limit of infinite s is given by the exponential moment of the maximum of the normalized Brownian excursion. Using the convergence $\mathrm{B}(x) = \lim_{s\to\infty} \mathrm{B}_s(x)$ [12] and elementary relation

$$\frac{n\mathrm{t}_{s_n}}{4^{s_n}} = \frac{n\,(2s_n)!}{4^{s_n}\, s_n!\,(s_n+1)!} = \frac{1}{\sqrt{\pi\chi^3}}(1 + o(1)), \quad s_n = \chi n^{2/3},$$

that follows from the Stirling formula, we conclude that

$$\limsup_{n\to\infty} \mathrm{Z}_{2s_n}^{(1,1)} \le \frac{1}{\sqrt{\pi\chi^3}}\, \mathrm{B}(12\chi^{3/2})\, e^{4\chi^3}.$$

Let us consider the sub-sum $Z_{2s}^{(1,2)}$. Applying the upper bound (3.7) to the right-hand side of (4.3), using the analog of (4.4) and replacing the sum over

$\langle \mathcal{S}^{(1,2)} \rangle$ by the sum over all possible values of its variables, we can write the inequality

$$Z_{2s}^{(1,2)} \le n V_2^s \sum_{u=1}^{s} |\Theta_{2s}^{(u)}| \sum_{r=0}^{s} \frac{1}{r!} \left(\frac{12su}{n} \right)^r \sum_{\mu_2''=0}^{s} \exp\left\{ -\frac{s^2}{2n} \right\} \frac{1}{\mu_2''!} \left(\frac{s^2}{2n} e^{s/n} \right)^{\mu_2''}$$

$$\times \sum_{\mu_3''=0}^{s} \frac{1}{\mu_3''!} \left(\frac{3s^3}{n^2} \right)^{\mu_3''} \sum_{u_2+u_3+|\bar\nu|_1 \ge 1} \frac{1}{u_2!} \left(\frac{16k_0^4 sr \hat U_n^2}{\rho} \right)^{u_2}$$

$$\times \frac{1}{u_3!} \left(\frac{32k_0^5 s\mu_3'' \hat U_n^2}{\rho} \right)^{u_3} \prod_{k=k_0+1}^{s} \frac{1}{\nu_k!} \left(n \left(\frac{C_1 s \hat U_n^2}{\rho} \right)^k \right)^{\nu_k}, \tag{4.6}$$

where we denoted $C_1 = \sup_{k\ge 2} \frac{2k}{(k!)^{1/k}}$ and used the relation $e^{s/n} \le 2$.

Regarding the last product of (4.6), we can write that

$$n \left(\frac{C_1 s \hat U_n^2}{\rho} \right)^k = A_n^{(k_0)} \left(\frac{C_1 s \hat U_n^2}{\rho} \right)^{k-k_0}, \quad A_n^{(k_0)} = n \left(\frac{C_1 s \hat U_n^2}{\rho} \right)^{k_0}.$$

It follows from (4.1) that $A_n^{(k_0)} \le \left(\frac{C_1 \chi}{V_2} \right)^{k_0} n^{-2\epsilon}$ with $\epsilon = (\varepsilon - \varepsilon_0)/12$ and therefore

$$n \left(\frac{C_1 s \hat U_n^2}{\rho} \right)^k \le \left(\frac{C_1 \chi}{V_2} \right)^{k_0} n^{-2\epsilon} \left(\frac{C_1 \chi}{V_2} n^{-4\epsilon} \right)^{k-k_0} \tag{4.7}$$

where we have taken into account that $n^{2/3} U_n^2/\rho \le n^{-4\epsilon}$. Then the following relation is true:

$$\sum_{|\bar\nu|_1 = \jmath_1}^{s} \prod_{k'=1}^{s} \frac{1}{\nu_{k_0+k'}!} \left(A_n^{(k_0)} \left(\frac{C_1 s \hat U_n^2}{\rho} \right)^{k'} \right)^{\nu_{k_0+k'}}$$

$$\le \exp\left\{ \left(\frac{C_1 \chi}{V_2} \right)^{k_0} n^{-2\epsilon} \sum_{k'=1}^{\infty} \left(\frac{C_1 \chi}{V_2} n^{-4\epsilon} \right)^{k'} \right\} - \jmath_1 \le \exp\left\{ \left(\frac{C_1 \chi}{V_2} \right)^{k_0} n^{-2\epsilon} \right\} - \jmath_1,$$

where $\jmath_1$ takes the values 0 or 1. Denoting $C_2 = 64 k_0^5 \chi / V_2$, we can also write that

$$\sum_{u_3=\jmath_2}^{s} \frac{1}{u_3!} \left(\frac{32k_0^5 s\mu_3'' \hat U_n^2}{\rho} \right)^{u_3} \le \begin{cases} \exp\{C_2\, \mu_3''\, n^{-4\epsilon}\}, & \text{if } \jmath_2 = 0, \\ C_2\, n^{-4\epsilon}\, e^{\mu_3''}, & \text{if } \jmath_2 = 1 \end{cases}$$

for sufficiently large values of n. A similar computation shows that

$$\sum_{u_2=\jmath_3}^{s} \frac{1}{u_2!} \left(\frac{16k_0^4 sr \hat U_n^2}{\rho} \right)^{u_2} \le \begin{cases} \exp\{C_2\, r\, n^{-4\epsilon}\}, & \text{if } \jmath_3 = 0, \\ C_2\, n^{-4\epsilon}\, e^{r}, & \text{if } \jmath_3 = 1. \end{cases}$$

Using the elementary inequality

$$\sum_{j_1+j_2+j_3\ge 1}\{\cdot\} \le \sum_{j_1\ge 1, j_2\ge 0, j_3\ge 0}\{\cdot\} + \sum_{j_1\ge 0, j_2\ge 1, j_3\ge 0}\{\cdot\} + \sum_{j_1\ge 0, j_2\ge 0, j_3\ge 1}\{\cdot\}$$

and accepting that n is such that $\exp\{s/n + C_2 n^{-4\epsilon}\} \le 2$, we can see that relation (4.6) implies the following inequality:

$$Z_{2s}^{(1,2)} \le nV_2^s \sum_{u=1}^{s} |\Theta_{2s}^{(u)}| \exp\left\{12e\chi^{3/2}\frac{u}{\sqrt{s}}\right\}$$

$$\times \exp\left\{\frac{s^2}{2n}\left(e^{s/n}-1\right) + 3e\chi^3 + 2C_3 n^{-2\epsilon}\right\}\left(2C_2 n^{-4\epsilon} + 2C_3 n^{-2\epsilon}\right),$$

where $C_3 = 2C_1\chi/V_2$. Now it is clear that $Z_{2s}^{(1,2)} = o(1)$, $\quad (n,s)_\chi \to \infty$, and therefore

$$\limsup_{(n,s)_\chi\to\infty} Z_{2s}^{(1)} \le \frac{1}{\sqrt{\pi\chi^3}}\ \mathrm{B}(12\chi^{3/2})\ e^{4\chi^3}. \tag{4.8}$$

4.2. Estimate of $Z_{2s}^{(2)}$

Rewriting relation (4.3) with $\mathcal{S}^{(1)}$ replaced by $\mathcal{S}^{(2)} = (r, p, q, \mu_2'', u_2; \mu_3', \mu_3'', u_3; \bar\nu)$ and using the result of Lemma 3.3 together with (4.4), we can write that

$$Z_{2s}^{(2)} \le nV_2^s \sum_{\mathrm{D}=1}^{n^\epsilon}\sum_{u=1}^{s}|\Theta_{2s}^{(u)}| \sum_{\mu_2''=0}^{s} \exp\left\{-\frac{s^2}{2n}\right\}\frac{1}{\mu_2''!}\left(\frac{s^2}{2n}e^{s/n}\right)^{\mu_2''}\sum_{r=0}^{s}\frac{1}{r!}\left(\frac{12su}{n}\right)^{r}$$

$$\times \sum_{\substack{p,\mu_3':\\ p+\mu_3'\ge 1}} \frac{1}{p!}\left(\frac{6s\mathrm{D}\hat U_n^2}{\rho}\right)^{p}\frac{1}{\mu_3'!}\left(\frac{9(\mathrm{D}+k_0)s^2\hat U_n^2}{n\rho}\right)^{\mu_3'}\sum_{u_2=0}^{s}\frac{1}{u_2!}\left(\frac{32k_0^4 sr\hat U_n^2}{\rho}\right)^{u_2}$$

$$\times \sum_{q=0}^{s}\frac{1}{q!}\left(\frac{6sk_0\hat U_n^2}{\rho}\right)^{q}\sum_{\mu_3''=0}^{s}\frac{1}{\mu_3!}\left(\frac{3s^3}{n^2}\right)^{\mu_3''}\sum_{u_3=0}^{s}\frac{1}{u_3!}\left(\frac{64k_0^5 s\mu_3\hat U_n^2}{\rho}\right)^{u_3}$$

$$\times \sum_{|\bar\nu|\ge 0}\prod_{k'=1}^{s}\frac{1}{\nu_k!}\left(A_n^{(k_0)}\left(\frac{C_1 s\hat U_n^2}{\rho}\right)^{k'}\right)^{\nu_k}. \tag{4.9}$$

Taking into account that

$$\frac{s\mathrm{D}U_n^2}{V_2\rho} \le \frac{\chi}{V_2}n^{-3\epsilon} \quad \text{and} \quad \frac{\mathrm{D}s^2U_n^2}{V_2 n\rho} \le \frac{\chi^2}{V_2}n^{-3\epsilon}6\chi^2 n^{-1/3} \le \frac{\chi^2}{V_2}n^{-3\epsilon}, \tag{4.10}$$

and repeating the computations of the previous subsection, we deduce from (4.9) the following estimate:

$$Z_{2s}^{(2)} \leq \frac{48\chi}{V_2} n^{-2\epsilon} \, nV_2^s \sum_{u=1}^{s} |\Theta_{2s}^{(u)}| \exp\left\{24\chi^{3/2} \frac{u}{\sqrt{s}} + 8\chi^3 + 2A_n^{(k_0)}\right\}.$$

Then

$$Z_{2s}^{(2)} = o(1), \quad \text{as} \quad (n,s)_\chi \to \infty. \tag{4.11}$$

4.3. Estimate of $Z^{(3)}$

In this subsection we estimate the sub-sum of (2.5) corresponding to the walks that have a vertex $\breve{\beta}$ of large exit degree D. In previous works (see, e.g., [7, 8, 25]), it is observed that in the case of L arrival cells at $\breve{\beta}$, the underlying Dyck path and the corresponding tree $\mathcal{T}_s$ have to have at least one vertex v whose exit degree is not less than D/L. Then the collection of these trees has an exponentially small cardinality with respect to the total number of trees, with the exponential factor determined by a value proportional to D/L.

In the case of dilute random matrices, this observation is not sufficient for getting the needed estimates of the cardinality of the set of such walks. Roughly speaking, our aim is to get the upper bound related with the exponential factor determined by the values proportional to D. Let us briefly outline the main idea of the proof of corresponding bounds.

We consider the families of walks such that their vertex of maximal exit degree $\breve{\beta}$ is characterized by a collection of certain parameters $\mathcal{P}$. This set of parameters described in Section 3 involves the marked BTS-instants τ_i and the corresponding descending lengths ℓ, φ, ψ. Given $\mathcal{P}$, the positions of the nest cells in the tree $\mathcal{T}_s$ with given exit sub-clusters are uniquely determined. This implies a fairly strong exponential estimate for the number of corresponding trees (see D-lemma of Section 6). On the other hand, it appears that the collection of the parameters $\mathcal{P}$ can be naturally inserted into the diagrams $\mathcal{G}$ that we use to estimate the number of walks in the corresponding classes.

4.3.1. Diagrams and classes of walks. Given u, D and $\mathcal{S}$, we consider a family $\mathbb{W}_{2s}^{(u)}(\mathrm{D}; \mathcal{S}, \langle \mathcal{P}_R \rangle)$ given by the walks such that their vertex of maximal exit degree $\breve{\beta}$ is characterized by the following set of numerical data (see subsection 3.4):

$$\langle \mathcal{P}_R \rangle = \left\{ (\bar{x}, \bar{m})_I, (\bar{z}, \Phi, \bar{f}')_K, (\bar{y}, \bar{\ell}, \Psi, \bar{f}'')_J \right\}, \quad R = I + M + K + 2J + F. \tag{4.12}$$

It is convenient to write that $\langle \mathcal{P}_R \rangle = (\langle \mathcal{Q}_R \rangle, \langle \mathcal{H}_R \rangle)$ with $\langle \mathcal{Q}_R \rangle = \{\bar{x}_I, \bar{z}_K, \bar{y}_J\}$ and $\langle \mathcal{H}_R \rangle = \left\{ \bar{m}_I, (\Phi, \bar{f}')_K, (\bar{\ell}, \Psi, \bar{f}'')_J \right\}$. In what follows, when no confusion can arise, we omit the angles in the denotations of $\mathcal{P}_R$ (4.12), as well as in $\mathcal{Q}_R$ and $\mathcal{H}_R$, that denote numerical realizations of the sets of the parameters $\mathcal{P}_R$, $\mathcal{Q}_N$ and $\mathcal{H}_R$, respectively. In denotations $\mathcal{Q}_R$ and $\mathcal{H}_R$, we keep the subscript R to indicate that these values are taken from one common set (4.12).

Let us describe the construction of the family $\mathbb{W}_{2s}^{(u)}(\mathrm{D}; \mathcal{S}, \mathcal{P}_R)$. Given $\mathcal{S}$, we build a diagram $\mathcal{G}(\mathcal{S})$ as it is done in Lemma 3.2 (see Section 3). Regarding the set of vertices $\mathcal{V}(\mathcal{G})$, we add to it a vertex $\breve{v}$ of the self-intersection degree $\varkappa(\breve{v}) = N = I + K$, i.e., a vertex with $I + K$ edge-windows attached. We denote the diagram obtained by $\mathcal{G}^*(\mathcal{S})$.

The next step is to distribute the J labels over the edge-windows of $\mathcal{G}$ that will be filled by the values $\bar{y}_J$. We refer to these labels as to the *y-labels*. The windows with y-labels will represent the marked instants of the remote BTS-pairs. Therefore one cannot use the first arrival edges attached to the vertices of $\mathcal{G}$. The edge-windows attached to the vertices of $\mathbb{M}_2''$ cannot also be used. Thus, we have $\mu_2' + u_2 + \mu_3 + u_3 + |\bar{\nu}|_1$ windows of $\mathcal{G}$ in our disposition and then the number of the ways to distribute J labels is bounded from above by $\binom{\mu_2'+u_2+\mu_3+u_3+|\bar{\nu}|_1}{J}$. The following upper bound will be useful for us:

$$\binom{\mu_2' + u_2 + \mu_3 + u_3 + |\bar{\nu}|_1}{J} \le \frac{(\mu_2' + u_2 + \mu_3 + u_3 + |\bar{\nu}|_1)^J}{J!}$$

$$\le \frac{1}{h^J} \exp\{h(\mu_2' + u_2 + \mu_3 + u_3 + |\bar{\nu}|_1)\}, \quad h \ge h_0 > 1. \tag{4.13}$$

We denote by $\mathcal{G}_{\langle\mathcal{Q}\rangle}(\mathcal{S}) = \mathcal{G}_{\mathcal{Q}}(\mathcal{S})$ a diagram obtained from $\mathcal{G}(\mathcal{S})$ by insertion of the marked instants $\langle \mathcal{Q}_R\rangle$ into the chosen J edge-windows of $\mathcal{G}(\mathcal{S})$.

The collection of blue, green and black edge-windows of $\mathcal{G}_{\mathcal{Q}}(\mathcal{S})$ that remain free will be denoted by $\overset{\diamond}{\mathcal{G}}_{\mathcal{Q}}(\mathcal{S})$. We denote by

$$\langle \overset{\diamond}{\mathcal{G}}_{\mathcal{Q}}(\mathcal{S})\rangle_s = \langle \mathcal{G}_{\triangleleft}\rangle_s$$

a realization of the values in these blue, green and black windows of $\mathcal{G}_{\langle\mathcal{Q}\rangle}(\mathcal{S})$. This gives the realization of the diagram $\mathcal{G}^*(\mathcal{S})$ that we denote as follows:

$$\langle \mathcal{G}^\star\rangle_s = \langle \breve{v}\rangle \uplus \langle \mathcal{G}_{\triangleleft}\rangle_s,$$

where $\langle \breve{v}\rangle$ denotes the realization of the edge-windows attached to $\breve{v}$ given by the values $(\bar{x}_I, \bar{z}_K)$. The red f-edges of $\mathcal{G}_{\mathcal{Q}}(S)$ are denoted by $\mathcal{G}_\circ$ as before.

The maximal exit degree of a walk $\mathcal{W}_{2s} \in \mathbb{W}_{2s}^{(u)}(\mathrm{D}; \mathcal{S}, \breve{\beta}, \mathcal{P}_R)$ can be represented as follows: $\mathrm{D} = \breve{D} + \tilde{D}$, where $\breve{D}$ is the number of marked edges of the form $(\breve{\beta}, \gamma)$ that belong to the reduced walk $\breve{\mathcal{W}}$ (3.9). It is known that the number of marked exits from $\breve{\beta}$ is equal to the number of non-marked arrival steps at $\breve{\beta}$,

$\breve{D} = M + F + J$ and that $F \le K$ (see [14] and also Lemma 5.1 of [8]). It is not hard to see that $M \le I - 1$. Then we can write that

$$\tilde{D} = \mathrm{D} - M - F - J \ge \mathrm{D} - I - K - J. \tag{4.14}$$

We can see that in a particular $\mathcal{W}_{2s}$, $\tilde{D}$ edges are attributed to R proper and imported cells at $\breve{\beta}$ and $R \le 2(I + J + K)$. Let us denote by $\bar{d}_R = (d_1, \dots, d_R)$ a particular distribution of $\tilde{D}$ balls over R ordered boxes, $\sum_{i=1}^{d} d_i = \tilde{D} = D_R$. Then the total number of different sets $\bar{d}_R$ is bounded by

$$\binom{D_R + R - 1}{R - 1} \le \binom{D_R + L - 1}{L - 1}, \quad L = 2(I + J + K). \tag{4.15}$$

Having determined all of the values described above, we denote by

$$\mathbb{W}_{2s}^{(u)}(\mathrm{D}, \bar{d}_R; \langle \mathcal{G}^{\star} \rangle_s, \langle \mathcal{H}_R \rangle, \Upsilon) = \mathbb{W}_{2s}^{(u)}(\bar{d}_R; \langle \mathcal{G}^{\star} \rangle_s, \mathcal{H}_R, \Upsilon) \tag{4.16}$$

the family of walks $\mathcal{W}_{2s}$ with $\theta^*(\mathcal{W}_{2s}) = u$ and such that $\mathcal{W}_{2s}$ follow the given rule Υ of non-marked continuations.

4.3.2. Exponential estimate and the upper bound. It is clear that the number of ways to attribute F' imported cells to K marked instants and F'' imported cells to J marked instants are bounded by

$$\binom{F' + K - 1}{K - 1} \le 2^{F'+K} \quad \text{and} \quad \frac{F'' + J - 1}{J - 1} \le 2^{F''+J},$$

respectively,and the number of ways to distribute $M \le I$ mirror cells over I marked arrivals at $\breve{\beta}$ is bounded by 2^{2I}. The sub-sum $Z_{2s}^{(3)}$ can be bounded by the following expression (cf. (4.3)):

$$\begin{aligned} Z_{2s}^{(3)} \le & \sum_{\mathrm{D}=\lfloor n^{\epsilon} \rfloor}^{s} \sum_{u=1}^{s} \sum_{I,K:\, I+K \ge 1} \sum_{M=0}^{I} \sum_{J=0}^{s} \sum_{F',F'' \ge 0} \sum_{\bar{x}_I, \bar{y}_J, \bar{z}_K} 2^{F'+F''+K+J+2I} \\ & \times \sum_{\langle \mathcal{S} \rangle} \sum_{\mathcal{G}(\mathcal{S})} \sum_{\langle \mathcal{Q}_R \rangle \in \mathcal{G}(\mathcal{S})} \sum_{\langle \mathcal{G}_{\triangleleft} \rangle_s} \sum_{\bar{d}_R} \sum_{\Upsilon} \sum_{\langle \mathcal{H}_R \rangle} \Pi(\breve{\beta}) \, |\Upsilon_{\breve{\beta}}| \\ & \times \sum_{\mathcal{W}_{2s} \in \mathbb{W}_{2s}^{\star}} \sum_{\mathcal{I}_{2s} \in \mathcal{C}(\mathcal{W}_{2s})} \hat{\Pi}_a(\mathcal{I}_{2s}) \, \Pi_b(I_{2s}), \end{aligned} \tag{4.17}$$

where $\Pi(\breve{\beta})$ is the weight of the edges attached to $\breve{\beta}$, and $|\Upsilon_{\breve{\beta}}|$ stands for the estimate of the local rule of non-marked passages. In (4.17), we also introduce a denotation $\mathbb{W}_{2s}^{\star} = \mathbb{W}_{2s}^{(u)}(\bar{d}_R; \langle \mathcal{G}^{\star} \rangle_s, \mathcal{H}_R, \Upsilon)$. Assuming that $\mathcal{G}^{\star}$ is such that

$\mu_2' = r + p + q$, we use the following upper bound (see Lemma 6.3 of Section 6):

$$\sum_{u=1}^{s} \sum_{\mathcal{H}_R} |\mathbb{W}_{2s}^{(u)}(\mathrm{D}, \bar{d}_R; \langle \mathcal{G}^\star \rangle_s, \mathcal{H}_R, \Upsilon)| \le s^{r/2}\, B_r\, 2^r\, \mathrm{D}^p\, k_0^q\, (s(\mathrm{D} + k_0))^{\mu_3'}$$

$$\times\, 4^R\, (D_R + 1)\, e^{-\eta D_R}\, \mathrm{t}_s, \tag{4.18}$$

where $B_r = \sup_{s\ge 1} \mathbf{E}_s((\theta^*/\sqrt{s})^r)$ (see (4.5)) and $\Delta_R = \tilde{D} = \sum_{r=1}^{R} d_R$. Let us note that (4.14) implies that

$$4^R\, (D_R + 1)\, e^{-\eta \mathrm{D}_R} \le s\, 4^L\, e^{\eta(I+J+K)}\, e^{-\eta \mathrm{D}}. \tag{4.19}$$

It follows from (4.15) that $\sum_{\bar{d}_R} 1 \le \binom{\mathrm{D}+L-1}{L-1}$, and elementary analysis shows that

$$\sup_{L\ge 1} \frac{4^L}{h_0^L}\, \frac{(\mathrm{D} + L - 1)!}{\mathrm{D}!\,(L-1)!} \le \exp\left\{ \frac{4e\mathrm{D}}{h_0} \right\}. \tag{4.20}$$

The product $\Pi(\breve{\beta})\, |\Upsilon_{\breve{\beta}}|$ can be bounded with the help of the following inequalities that are true for $h \ge 1$ and sufficiently large values of n (see Section 5):

$$e^{h(I+K)}\, \Pi(\breve{\beta})\, |\Upsilon_{\breve{\beta}}| \le\, e^{h(I+K)} \frac{s^{I+K} (C_1 U^2)^{I+K}}{\rho^{I+K-1}}$$

$$\le \frac{1}{h^{I+K}} \begin{cases} (hse^h)^{k_0}, & \text{if } I + K \le k_0, \\ e^{hk_0}\, n^{-\epsilon} (he^h n^{-\epsilon/2})^{I+K-k_0}, & \text{if } I + K \ge k_0 + 1. \end{cases} \tag{4.21}$$

Finally, it is not difficult to see that

$$\sum_{\bar{y}_J} \sum_{\langle \mathcal{G}_\triangleleft \rangle} 1 = \sum_{\langle \mathcal{G}_\Diamond \rangle} 1\,, \tag{4.22}$$

where the last sum is estimated by the right-hand side of inequality $(3.5a)$.

Taking into account the upper bound $(3.5a)$ and relations (4.4), (4.13), (4.18)–(4.22) and accepting that $n^\epsilon \ge e^{hk_0}$, we deduce from (4.17) the inequality

$$Z_{2s}^{(3)} \le\; (hse^h)^{k_0+3} \sum_{\mathrm{D}=\lfloor n^\epsilon \rfloor}^{s} \sum_{I,K:\, I+K\ge 1} \sum_{J=0}^{s} \frac{(8h_0)^{2(I+J+K)}}{h^{I+J+K}} \exp\left\{ -\eta \mathrm{D} + \frac{4e\mathrm{D}}{h_0} \right\}$$

$$\times \exp\left\{ \frac{s^2}{2n} \left(e^{s/n} - 1 \right) \right\} \sum_{r,p,q\ge 0} B_r \cdot \mathrm{H}_{(\mathcal{S},2)}^{(\sqrt{s},\mathrm{D},k_0)}(e^h)$$

$$\times \sum_{\mu_3', \mu_3'', u_3 \ge 0} \mathrm{H}_{(\mathcal{S},3)}^{(\mathrm{D},k_0)}(e^h) \sum_{|\bar{\nu}|_1 \ge 0} \mathrm{H}_{(\mathcal{S},\bar{\nu})}^{(k_0+1)}(e^h), \tag{4.23}$$

where denotations (3.7) are used.

Assuming that

$$\lim_{s\to\infty} \sum_{r=0}^{s} \frac{(2\chi^{3/2}e^h)^r}{r!}\, B_r = \mathrm{B}(2\chi^{3/2}e^h), \tag{4.24}$$

the choice of $h = (8h_0+1)^2+1$ and $h_0 = 8/\eta$ in the right-hand side of (4.23) is sufficient for the relation

$$Z_{2s}^{(3)} = O\left(s^{k_0+4}\exp\{-\eta n^{\epsilon}/2\}\right) = o(1) \tag{4.25}$$

to hold in the limit $(n,s)_\chi \to \infty$.

Relation (4.24) can be proved provided $B_r = \sup_{s\ge 1} B_r^{(s)} = \lim_{s\to\infty} B_r^{(s)}$. This statement would follow from the monotonicity of $B_r^{(s)}$ with respect to s. We did not find any related reference in the literature but assume that this monotonicity is true. We do not study this question in the present paper. In fact, the existence of an upper bound of the left-hand side of (4.24) would be sufficient for us.

4.4. Proof of Theorem 2.1 and Theorem 2.2

Using the standard arguments of the probability theory, we can write that

$$\mathbf{P}\left(\left\{\omega : \hat\lambda_{\max}^{(n,\rho_n)} > 2v\left(1+\frac{x}{n^{2/3}}\right)\right\}\right) \le \frac{\mathbf{E}\,\mathrm{Tr}(\hat H^{(n,\rho_n)})^{2s_n}}{\left(2v(1+xn^{-2/3})\right)^{2s_n}}, \tag{4.26}$$

where $\hat\lambda_{\max}^{(n,\rho_n)}$ is the maximal in absolute value eigenvalue of $\hat H^{(n,\rho_n)}$.

Relations (4.2), (4.8), (4.11) and (4.25) show that

$$\limsup_{n\to\infty} \mathbf{E}\,\mathrm{Tr}(\hat H^{(n,\rho_n)})^{2s_n} = \limsup_{n\to\infty}\left(Z_{2s_n}^{(1,1)} + Z_{2s_n}^{(1,2)} + Z_{2s_n}^{(2)} + Z_{2s_n}^{(3)}\right)$$

$$\le \frac{1}{\sqrt{\pi\chi^3}}\, e^{4\chi^3}\,\mathrm{B}(6\chi^{3/2}) = \mathfrak{M}(\chi). \tag{4.27}$$

It follows from (4.26) and (4.27) that

$$\limsup_{n\to\infty} \mathbf{P}\left\{\hat\lambda_{\max}^{(n)} > 2v\left(1+\frac{x}{n^{2/3}}\right)\right\} \le \inf_{\chi>0} \mathfrak{M}(\chi)e^{-x\chi} = \mathfrak{P}(x). \tag{4.28}$$

Let us consider the subset $\Lambda_n \subseteq \Omega$, $\Lambda_n = \cap_{1\le i<j\le n}\{\omega : |a_{ij}(\omega)| \le U_n\}$. Denoting $\hat\lambda_{\max}^{(n,\rho_n)} = \lambda_{\max}(\hat H^{(n,\rho_n)})$, we can write that

$$\mathbf{P}\left(\left\{\omega : \lambda_{\max}^{(n,\rho_n)} > y\right\}\right) = \mathbf{P}\left(\left\{\omega : \hat\lambda_{\max}^{(n,\rho_n)} > y\right\}\right) + \mathbf{P}\left(\left\{\omega : \lambda_{\max}^{(n,\rho_n)} > y\right\} \cap \bar\Lambda_n\right), \tag{4.29}$$

where $\bar\Lambda_n = \Omega \setminus \Lambda_n$. Clearly,

$$\mathbf{P}\left\{\bar\Lambda_n\right\} \le \sum_{1\le i<j\le n} \mathbf{P}\left\{|a_{ij}| > U_n\right\} \le n^2 \frac{\mathbf{E}|a_{ij}|^{12+2\phi}}{U_n^{12+2\phi}}. \tag{4.30}$$

The choice of δ given by (4.1) is sufficient for the right-hand side of (4.30) to vanish in the limit $n \to \infty$. Then we can deduce from (4.28) and (4.29) the following estimate, (cf. (2.3)),

$$\limsup_{n\to\infty} \mathbf{P}\left\{\lambda_{\max}^{(n,\rho_n)} > 2v\left(1+\frac{x}{n^{2/3}}\right)\right\} \le \mathfrak{P}(x). \tag{4.31}$$

Theorem 2.1 is proved. ∎

Returning to (4.27), we observe that $Z^{(1,2)}_{2s_n} + Z^{(2)}_{2s_n} + Z^{(3)}_{2s_n} = o(1)$ as $n\to\infty$. This means that the upper bound $\mathfrak{M}(\chi)$ is the same as if one considers the moments of Wigner random matrices $H^{(n,n)}$ (2.1) provided $V_{12+2\phi} < \infty$. The same is true for the random matrices A_n of Gaussian Orthogonal Ensemble [16]. The limit of the corresponding sub-sum $Z^{(1,1)}_{2s_n}(A_n)$ exists [25], $\lim_{n\to\infty} Z^{(1,1)}_{2s_n}(A_n) = \mathcal{M}_{\rm GOE}(\chi)$, and therefore we conclude about the existence of the limit

$$\lim_{n\to\infty} \mathbf{E}\,\mathrm{Tr}(\hat H^{(n,\rho_n)})^{2s_n} = \mathcal{M}_{\rm GOE}(\chi). \tag{4.32}$$

Let us prove Theorem 2.3. Regarding the moments of random matrices $\tilde H^{(n,\rho_n)}$, we can use the same representation as before and write that

$$\tilde{\mathrm{M}}^{(n,\rho_n)}_{2s_n} = \mathbf{E}\,\mathrm{Tr}\left(\tilde H^{(n,\rho_n)}\right)^{2s_n} = \tilde Z^{(1,1)}_{2s_n} + \tilde Z^{(1,2)}_{2s_n} + \tilde Z^{(2)}_{2s_n} + \tilde Z^{(3)}_{2s_n}, \tag{4.33}$$

where $\tilde Z_{2s_n}$ are determined in the same way as it is done in (4.2). It is easy to see that the estimate of $\tilde Z^{(1,1)}_{2s_n}$ is the same as that of $Z^{(1,1)}_{2s_n}$.

To get the estimate of $\tilde{\mathrm{Z}}^{(1,2)}_{2s_n}$, we repeat the computations used to estimate $Z^{(1,2)}_{2s_n}$ with the only difference that (4.7) is replaced by the expression

$$n\frac{(C_1U^2s)^k}{(v^2\rho)^k} = \left(\frac{C_1U^2}{v^2}\right)^k n^{1-2\varepsilon k_0/3-2\varepsilon(k-k_0)/3}, \quad k\ge k_0+1.$$

Then the choice of the technical value $k_0 = \lfloor \frac{3}{2\varepsilon}\rfloor + 1$ (cf. (4.1)) is sufficient to conclude that $\tilde Z^{(1,2)}_{2s_n} = o(1)$ as $n\to\infty$. The same concerns the estimates of $\tilde Z^{(2)}_{2s_n}$ and $\tilde Z^{(3)}_{2s_n}$.

These arguments show that

$$\limsup_{n\to\infty} \tilde{\mathrm{M}}^{(n,\rho_n)}_{2s_n} \le \frac{1}{\sqrt{\pi\chi^3}}\,\mathrm{B}(6\chi^{3/2})\, e^{4\chi^3} \tag{4.34}$$

for any positive $\varepsilon \in (0, 1/2]$. It is easy to see that (4.34) implies (2.4). Theorem 2.2 is proved. ■

In conclusion, let us say that the upper bounds (4.27) and (4.34) obtained are not optimal. Regarding the estimate of $Z_{2s}^{(1,1)}$ (see Section 4), we observe that the leading contribution to this sum comes from the walks with the simple self-intersections (open and not open ones) and the triple self-intersections that are not open. The last group of intersections gives the factor $\exp\{\chi^3\}$ instead of $\exp\{4\chi^3\}$.

The simple open self-intersections with no BTS-pairs should be included into the term μ_2'' and their contribution is normalized by the term $\exp\{-s^2/(2n)\}$. Thus the factor B of (4.27) and (4.3) should take into account two kinds of BTS-pairs and is to be replaced by $\mathrm{B}(2\chi^{3/2})$. Therefore one could expect the explicit form of the relation like (4.32) to be as follows:

$$\lim_{n\to\infty} \mathbf{E}\,\mathrm{Tr}(\tilde{H}^{(n,\rho_n)})^{2s_n} = \mathcal{M}_{\mathrm{GOE}}(\chi) = \frac{1}{\sqrt{\pi\chi^3}}\,\mathrm{B}(2\chi^{3/2})\, s e^{2\chi^3}. \tag{4.35}$$

It is not hard to observe that in the case of hermitian random matrices, only one type of BTS-pair is present in the simple open self-intersections. Therefore we expect the limit

$$\lim_{n\to\infty} \mathbf{E}\,\mathrm{Tr}(\tilde{H}^{(n,\rho_n)})^{2s_n} = \mathcal{M}_{\mathrm{GUE}}(\chi) = \frac{1}{\sqrt{\pi\chi^3}}\,\mathrm{B}(\chi^{3/2})\, e^{2\chi^3} \tag{4.36}$$

to hold.

Let us note that the limiting expressions $\mathcal{M}_{\mathrm{GOE}}(\chi)$ and $\mathcal{M}_{\mathrm{GUE}}(\chi)$ are well known in the spectral theory of random matrices. They can be computed with the help of orthogonal polynomial technique (see [25] and references therein). By using the Laplace transform method, it can be shown that these values are related with the Tracy-Widom distributions $F(t) = \mathbf{P}(\lambda_{\max}^{(n)} \le 1 + t/n^{2/3})$ of GOE and GUE, respectively [1, 25, 26]. Then one may conclude also about the convergence of the left-hand side of (2.3) to the Tracy-Widom law. The discussion of these questions goes out of the range of the present paper.

5. Auxiliary Statements

5.1. Proof of Lemma 3.2

Let us introduce the following denotations. A self-intersection with the marked instants $\tau_1 < \ldots < \tau_k \le s$ will be denoted by $\mathfrak{i}^{(k)}(\tau_1, \ldots, \tau_k)$. We denote the vertex of self-intersection $\mathfrak{i}^{(k)}$ by $v = v(\mathfrak{i}^{(k)})$. A collection of self-intersections $\{\mathfrak{i}^{(k_j)}\}, j \ge 1$ with given marked instants $\tau_i \in [1, \ldots, s]$ will be denoted by $\langle \mathfrak{I} \rangle_s$.

Given $\theta = \theta_{2s}$, $\langle \mathfrak{I} \rangle_s$ and a rule of non-marked passages Υ, the corresponding walk $\mathcal{W}_{2s}$, if it exists, is unique and completely determined by the chronological run along θ.

We will also use the following denotation of the self-intersection with a number of free instants: $\tilde{\mathfrak{i}}^{(k)}_l = \tilde{\mathfrak{i}}^{(k)}(\mathfrak{O}_l, \tau^{(l)}_k)$, where $\mathfrak{O}_l = (\mathfrak{o}_1, \ldots, \mathfrak{o}_l)$, $\tau^{(l)}_k = (\tau_{l+1}, \ldots \tau_k)$ and

$$\tilde{\mathfrak{i}}^{(k)}(\mathfrak{o}_1, \ldots, \mathfrak{o}_l, \tau_{l+1}, \ldots, \tau_k) = \sqcup_{x_1 < \cdots < x_l < \tau_{l+1}} \mathfrak{i}^{(k)}(x_1, \ldots, x_l, \tau_{l+1}, \ldots, \tau_k).$$

We say that $\mathfrak{o}_j$ are the *windows* of self-intersection. In what follows, we will omit the tilde sign and the subscript l in $\tilde{\mathfrak{i}}^{(k)}_l(\mathfrak{O}_l, \tau^{(l)}_k)$ when no confusion can arise.

Let us denote by $\mathbb{W}^{[\theta]}_{2s}(\langle \mathfrak{I} \rangle_s, \tilde{\mathfrak{i}}^{(k)}_l; \Upsilon)$ a family of walks that have a Dyck structure θ, the set of given self-intersections $\langle \mathfrak{I} \rangle_s$, follow the rule Υ, and have an additional k-fold self-intersection $\tilde{\mathfrak{i}}^{(k)}_l$. We consider all possible walks of this kind with all possible values of the free instants of self-intersection $\tilde{\mathfrak{i}}^{(k)}$. Given $(\theta, \langle \mathfrak{I} \rangle_s, \Upsilon)$, we omit these variables and indicate their presence by an asterisk in the denotation $\overset{*}{\mathbb{W}}_{2s}$. The next sub-section represents a rigorous formulation of the observations used in [22] and [25]. The tools developed will give us means to proceed in more complicated cases.

5.1.1. Walks with $\beta \in \mathbb{M}'_2$. Let us consider a family of walks $\overset{*}{\mathbb{W}}_{2s}\left(\mathfrak{i}^{(2)}(\mathfrak{o}_1, \tau_2)\right)$ such that the instant of the second marked arrival $\bar{\mathfrak{a}}_2 = \mathfrak{a}_2$ at $v = v(\mathfrak{i}^{(2)})$ is given by τ_2. The set $\overset{*}{\mathbb{W}}_{2s}\left(\mathfrak{i}^{(2)}(\mathfrak{o}_1, \tau_2)\right)$ can be constructed as follows. Regarding a sub-walk $\mathcal{W}_{[0, \xi_{\tau_2}-1]}$ that is completely determined by $(\theta, \langle \mathfrak{I} \rangle_s, \Upsilon)$, we set $\mathcal{W}_{2s}(\xi_{\tau_2}) = \beta$, $\beta \in g(\mathcal{W}_{[0,\xi_{\tau_2}-1]})$. Then, respecting the rules of $\theta, \langle \mathfrak{I} \rangle_s$ and Υ, we continue $\mathcal{W}$ to the time interval $[\xi_{\tau_2}+1, 2s]$. If such a walk exists, we add it to the list of the elements of $\overset{*}{\mathbb{W}}_{2s}\left(\mathfrak{i}^{(2)}(\mathfrak{o}_1, \tau_2)\right)$. In this case, the value τ' that is the first arrival instant at β fills the window $\mathfrak{o}_1$. We say that the value $\tau_1 = \tau'$ of the simple self-intersection is a realization of the window $\mathfrak{o}_1$ given by the walk $\mathcal{W}_{2s}$. We denote this realization by $\tau_1 = \langle \mathfrak{o}_1 \rangle_{\mathcal{W}_{2s}}$.

In this construction, the only condition on τ' at the instant τ_2 of the second arrival $\mathfrak{a}_2$ is such that there is no other marked arrivals at β than the first one, τ'. Let us say that this situation describes the *unconstraint* simple self-intersection, when there is no other special condition imposed on $\mathfrak{a}_2$.

Let us denote by $\mathfrak{i}^{(2)}\left(\mathfrak{o}_1, \left[\overset{\tau_2}{\mathfrak{u}_2}\right]\right)$ a simple self-intersection $\mathfrak{i}^{(2)(\mathfrak{o}_1, \tau_2)}$ with a certain condition $\mathfrak{u}_2$ imposed on the second arrival instant $\mathfrak{a}_2$ at $v(\mathfrak{i}^{(2)})$.

We consider first the construction of walks $\mathcal{W}_{2s} \in \overset{*}{\mathbb{W}}_{2s}\left(\mathfrak{i}^{(2)}\left(\mathfrak{o}_1, \left[\overset{\tau_2}{\mathfrak{u}_2}\right]\right)\right)$ with open simple self-intersection $\mathfrak{i}^{(2)}$ that corresponds to condition (c) of subsection 3.2 with denotation $\mathfrak{u}_2 = (o)$. Starting with a uniquely determined sub-walk

$\mathcal{W}_{[0,\xi_{\tau_2}-1]}$ that follows the rules of θ, $\langle\mathfrak{I}\rangle_s$ and Υ, we can see that the vertex $v=\gamma$ of the open simple self-intersection $\mathfrak{i}^{(2)}$ can be chosen from the subset of vertices $\mathbb{V}_{t'}^{(o)} \in \mathbb{V}(g)$, $g(\mathcal{W}_{[0,\xi_{\tau_2}-1]})$ that are open at the instant of time $t'=\xi_{\tau_2}-1$, i.e., have at least one open non-oriented edge attached to γ. We will say that $\mathbb{V}_{t'}^{(o)}$ is the set of t'-*open* vertices. Choosing one of the admissible vertices, we continue the run of the walk according to the rules θ, $\langle\mathfrak{I}\rangle_s$ and Υ, if it is possible.

A simple but important observation is that for any instant of time ξ_{τ_2}, the following upper bound holds: $\max_t |\mathbb{V}^{(o)}(\mathcal{W}_{[0,t]})| \le 2\theta^*$, where θ^* is the height of the Dyck path θ_{2s} [23]. Therefore we can write that

$$\#\left\{\tau_1:\ \tau_1=\langle\mathfrak{o}_1\rangle_{\mathcal{W}_{2s}},\ \mathcal{W}_{2s}\in \overset{*}{\mathbb{W}}_{2s}\left(\mathfrak{i}^{(2)}\left(\mathfrak{o}_1,\left[\begin{smallmatrix}\tau_2\\ \mathfrak{u}_2\end{smallmatrix}\right]\right)\right)\right\} \le 2\theta^*, \quad \mathfrak{u}_2=(o). \tag{5.1}$$

Clearly,

$$\overset{*}{\mathbb{W}}_{2s}\left(\mathfrak{i}^{(2)}\left(\mathfrak{o}_1,\left[\begin{smallmatrix}\tau_2\\ \mathfrak{u}_2\end{smallmatrix}\right]\right)\right) \subseteq \overset{*}{\mathbb{W}}_{2s}\left(\mathfrak{i}^{(2)}\left(\mathfrak{o}_1,\tau_2\right)\right),$$

and we can say that the set $\overset{*}{\mathbb{W}}_{2s}\ (\mathfrak{i}^{(2)}(\mathfrak{o}_1,\tau_2))$ is *filtered* by the condition $\mathfrak{u}_2$. With a certain abuse of language, we can say that the possible values τ' at the window $\mathfrak{o}_1$ are *restricted* or *filtered* by the condition $\mathfrak{u}_2$ imposed on the second arrival $\mathfrak{a}_2$. In a brief form, we will say that the first arrival $\mathfrak{a}_1$ is filtered by the property $\mathfrak{u}_2$ of the second arrival $\mathfrak{a}_2$.

The next case of the simple self-intersection with filtered first arrival is given by condition (b) of subsection 3.2. The corresponding self-intersection produces a p-edge of the form (γ,v), $v=v(\mathfrak{i}^{(2)})$. In this case, it is necessary that v belong to the exit cluster $\Delta(\alpha)$ of $g(\mathcal{W}_{[0,\xi_{\tau_2}-1]})$ with $\alpha=\mathcal{W}(\xi_{\tau_2}-1)$ that is uniquely determined. We denote this condition by $\mathfrak{u}_2=(\Delta)$.

The condition $\mathfrak{u}_2=(\Delta)$ means that we can continue the sub-walk $\mathcal{W}_{[0,\xi_{\tau_2}-1]}$ to the instant of time ξ_{τ_2} with a letter (a vertex) that belongs to $\Delta(\alpha)$. To choose this letter is also to choose the marked instant of the first arrival at this vertex. Therefore, if one considers the family of the walks $\overset{*}{\mathbb{W}}_{2s}\left(\mathrm{D};\mathfrak{i}^{(2)}\left(\mathfrak{o}_1,\left[\begin{smallmatrix}\tau_2\\ \mathfrak{u}_2\end{smallmatrix}\right]\right)\right)$ such that $|\mathcal{D}(\mathcal{W}_{2s})|=\mathrm{D}$, then

$$\#\left\{\tau_1:\ \tau_1=\langle\mathfrak{o}_1\rangle_{\mathcal{W}_{2s}},\ \mathcal{W}_{2s}\in \overset{*}{\mathbb{W}}_{2s}\left(\mathrm{D};\mathfrak{i}^{(2)}\left(\mathfrak{o}_1,\left[\begin{smallmatrix}\tau_2\\ \mathfrak{u}_2\end{smallmatrix}\right]\right)\right)\right\} \le \mathrm{D}, \quad \mathfrak{u}_2=(\Delta). \tag{5.2}$$

Thus, $\overset{*}{\mathbb{W}}_{2s}\ (\mathfrak{i}^{(2)}(\mathfrak{o}_1,\tau_2))$ is filtered by the condition $\mathfrak{u}_2=(\Delta)$. We will say that $\mathfrak{i}^{(2)}$ of this type produces a multiple edge in $\bar{\mathbb{E}}_g^{(k_0)}$.

The second type of filtering related with the multiple edges in $\bar{\mathbb{E}}_g^{(k_0)}$ is given by condition (a) of subsection 3.2. In this case, a necessary condition on $\langle\mathfrak{o}_1\rangle_{\mathcal{W}_{2s}}=\tau_1$ is that the marked edge $(v(\mathfrak{i}^{(2)}),\alpha)=(\mathcal{W}_{2s}(\xi_{\tau_1}),\mathcal{W}_{2s}(\xi_{\tau_2}-1))$ exists and belongs to $\bar{\mathbb{E}}_g^{(k_0)}$. The latter condition implies that $v(\mathfrak{i}^{(2)})\in\Lambda(\alpha)$ and means that the vertex $v(\mathfrak{i}^{(2)})$ is the μ-vertex. We denote this necessary condition by $\mathfrak{u}_2 = (\Lambda^{(k_0)})$. The number of vertices in $\Lambda(\alpha)$ bounded by k_0, we can write that

$$\# \left\{ \tau_1 : \ \tau_1 = \langle \mathfrak{o}_1 \rangle_{\mathcal{W}_{2s}}, \ \mathcal{W}_{2s} \in \overset{*}{\mathbb{W}}_{2s} \left(\mathfrak{i}^{(2)} \left(\mathfrak{o}_1, \left[\begin{smallmatrix} \tau_2 \\ \mathfrak{u}_2 \end{smallmatrix} \right] \right) \right) \right\} \leq k_0, \quad \mathfrak{u}_2 = (\Lambda^{(k_0)}). \tag{5.3}$$

In this case, the values τ' at the window $\mathfrak{o}_1$ are filtered by the $\Lambda^{(k_0)}$-condition.

It is obvious that the upper bounds (5.1), (5.2) and (5.3) are true in the case when $\beta \in \mathbb{M}'_2$, i.e., when $\kappa(\beta) = m \geq 2$. To prove this, it is sufficient to consider the walks with the self-intersection $\tilde{\mathfrak{i}}_1^{(m)} = \tilde{\mathfrak{i}}^{(m)}(\mathfrak{o}_1, \tau_2, \tau_3, \ldots, \tau_m)$.

Let us note that in the reasonings above, we did not make any difference between two procedures, the first one related with the continuation of the sub-walk $\mathcal{W}_{[0,\xi_{\tau_2}-1]}$ to the instant of time ξ_{τ_2} with some letter that verifies one or another condition and the second one given by the filtration of the set of walks $\overset{*}{\mathbb{W}}_{2s}$ $(\mathfrak{i}^{(i)}(\mathfrak{o}_1, \tau_2))$ with respect to one or another filtering condition. It is easy to see that these two procedures are similar and lead to the same results, i. e., to the same estimates of the cardinalities of families of walks. We will use this observation in the sub-sections that follow.

The last remark is that in the reasoning above the cases when $v(\mathfrak{i}^{(2)}) = \alpha$ are not excluded. This means that the estimates presented are valid in the cases of loops. This is also true with respect to the proofs of estimates that follow.

5.1.2. Walks with $\beta \in \mathbb{M}'_3$. Using the tools of the previous subsection, we can describe the properties of walks that belong to $\overset{*}{\mathbb{W}}_{2s}$ $(\mathfrak{i}^{(3)}(\mathfrak{o}_1, \mathfrak{o}_2, \tau_3))$, where the vertex of self-intersection $\mathfrak{i}^{(3)}$ is such that $\beta = v(\mathfrak{i}^{(3)}) \in \mathbb{M}'_3$. If the edge $e_3 = e(\xi_{\tau_3})$ of the third arrival $\bar{\mathfrak{a}}_3 = \mathfrak{a}_3$ is the p-edge, we will write that $e(\xi_{\tau_3}) \in \mathbb{V}^{(p)}(g)$. If $e_3 = e(\xi_{\tau_3})$ is the q-edge, we will write that $e(\xi_{\tau_3}) \in \mathbb{V}^{(q)}(g)$.

Lemma 5.1. *Let us denote by* $\overset{*}{\mathbb{W}}_{2s}$ $(\mathrm{D}; \mathfrak{i}^{(3)})$ *a family of walks with* $k_j \leq k_0$, $j \geq 1$ *and such that* $\mathcal{D}(\mathcal{W}_{2s}) = \mathrm{D}$. *Then*

$$\# \left\{ (\tau_1, \tau_2) = \langle (\mathfrak{o}_1, \mathfrak{o}_2) \rangle_{\mathcal{W}}, \mathcal{W} \in \overset{*}{\mathbb{W}}_{2s} \left(\mathrm{D}; \mathfrak{i}^{(3)} \left(\mathfrak{o}_1, \mathfrak{o}_2, \left[\begin{smallmatrix} \tau_3 \\ \mathfrak{u}_3 \end{smallmatrix} \right] \right) \right) \right\}$$
$$\leq \begin{cases} s\mathrm{D}, & \textit{if } \mathfrak{u}_3 = (\Delta), \\ sk_0, & \textit{if } \mathfrak{u}_3 = (\Lambda^{(k_0)}). \end{cases} \tag{5.4}$$

The statement of Lemma 5.1 seems to be a simple generalization of relations (5.2) and (5.3) of the previous sub-section. However, the proof of Lemma 5.1 crucially depends on the classification of vertices of $\mathbb{M}'_2$ and $\mathbb{M}'_3$ we introduced and therefore represents a key point in the general method of the proof of Lemmas 3.2 and 3.3. It should be emphasized that the estimates like (5.4) were not considered in papers [22, 23] and [25].

P r o o f of Lemma 5.1. Let us start with the case $e_3 \in \mathbb{V}^{(p)}(g)$. Then the edge $e_3 = (\alpha, \beta)$ is such that the marked edge $e' = (\alpha, \beta)$ exists in the graph of the sub-walk $\mathcal{W}_{[0,\xi_{\tau_3}-1]}$, and $\alpha = \mathcal{W}(\xi_{\tau_3} - 1)$. By the definition, $\beta \notin \mathbb{M}_2'$. This means that the edge e' represents either the first marked arrival at β, $e' = e(\mathfrak{a}_1)$ or the second one, $e' = e(\mathfrak{a}_2)$, and that the situation when $e(\mathfrak{a}_1) = e(\mathfrak{a}_2) = (\alpha, \beta)$ is excluded.

We are going to construct the walks of $\overset{*}{\mathbb{W}}_{2s}$ $(\mathrm{D}; \mathfrak{i}^{(3)}(\mathfrak{o}_1, \mathfrak{o}_2, \tau_3))$ with $v(\mathfrak{i}^{(3)}) \in \mathbb{M}_3'$ such that $e_3 = e(\xi_{\tau_3})$ is a p-edge. We denote the exit cluster of $\alpha = \mathcal{W}(\xi_{\tau_3} - 1)$ by $\Delta(\alpha) = \{\beta_1, \dots \beta_l\}$ with $l \le \mathrm{D}$. Let us assume for the moment that $\varkappa(\beta_i) = 1$ for all $i \in [1, \dots, l]$. We choose one of the vertices β_i and take the marked instant τ' such that $\mathcal{W}_{[0,\xi_{\tau_3}-1]}(\xi_{\tau'}) = \beta_i$. The sub-walk $\mathcal{W}_{[0,\xi_{\tau_3}-1]}$ will be continued at the instant of time ξ_{τ_3} with the letter β_i.

Our aim is to construct a sub-walk $\bar{\mathcal{W}}_{[0,\xi_{\tau_3}-1]}$ that has a self-intersection $(\cdot, \cdot)_1$ with the participation of τ'. The subscript 1 indicates the fact that this simple self-intersection is not present in $\mathcal{W}_{[0,\xi_{\tau_3}-1]}$. We proceed as follows: we take any marked instant of time τ'', not involved into the self-intersections already present, and consider the letter $\gamma'' = \mathcal{W}_{[0,\xi_{\tau_3}-1]}(\xi_{\tau''})$. If $\tau'' < \tau'$, then we replace β_i by γ''.

The walk $\bar{\mathcal{W}}_{[0,\xi_{\tau_3}-1]}$ performs a self-intersection (τ'', τ'), and in general situation this fact changes the run of the walk after $\xi_{\tau'}$ with respect to the run of the initial sub-walk $\mathcal{W}_{[0,\xi_{\tau_3}-1]}$. Therefore the exit cluster $\tilde{\Delta}(\alpha)$ can differ from $\Delta(\alpha)$. Indeed, one can easily find the examples of sub-walks $\bar{\mathcal{W}}$ and $\mathcal{W}$ such that $\tilde{\Delta}(\alpha) \cap \Delta(\alpha) = \emptyset$.

As a consequence, it may happen that $e(\tau') \notin \bar{\Delta}(\alpha)$. Moreover, if one considers another value $\tilde{\tau}'$ such that $e(\tilde{\tau}') \notin \Delta(\alpha)$, one cannot guarantee that $e(\tilde{\tau}') \notin \bar{\Delta}'(\alpha)$. Thus, without additional considerations, we cannot say that given τ_3 and τ'', the set of all possible values of τ' has a cardinality bounded by the maximal exit degree D.

An important observation here is that the changes of the exit cluster of α are possible only in the cases when the sub-walk performs a BTS-couple $(\xi_{\tau'}, \xi_{\tau'} + 1)$. If it is not the case, all cells at α created up to the instant $\xi_{\tau_3} - 1$ remain the same and therefore the edges of $\Delta(\alpha)$ are determined uniquely by θ_{2s}. The key point is that the relation $\beta \in \mathbb{M}_3'$ implies that $\beta \notin \mathbb{M}_2'$. This means that the second arrival $\mathfrak{a}_2$ does not verify condition (c) mentioned above, and we cast off those walks that arrive by $\mathfrak{a}_2$ at an open vertex. Therefore $\mathfrak{a}_2$ cannot represent the BTS-pair, and the set $\Delta(\alpha)$ is determined uniquely by the parameters θ, $\langle \mathfrak{I} \rangle_s$ and Υ. The exit cluster $\Delta(\alpha)$ does not change when the values τ'' and τ' vary. Then the edge $e(\tau') = (\alpha, \beta_i) = (\alpha, \gamma'')$ always belongs to $\Delta(\alpha)$.

If $\tau' < \tau''$, then we replace the letter γ'' by β_i. Again, the fact that $(\alpha, \beta_i) \in \Lambda(\alpha)$ remains true. The number of walks that verify the conditions imposed is bounded by $(s-1)D$. Thus the following upper bound holds:

$$\#\left\{(\tau_1,\tau_2)_1=\langle(\mathfrak{o}_1,\mathfrak{o}_2)\rangle_{\mathcal{W}}, \mathcal{W}\in \overset{*}{\mathbb{W}}_{2s}\left(D;\mathfrak{i}^{(3)}\left(\mathfrak{o}_1,\mathfrak{o}_2,\left[\begin{smallmatrix}\tau_3\\ \mathfrak{u}_3\end{smallmatrix}\right]\right)\right)\right\}\le(s-1)\mathrm{D},\ \mathfrak{u}_3=(\Delta).$$

Let us consider the case when $\varkappa(\beta_i)=2$ in the graph of the sub-walk $\mathcal{W}_{[0,\xi_{\tau_3}-1]}$. Then we can write that

$$\#\left\{(\tau_1,\tau_2)_1=\langle(\mathfrak{o}_1,\mathfrak{o}_2)\rangle_{\mathcal{W}}, \mathcal{W}\in \overset{*}{\mathbb{W}}_{2s}\left(D;\mathfrak{i}^{(3)}\left(\mathfrak{o}_1,\mathfrak{o}_2,\left[\begin{smallmatrix}\tau_3\\ \mathfrak{u}_3\end{smallmatrix}\right]\right)\right)\right\}\le \mathrm{D},\ \mathfrak{u}_3=(\Delta).$$

These two estimates give (5.4) with $\mathfrak{u}_3=(\Delta)$.

Let us construct the walks of $\overset{*}{\mathbb{W}}_{2s}$ $(\mathfrak{i}^{(3)}(\mathfrak{o}_1,\mathfrak{o}_2,\tau_3))$ with $v(\mathfrak{i}^{(3)})\in\mathbb{M}_3'$ such that $e_3=e(\xi_{\tau_3})\in\mathbb{V}^{(q)}(g)$. Given θ, $\langle\mathfrak{I}\rangle_s$ and Υ, the enter cluster of $\alpha=\mathcal{W}_{[0,\xi_{\tau_3}-1]}(\xi_{\tau_3}-1)$ that is $\Lambda(\alpha)=\{\beta_1,\dots\beta_l\}$, $l\le k_0$, is uniquely determined. Let us assume for the moment that $\varkappa(\beta_i)=1$ for all $i\in[1,\dots,l]$. We choose one of the vertices β_i and take the marked instant τ' such that $\mathcal{W}_{[0,\xi_{\tau_3}-1]}(\xi_{\tau'})=\beta_i$. The sub-walk will be continued at the instant of time ξ_{τ_3} with the letter β_i.

We are going to construct a sub-walk $\tilde{\mathcal{W}}_{[0,\xi_{\tau_3}-1]}$ which has a self-intersection $(\cdot,\cdot)_1$ with the participation of τ'. The subscript 1 indicates the fact that this simple self-intersection is not present in $\mathcal{W}_{[0,\xi_{\tau_3}-1]}$. We proceed as follows: we take any marked instant of time τ'' , not involved into the self-intersections already present, and consider the letter $\gamma''=\mathcal{W}_{[0,\xi_{\tau_3}-1]}(\xi_{\tau''})$. If $\tau''<\tau'$, then we replace β_i by γ''. The fact that $\mathcal{W}_{[0,\xi_{\tau_3}-1]}$ performs this additional self-intersection (τ'',τ') does not change the set $\Lambda(\alpha)$ because we reject the walks such that the intersection (τ'',τ') is the open self-intersection. Therefore the edge (β_i,α) still belongs to $\Lambda(\alpha)$. If $\tau'<\tau''$, then we replace the letter γ'' by β_i. Again, the fact that $(\beta_i,\alpha)\in\Lambda(\alpha)$ remains true. The choice of β_i is bounded by k_0, we get not more than $(s-1)k_0$ different walks of this kind,

$$\#\left\{(\tau_1,\tau_2)_1=\langle(\mathfrak{o}_1,\mathfrak{o}_2)\rangle_{\mathcal{W}}, \mathcal{W}\in \overset{*}{\mathbb{W}}_{2s}\left(D;\mathfrak{i}^{(3)}\left(\mathfrak{o}_1,\mathfrak{o}_2,\left[\begin{smallmatrix}\tau_3\\ \mathfrak{u}_3\end{smallmatrix}\right]\right)\right)\right\}$$
$$\le(s-1)k_0,\ \mathfrak{u}_3=(\Lambda^{(k_0)}).$$

Let us consider the case when $\varkappa(\beta_i)=2$ in the graph of the sub-walk $\mathcal{W}_{[0,\xi_{\tau_3}-1]}$. Then we can write that

$$\#\left\{(\tau_1,\tau_2)_1=\langle(\mathfrak{o}_1,\mathfrak{o}_2)\rangle_{\mathcal{W}}, \mathcal{W}\in \overset{*}{\mathbb{W}}_{2s}\left(D;\mathfrak{i}^{(3)}\left(\mathfrak{o}_1,\mathfrak{o}_2,\left[\begin{smallmatrix}\tau_3\\ \mathfrak{u}_3\end{smallmatrix}\right]\right)\right)\right\}\le k_0,\ \mathfrak{u}_3=(\Lambda^{(k_0)}).$$

Gathering these two bounds, we obtain the estimate (5.4) with $\mathfrak{u}_3=(\Lambda^{(k_0)})$. Lemma 5.1 is proved. ■

It is not hard to generalize these considerations to the case of walks with the self-intersection of the form $\tilde{\mathfrak{i}}_2^{(k)}(\mathfrak{o}_1,\mathfrak{o}_2,\tau_3,\dots,\tau_k)$ and obtain here the same upper bounds (5.4).

5.1.3. General filtration procedure. The general filtration procedure is as follows. Let us consider a diagram $\mathcal{G}$ and recall that $\langle \mathcal{G}_\diamondsuit \rangle_s$ denotes a realization of blue, green and black half-edges of $\mathcal{G}$. This procedure will be considered in more details in the next sub-section. Regarding the first vertex v of $\mathcal{G}$ that has at least one red f-edge attached, we consider the value τ in the blue edge-window attached at v as the value τ_2 or τ_3, in dependence whether $\varkappa(v)$ is equal to 2 or 3. Let us consider the edges of $\mathcal{V}(\mathcal{G}) \setminus v$ such that the integers in their windows lie to the left of τ. They form a family of self-intersections that we regard as the family $\langle \mathfrak{I} \rangle_s$ introduced above.

Now we can use the filtration procedure of the previous sub-sections and perform a chronological run $\mathfrak{R}_{\mathcal{T}}(t)$ along $\mathcal{T}_s = \mathcal{T}(\theta_{2s})$. If the next in turn step $t' = \xi_{\tau'}$ is marked and τ' does not belong to $\langle \mathfrak{I} \rangle_s$ and to the realizations of the red edges seen by $\mathcal{W}$, the walk $\mathcal{W}$ creates a new, next in turn, letter from the alphabet $\mathcal{A}$. If τ' belongs to $\langle \mathfrak{I} \rangle_s$, we follow the rules of $\langle \mathfrak{I} \rangle_s$ and Υ; we continue these actions till the instant $\xi_\tau - 1$. Then we consider all possible continuations of the sub-walk $\mathcal{W}_{[0,\xi_t-1]}$ to the instant of time ξ_t, choose one of them and continue the run of the sub-walk till it meets the second blue window of the vertex that has red edges attached. Then the procedure is repeated.

It is clear that the set of walks $\mathbb{W}_{2s}^{[\theta]}(\mathrm{D}; \mathcal{S}, \langle \mathcal{G}_\diamondsuit \rangle_s)$ with given $\mathcal{G}(\mathcal{S}) = \mathcal{G}$ has a cardinality equal to the number of realizations of all red edge-windows $\mathcal{G}_\circ$ of $\mathcal{G}$ and that
$$|\mathcal{G}_\circ| \le (2\theta^*)^r \, \mathrm{D}^p \, k_0^q \cdot (s(D + k_0))^{\mu_3'}. \tag{5.5}$$
Relation (3.5b) follows from (5.5).

5.1.4. Values in blue, green and black edge-windows. Given $\mathcal{G}$, let us fill its blue edge-windows $\mathcal{E}_\mathrm{b}$ first. We consider the set of integers $[1, \ldots, s]$ and choose the values for r groups of one element, p groups of one element, q groups of one element, μ_2'' groups of two elements, μ_3' groups of one element, μ_3'' groups of three elements and ν_k groups of k elements, $k_0 + 1 \le k \le s$.

The number of ways to choose these groups of subsets of $[1, \ldots, s]$ is given by the expression
$$|\langle \mathcal{E}_\mathrm{b} \rangle_s| = \frac{s!}{r!\,p!\,q!\ \mu_3'!\ \mu_2''!\,(2!)^{\mu_2''}\ (3!)^{\mu_3''}\ \mu_3''!} \ \prod_{k=k_0+1}^{s} \frac{1}{\nu_k!\ (k!)^{\nu_k}} \cdot \frac{1}{(s-E)!},$$
where $E = r + p + q + 2\mu_2'' + \mu_3' + 3\mu_3'' + \|\bar{\nu}\|$. Clearly,
$$|\langle \mathcal{E}_\mathrm{b} \rangle_s| \le \frac{s^E}{r!\,p!\,q!\ \mu_3'!\ \mu_2''!\,(2!)^{\mu_2''}\ (3!)^{\mu_3''}\ \mu_3''!} \ \prod_{k=k_0+1}^{s} \frac{1}{\nu_k!\ (k!)^{\nu_k}}. \tag{5.6}$$
The values in the green edge-windows $\mathcal{E}_\mathrm{g}$ are chosen from the set $(1, \ldots, s) \setminus \langle \mathcal{E}_\mathrm{b} \rangle_s$. Thus,
$$|\langle \mathcal{E}_\mathrm{g} \rangle_s| \le s^{u_2 + u_3}. \tag{5.7}$$

The number of realizations $\langle \mathcal{G}_{\Diamond} \rangle_s$ is bounded by the product of the right-hand sides of (5.6) and (5.7). This gives (3.5a). Lemma 3.2 is proved. ∎

5.2. Proof of Lemma 3.3

5.2.1. The number of diagrams $\mathcal{G}(\mathcal{S})$. As it is pointed out, given $\mathcal{S} = (r, p, q, \mu_2'', u_2; \mu_3', \mu_3'', u_3; \bar{\nu})$, the diagrams $\mathcal{G} \in \mathbb{G}(\mathcal{S})$ differ by the positions of the green edges attached to the vertices of $\mathbb{M}_2' = \mathcal{V}_2'(\mathcal{G})$ and $\mathbb{M}_3 = \mathcal{V}_3(\mathcal{G})$. According to the last remark of the previous sub-section, we can consider the vertices of $\mathcal{V}_2'$ and $\mathcal{V}_3$ as the ordered ones. By the construction, $|\mathcal{V}_2'| = \mu_2'$ and $|\mathcal{V}_3| = \mu_3$. So, we can consider $\mathcal{G}$ as a union of three parts, $\mathcal{G} = \mathcal{G}_2 \uplus \mathcal{G}_3 \uplus \mathcal{G}_{(k_0+1)}$

We take the u_2 green edges and distribute them over the vertices $\mathcal{V}_2'$. We draw the green edges to the right of the blue edges.

Let us denote by $u_2^{(i)}$, $1 \le i \le k_0 - 2$, the number of vertices that have i green edges attached, $\sum_{i=1}^{k_0-2} i u_2^{(i)} = u_2$. Given $\mu_2' = r + p + q$ and $\bar{u}_2 = (u_2^{(1)}, \dots, u_2^{(k_0-2)})$, the number of all possible diagrams $\mathcal{G}_2$ is equal to

$$\binom{\mu_2'}{u_2^{(1)}, \ \dots \ , u_2^{(k_0-2)}} = \frac{\mu_2'!}{u_2^{(1)}! \ \cdots \ u_2^{(k_0-2)}! \ \ (\mu_2' - |\bar{v}_2|)!} \ .$$

Therefore the cardinality of the set $\mathbb{G}_2 = \{\mathcal{G}_2\}$ is bounded as follows:

$$|\mathbb{G}_2| \le \prod_{i=1}^{k_0-2} \frac{(\mu_2')^{u_2^{(i)}}}{u_2^{(i)}!}. \tag{5.8}$$

Here the elementary bound $\binom{a}{b} \le a^b/b!$ is used.

The u_3 green edges are distributed over the vertices of $\mathcal{V}_3$ and placed to the right of the blue edges at each vertex. Assuming that the number of vertices that have j green edges is given by $u_3^{(j)}$, $1 \le j \le k_0 - 3$ with $\sum_{j=1}^{k_0-3} j u_3^{(j)} = u_3$, we can see that for given values of the parameters μ_3 and $\bar{u}_3 = (u_3^{(1)}, \dots, u_3^{(k_0-3)})$ one obtains the following upper bound for the cardinality of the set $\mathbb{G}_3 = \{\mathcal{G}_3\}$:

$$|\mathbb{G}_3| \le \prod_{j=1}^{k_0-3} \frac{(\mu_3)^{u_3^{(j)}}}{u_3^{(j)}!}. \tag{5.9}$$

5.2.2. Sum over rules Υ and diagrams $\mathcal{G}(\mathcal{S})$. In this subsection we prove Corollary of Lemma 3.2. Let us denote by $\mathbb{Y} = \mathbb{Y}(\mathcal{S})$ the family of all possible rules of continuation for the class of walks with the set of parameters $\mathcal{S}$, $\mathbb{Y}(\mathcal{S}) = \{\Upsilon(\mathcal{S})\}$. To do this, we have to consider a contribution of each vertex β such that the non-marked depart from β can be performed in a number of different ways. We denote by Υ_β this local rule of passage. Let ξ_τ be a marked instant time of the i-th arrival at a vertex β of self-intersection, and the instant $\xi_\tau + 1$ be a non-marked one. If the vertex β is closed at the instant of time $\xi_\tau - 1$, then there is only one possibility to continue the run of the walk at the non-marked instant of time $\xi_\tau + 1$.

Regarding a vertex of simple open self-intersection with the second arrival at the marked instant ξ_τ, we can see that the number of ξ_τ-open edges attached to β is bounded by 3. Then there are not more than 3 possible continuations of the run of the walk at the non-marked instant of time. If there are r vertices of simple open self-intersection, then the contribution of these vertices to $\mathbb{Y}$ is bounded by 3^r. The same concerns the vertices from $\mathbb{M}_2'$ that have p-edges and q-edges attached. They can produce the open self-intersections too. Thus, the total contribution of $o-$, $p-$ and q-arrivals at the vertices of $\mathbb{M}_2'$ such that $\varkappa(\beta) = 2$ is bounded by 3^{r+p+q}.

It is proved in [14] (see also [8]) that any vertex β of the self-intersection degree k has k non-marked departures from β and that at any instant of time there is not more than $2k$ open edges attached to β.

Using this simple but important observation, we conclude that for any vertex $\beta \in \mathbb{M}_2'$ that has $u \ge 1$ green edges attached, the upper bound for the number of continuations is given by

$$(2(2+u))^{2+u} \le (2k_0)^{2+u} \le (2k_0)^{3u}.$$

Let us consider the vertex $\beta \in \mathbb{M}_3''$. If $\varkappa(\beta) = 3$, then the number of non-marked departures from β with the choice of edges to close is not greater than 1, and the number of possible continuations is bounded by 3. This departure can be performed after the third marked arrival at β.

Regarding the vertex β of $\mathbb{M}_3'$ such that $\varkappa(\beta) = 3 + u$ with $u \ge 1$, we get the following upper bound for the total number of continuations at this vertex:

$$(2(3+u))^{3+u} \le (2k_0)^{4u}.$$

Any vertex $\beta \in \mathbb{M}_3''$ such that $\varkappa(\beta) = 3$ produces not more than 9 possible continuations.

Finally, any ν-vertex β such that $\varkappa(\beta) = k$ contributes to the number of possible continuations with a factor bounded from above by $(2k)^k$.

Combining the upper bounds obtained, we get the inequality

$$|\mathbb{Y}(\mathcal{S})| \le 3^{r+p+q}\, 9^{\mu_3}\, (2k_0)^{3u_2+4u_3} \cdot \prod_{k=k_0+1}^{s} (2k)^{k\nu_k}. \tag{5.10}$$

Gathering estimates (5.6)–(5.9) and (5.10) and combining them with the result of Lemma 3.2, we can estimate the number of walks for a class with given diagram $\mathcal{G}(\mathcal{S})$ with $\mathcal{S} = (\mu_2'', r, p, q, \bar{u}_2; \mu_3', \mu_3'', \bar{u}_3; \bar{\nu})$ by the following expression:

$$\frac{1}{\mu_2''!}\left(\frac{s^2}{2}\right)^{\mu_2''} \frac{(6s\theta^*)^r}{r!}\frac{(3sD)^p}{p!} \cdot \frac{(3sk_0)^q}{q!}\frac{1}{\mu_3''!}\left(\frac{9s^3}{3!}\right)^{\mu_3''} \times \frac{(9s^2(D+k_0))^{\mu_3'}}{\mu_3'!}$$

$$\times\ ((2k_0)^3 s)^{u_2}((2k_0)^4 s)^{u_3} \prod_{i=1}^{k_0-2} \frac{(\mu_2')^{u_2^{(i)}}}{u_2^{(i)}!} \prod_{j=1}^{k_0-3} \frac{(\mu_3)^{u_3^{(j)}}}{u_3^{(j)}!}. \tag{5.11}$$

The sum of (5.11) over all possible values of $\bar{u}_2$ gives the upper bound

$$((2k_0)^3 s)^{u_2} \sum_{u_2^{(1)}+\cdots+u_2^{(k_0-2)}=u_2} \prod_{i=1}^{k_0-2} \frac{(\mu_2')^{u_2^{(i)}}}{u_2^{(i)}!}$$

$$= \frac{((2k_0)^3 s\mu_2')^{u_2}}{u_2!} \sum_{u_2^{(1)}+\cdots+u_2^{(k_0-2)}=u_2} \frac{u_2!}{u_2^{(1)}!\cdots u_2^{(k_0-2)}!} = \frac{\left((2k_0)^3(k_0-2)s\mu_2'\right)^{u_2}}{u_2!}. \tag{5.12}$$

When deriving (5.12), we use the multinomial theorem.

The sum of (5.11) over all possible values of $\bar{u}_3$ gives the following upper bound:

$$((2k_0)^4 s)^{u_3} \sum_{u_3^{(1)}+\cdots+u_3^{(k_0-3)}=u_3} \prod_{j=1}^{k_0-3} \frac{(\mu_3)^{u_3^{(j)}}}{u_3^{(j)}!} \le \frac{\left((2k_0)^4(k_0-3)s\mu_3\right)^{u_3}}{u_3!}. \tag{5.13}$$

Combining relations (5.11), (5.12), (5.13) with the result of Lemma 3.2, we obtain inequality (3.6). Corollary of Lemma 3.2 is proved.

To complete the proof of Lemma 3.3, it remains to estimate the number of trajectories in the class of equivalence $\mathcal{C}_{\mathcal{W}}$ (3.1). It is easy to see that given $\mathcal{W}_{2s}$ of the class $\mathcal{G} = \mathcal{G}(\mathcal{S})$, we have the equality

$$|\mathbb{U}(\mathcal{I}_{2s}; 2s)| = |\mathbb{V}_g| = s - \sigma + 1, \tag{5.14}$$

where $\sigma = \mu_2 + 2\mu_3 + u_2 + u_3 + |\bar{\nu}|_1$. Then [22]

$$|\mathcal{C}_{\mathcal{W}}| \ = \prod_{k=1}^{s-\sigma}\left(1 - \frac{k}{n}\right) \ \le\ \exp\left\{-\frac{(s-\sigma)^2}{2n}\right\}. \tag{5.15}$$

This simple but important upper bound can be proved with the help of representation $1 - k/n = \exp\{\ln(1-k/n)\}$ and the use of the Taylor expansion. Now, combining (5.15) with (3.6) and the result of Lemma 3.1, we get inequality (3.7). Lemma 3.3 is proved. ∎

5.3. Walks with imported cells

As we have seen in Section 3, the proper and imported cells at the vertex of maximal exit degree $\breve{\beta}$ are characterized by the set of parameters $\mathcal{P}_R = (\mathcal{Q}_R, \mathcal{H}_R)$ (4.12). The aim of this sub-section is to describe the general principles of the study of the family of walks $\mathbb{W}_{2s}^{(u)}(\mathrm{D}, \bar{d}_R; \langle \mathcal{G}^\star \rangle_s, \mathcal{H}_R; \Upsilon)$ (4.16). In this subsection, we assume that $I = 0$ and there is neither proper nor mirror cells at $\breve{\beta}$.

Let us consider the walks that have only one imported cell generated by one BTS-instant determined by a couple (τ_1, ϕ), where the marked instant τ_1 is either z_1 or y_1 and denote it by $\mathbb{W}_{2s}^{(u)}(\mathrm{D}, d_1, d_2; \langle \mathcal{G}^\star_{\tau_1} \rangle_s, \phi, \Upsilon)$. We choose a tree of τ_1 edges $\mathcal{T}_{\tau_1}$ and perform over it a partial chronological run $\mathfrak{R}_{\mathcal{T}}^{(\tau_1)}$,

$$\mathfrak{R}_{\mathcal{T}}^{(\tau_1)} = \{\mathfrak{R}_{\mathcal{T}}(t),\ 1 \le t \le \xi_{\tau_1} - 1\}. \tag{5.16}$$

Going along $\mathfrak{R}_{\mathcal{T}}^{(\tau_1)}$, we construct a sub-walk $\mathcal{W}_{[0,\xi_{\tau_1}-1]}$ according to the self-intersections of $\langle \mathcal{G}^\star \rangle_s$. During this run, we choose the realizations of the values in red windows $\mathcal{G}_\circ$ with the help of general filtration procedure described in the proof of Lemma 3.2 (see subsections 5.1 and 5.2) and follow the rules Υ at the non-marked steps. Then we can write that

$$\mathcal{W}_{[0,t_1]} = \mathcal{W}\left(\mathcal{T}_{\tau_1}, \langle \mathcal{G}^\star_{\tau_1} \rangle_s, \langle \mathcal{G}_\circ \rangle_{[0,t_1]}, \Upsilon\right), \tag{5.17}$$

where $\langle \mathcal{G}_\circ \rangle_{[0,t_1]}$ indicates a realization of red edge-windows of $\mathcal{G}$ on the time interval $[0, t_1] = [0, \xi_{\tau_1} - 1]$.

5.3.1. Filtration of values ℓ, φ and ψ. Let us assume that the instant $\tau_1 = y_1$ fills the edge-window of the second arrival $\mathfrak{a}_2 = \mathfrak{a}_2(\mathfrak{i})$ of a simple self-intersection $\mathfrak{i}^{(2)} = \mathfrak{i}(\mathfrak{o}_1, y_1)$. As we have pointed out (see subsection 4.3), this edge-window can represent either o-edge or p-edge or q-edge of $\mathcal{G}$.

To construct a continuation of the sub-walk $\mathcal{W}_{[0,t_1]}$ at the marked instant of time ξ_{τ_1}, we have to choose one of the vertices γ_i of $g(\mathcal{W}_{[0,t_1]})$ that verify the condition that the chosen vertex γ is situated on the distance of ϕ non-marked steps from $\breve{\beta}$, where ϕ denotes either ℓ or ψ_1.

Let us consider the case when the second arrival $\mathfrak{a}_2$ has to verify the o-property. Then the collection $\Gamma = \Gamma_{t_1}^{(\ell,(o))}(\breve{\beta})$ of admissible vertices γ_i is such that

$$\bigsqcup_{\ell=1}^{u} \Gamma_{t_1}^{(\ell,(o))}(\breve{\beta}) = \mathbb{V}_{t_1}^{(o)},$$

where $\mathbb{V}_{t_1}^{(o)}$ is the set of t_1-open vertices. Clearly, $\sum_{\ell=1}^{u} |\Gamma_{t_1}^{(\ell,(o))}(\breve{\beta})| \le 2u$.

Let us consider the case when $\mathfrak{a}_2$ verifies the p-condition. Then the admissible vertices γ_i belong to the sub-set $\Delta^{(\ell)}(\alpha) \subseteq \Delta(\alpha)$, $\alpha = \mathcal{W}(\xi_{\tau_1} - 1)$, of vertices that are situated on the distance of ℓ non-marked steps from $\breve{\beta}$. Therefore,

$$\sum_{\ell=1}^{u} |\Gamma_{t_1}^{(\ell,(\Delta))}(\breve{\beta})| = \sum_{\ell=1}^{u} |\Delta^{(\ell)}(\alpha)| \le \mathrm{D}. \tag{5.18}$$

The same reasoning can be applied to the case of the q-condition imposed on $\mathfrak{a}_2$. Summing up, we can write that

$$\sum_{\ell=1}^{u} |\Gamma_{t_1}^{(\ell,\mathfrak{u}_2)}(\breve{\beta})| \le |\Gamma^{(\mathfrak{u}_2)}| = \begin{cases} 2u, & \text{if } \mathfrak{u}_2 = (o), \\ \mathrm{D}, & \text{if } \mathfrak{u}_2 = (\Delta), \\ k_0, & \text{if } \mathfrak{u}_2 = (\Lambda). \end{cases} \tag{5.19}$$

Let us consider the secondary imported cell generated by the remote BTS-instant $\tau_1 = y_1$. In the graph $g(\mathcal{W}_{[0,\xi_{y_1}-1]})$, the way from $\breve{\beta}$ to $\breve{\beta}$ by ψ_1 non-marked steps is uniquely determined by the rule Υ. Therefore we can write that

$$\sum_{\psi_1=1}^{u} |\Gamma_{t_1}^{(\psi_1)}(\breve{\beta})| \le 1. \tag{5.20}$$

The same is true for all subsequent secondary imported cells generated by y_1, and therefore (5.20) holds for all values of ψ_i with $1 \le i \le f_1''$.

Now it is clear that in the case when the BTS-instant represented by $\tau_1 = z_1$ is the local one, then for all imported cells generated by z_1 the following equality is valid:

$$\sum_{\varphi_k=1}^{u} |\Gamma_{t_1}^{(\varphi_k)}(\breve{\beta})| \le 1, \quad 1 \le k \le f_1'. \tag{5.21}$$

Let us consider the case when the y-label of (y_1, ℓ) is attributed to the edge-window of the third arrival $\mathfrak{a}_3$ at a vertex of $\mathbb{M}_3'$ of $\mathcal{G}$. Assume that $\mathfrak{a}_3$ verifies Δ-condition. Repeating the proof of Lemma 5.1, attribute to $\mathcal{W}(\xi_{y_1})$ a vertex γ that belongs to $\Delta^{(\ell)}(\alpha)$, $\alpha = \mathcal{W}(\xi_{y_1} - 1)$ and take a marked instant τ' that determines γ. Taking into account the last inequality of (5.18), we can see that after the sum over ℓ, the total number of admissible values τ' is bounded by D. The remaining part of the reasoning is the same as before. The case when $\mathfrak{a}_3$ verifies the Λ-condition can be studied by the similar argument. Then (cf. (5.4))

$$\sum_{\ell=1}^{u} \# \left\{ \langle(\mathfrak{o}_1,\mathfrak{o}_2)\rangle_{\mathcal{W}},\ \mathcal{W} \in \overset{*}{\mathbb{W}}_{2s} \left(\mathfrak{i}^{(3)} \left(\mathfrak{o}_1, \mathfrak{o}_2, \begin{bmatrix} (y_1,\ell) \\ \mathfrak{u}_3 \end{bmatrix} \right) \right) \right\} \le \begin{cases} s\mathrm{D}, & \text{if } \mathfrak{u}_3 = (\Delta), \\ sk_0, & \text{if } \mathfrak{u}_3 = (\Lambda). \end{cases} \tag{5.22}$$

It is clear that the sum over the values ψ_i attributed to y_1 put into the edge-window of $\mathfrak{a}_3$ gives the upper bounds of the form (5.20). The same is true for the sums over variables ϕ_i in the case of $\tau_1 = z_1$.

Finally, let us consider the cases when the y-label is attributed to an edge-window $\mathfrak{e}$ of $\mathcal{G}$ that represents the third arrival $\mathfrak{a}_3$ to the corresponding vertex v. If $v \in \mathbb{M}_2'$, then $\mathfrak{e}$ is the u-edge. If $v \in \mathbb{M}_3''$, then $\mathfrak{e}$ represents the blue μ-edge. If v is a ν-vertex, then $\mathfrak{e}$ is the black ν-edge. In all of these three cases, the vertex $\gamma = \mathcal{W}(\xi_{y_1})$ is determined uniquely by the sub-walk $\mathcal{W}_{[0,t_1]}$ and therefore

$$\sum_{\phi=1}^{u} |\Gamma_{t_1}^{(\phi)}(\beta)| \le 1. \tag{5.23}$$

The same is true in the cases when y_1 is attributed to the edge-windows of the k-th arrivals $\mathfrak{a}_k$.

Summing up, we can say that after the summation over ϕ the number of realizations of the values in red edge-windows is bounded by the same expressions (5.19) and (5.22) as in the ordinary case without y-labels considered in the previous sub-section.

Let us consider a walk constructed with the help of a diagram $\mathcal{G}^\star$ that has two y-labels with the values to (y_1, ℓ_1) and (y_2, ℓ_2), respectively, belonging to the same vertex $v \in \mathcal{V}(\mathcal{G}^\star)$. We assume that $y_1 < y_2$. In this case the filtering of the values of ℓ_1 is performed as before, and the sum over ℓ_1 gives one of the estimates (5.19)–(5.23) for the number of admissible vertices. When arrived at the instant of time $\xi_{y_2} - 1$, the vertex γ is determined by the construction of the sub-walk $\mathcal{W}_{[0,\xi_{y_2}-1]}$, and y_2 is attributed to the arrival α_k at v with $\kappa \ge 3$. Then the sum over ℓ_2 gives us the upper bound

$$\sum_{\ell_2=1}^{u} |\Gamma_{t_2}^{(\ell_2)}(\breve\beta)| \le 1. \tag{5.24}$$

To get the final account on the sums over variables ℓ, ψ and φ, let us assume that the diagram $\mathcal{G}_\mathcal{Q}$ contains $\breve\mu_2'$ y-labels attributed to $\breve r$ o-edges, $\breve p$ p-edges and $\breve q$ q-edges of $\mathcal{G}$, and $\breve\mu_3'$ y-labels attributed to the third arrivals at the vertices of $\mathbb{M}_3'$. Then

$$\prod_{j=1}^{J} \sum_{\ell_j=1}^{u} \prod_{i=1}^{f_j''} \sum_{\psi_i=1}^{u} \prod_{k=1}^{K} \prod_{l=1}^{f_k'} \sum_{\varphi_l=1}^{u} |\Gamma^{(\phi)}(\mathcal{G}_\mathcal{Q})| \le (2u)^{\breve r}\, \mathrm{D}^{\breve p}\, k_0^{\breve q}\, (2s(\mathrm{D}+k_0))^{\breve\mu_3'} = \breve{\mathfrak{F}}(\mathcal{G}), \tag{5.25}$$

where we denoted by $|\Gamma^{(\phi)}(\mathcal{G}_\mathcal{Q})|$ the product over all cardinalities of the sets of admissible vertices presented.

5.3.2. Underlying Dyck paths and trees.

Regarding a walk $\mathcal{W}_{2s} \in \mathbb{W}_{2s}^{(u)}(\mathrm{D}, d_1, d_2; \langle \mathcal{G}_{\tau_1}^\star \rangle_s, \phi, \Upsilon)$, let us denote by t_1 and t_2 the instants of time such that $\mathcal{W}_{2s}(t_i) = \breve\beta$, $\tau_1 = \xi_{\tau_1}$, and t_2 represents the arrival $\mathfrak{a}'$ at $\breve{\beta}$ by the corresponding imported cell. According to the definition of variables (τ_1, ϕ), the sub-walk $\mathcal{W}_{[t_1+1,t_2-1]}$ is such that it has ϕ non-marked steps and after each of these steps it has tree-type sub-walks $\tilde{\mathcal{W}}^{(k)}$ that are reduced by $\hat{\mathcal{R}}$ to the empty walks. This means that the Dyck structure of $\mathcal{W}_{2s}$ is such that the nest cell $\varpi' = \varpi(\tau_1, \phi)$ that corresponds to $\mathfrak{a}'$ is obtained after ϕ descending steps are performed in $\mathcal{T}_{\tau_1}$ from the vertex $\upsilon = \mathfrak{R}_{\mathcal{T}_{\tau_1}}(\xi_{\tau_1})$. Then the vertex υ' of $\mathcal{T}_s$ of $\mathcal{W}_{2s}$ is determined, where the exit sub-cluster of d_2 edges is to be placed.

Therefore $\mathcal{T}_s$ is to be of the following structure: consider a tree $\mathcal{T}_{\tau_1}^{[l]}$ that contains τ_1 edges and has the descending part of the length $l \geq \phi$ (see Fig. 1). We will say that the vertex $\upsilon_1 = \mathfrak{R}_{\mathcal{T}}(\xi_{\tau_1})$ is on the distance l from the root of the tree. On the l vertices of the descending part, construct a realization of l sub-trees of the total number of $s - \tau_1$ edges such that at the nest cell ϖ' there exists a tree with the root sub-cluster of d_2 edges (see Section 6 for more details). We denote such a family of trees by $\mathcal{T}_{s-\tau_1}^{\{l\}}(d_2, \tau_1, \phi)$.

The family of trees that corresponds to the Dyck paths generated by the elements of $\mathbb{W}_{2s}^{(u)}(d_2, \langle \mathcal{G}^\star \rangle_s, \phi, \Upsilon)$ is given by the expression

$$\bigsqcup_{l=1}^{\tau_1} \Big\{ \underbrace{\mathcal{T}_{\tau_1}^{[l]} \otimes \mathcal{T}_{s-\tau_1}^{\{l\}}(d_2, \tau_1, \phi)}_{u\text{-condition}} \Big\} = \mathbb{T}_s^*,$$

where we denoted

$$\mathbb{T}_s^* = \mathbb{T}_s^{(u)}(d_2, \tau_1, \phi). \tag{5.26}$$

The under-brace with u-condition means that we construct the sub-trees $\mathcal{T}_{\tau_1}^{[l]}$ and $\mathcal{T}_{s-\tau_1}^{\{l\}}(d_2, \tau_1, \phi)$ in the way that the height of the common tree obtained $\mathcal{T}_s$ attains u. In Section 6 we describe in details the set of these trees.

Ignoring the condition that the vertex $\breve{\beta} = \mathcal{W}_{2s}(x_1)$ has the first proper cell x_1 with d_1 edges of the exit sub-cluster, we can write that

$$\begin{aligned} |\mathbb{W}_{2s}^{(u)}(d_2, \langle \mathcal{G}^\star \rangle_s, \phi, \Upsilon)| = \sum_{l=1}^{\tau_1} \sum_{\mathcal{T}_{\tau_1}^{[l]}} \sum_{\langle \mathcal{G}_\circ \rangle_{[0,\xi_{\tau_1}-1]}} |\Gamma_{\tau_1}^{(\phi)}(\breve{\beta})| \\ \times \sum_{\mathcal{T}_{s-\tau_1}^{\{l\}}(d_2,\tau_1,\phi)} \sum_{\langle \mathcal{G}_\circ \rangle_{[\xi_{\tau_1}+1,2s]}^{(\tau_1,\phi)}} 1. \end{aligned} \tag{5.27}$$

In the last sum of (5.27), we have denoted by $\langle \mathcal{G}_\circ \rangle_{[\xi_{\tau_1}+1,2s]}^{(\tau_1,\phi)}$ a realization of the values in the red edge-windows of $\mathcal{G}$ on the time interval $[\xi_{\tau_1} + 1, 2s]$; this realization also depends on $\mathcal{T}_{\tau_1}^{[l]}$ and $\mathcal{T}_{s-\tau_1}^{\{l\}}(d_2, \tau_1, \phi)$. We denote by $r_1, p_1, q_1, 2\mu_3'(1)$ and $r_2, p_2, q_2, 2\mu_3'(2)$ the number of red windows to be determined during the time intervals $[0, \xi_{\tau_1} - 1]$ and $[\xi_{\tau_1} + 1, 2s]$, respectively. It is clear that $r_1 = r_2 + 1 = r$. The same equalities are verified by other red edge-windows.

Let us forget for the moment that the walks of the left-hand side of (5.26) are such that their Dyck paths have the height $\theta^* = u$. Then it is not hard to show that the following upper bound holds for any given value of ϕ:

$$\sum_{\mathcal{T}^{\{l\}}_{s-\tau_1}(d_2,\tau_1,\phi)} 1 \ \leq \ e^{-\eta d_2} \sum_{\mathcal{T}^{\{l\}}_{s-\tau_1}} 1, \tag{5.28}$$

where $\eta = \ln(4/3)$ (see Lemma 6.1 of the next section).

Taking into account that the upper bound (cf. (5.5))

$$\left| \left\{ \langle \mathcal{G}_\circ \rangle^{(\tau_1,\phi)}_{[\xi_{\tau_1}+1,2s]} \right\} \right| \leq |\mathcal{G}^{(2)}_\circ| = (2u)^{r_2} D^{p_2} k_0^{q_2} (s(D+k_0))^{\mu_3'(2)} \tag{5.29}$$

is uniform with respect to ϕ, we use (5.19) and deduce from (5.27) that

$$\sum_{\phi=1}^{u} |\mathbb{W}^{(u)}_{2s}(d_2, \langle \mathcal{G}^\star \rangle_s, \phi, \Upsilon)| \leq \sum_{l=1}^{\tau_1} \sum_{\mathcal{T}^{[l]}_{\tau_1}} \sum_{\langle \mathcal{G}_\circ \rangle_{[0,\xi_{\tau_1}-1]}} \sum_{\mathcal{T}^{\{l\}}_{s-\tau_1}} e^{-\eta d_2} \cdot |\mathcal{G}^{(2)}_\circ| \cdot \sum_{\phi=1}^{u} |\Gamma^{(\phi)}_{\tau_1}(\breve{\beta})| \,.$$

Taking into account that

$$\sum_{l=1}^{\tau_1} \sum_{\mathcal{T}^{[l]}_{\tau_1}} \sum_{\mathcal{T}^{\{l\}}_{s-\tau_1}} 1 \ = \mathrm{t}_s,$$

we finally get the upper bound

$$\sum_{\phi=1}^{u} |\mathbb{W}^{(u)}_{2s}(d_2, \langle \mathcal{G}^\star \rangle_s, \phi, \Upsilon)| \leq \ e^{-\eta d_2} \, \mathrm{t}_s \, |\mathcal{G}_\circ|, \tag{5.30}$$

where $|\mathcal{G}_\circ|$ is given by (5.5).

The u-condition of (5.26) makes the use of (5.28) more complicated. The proof of the exponential estimates of the form of (5.30) is given in Section 6.

6. Catalan trees and D-lemma

6.1. Exponential bound for Catalan trees

The following statement slightly improves the corresponding results of [7] and [8].

Lemma 6.1. *Consider the family of Catalan trees constructed with the help of s edges and such that the root vertex ϱ has d edges attached to it and denote by $\tilde{\mathrm{t}}_s(d)$ its cardinality. Then the upper bound*

$$\tilde{\mathrm{t}}_s(d) \le e^{-\eta d}\, \mathrm{t}_s, \quad \eta = \ln(4/3) \tag{6.1}$$

is true for any given integers d and s such that $1 \le d \le s$.

P r o o f. By the definition,

$$\tilde{\mathrm{t}}_s(d) = \sum_{u_1+\cdots+u_{d-1}+u_d = s-d} \mathrm{t}_{u_1} \cdots \mathrm{t}_{u_{d-1}}\, \mathrm{t}_{u_d}, \tag{6.2}$$

where the sum runs over integers $u_j \ge 0$. We will say that (6.2) represents the number of Catalan trees of s edges that have a *sub-cluster* attached to the root ϱ that contains d edges.

Using the fundamental recurrence relation

$$\mathrm{t}_{s+1} = \sum_{j=0}^{s} \mathrm{t}_j\, \mathrm{t}_{k-j}, \quad s \ge 0, \quad \mathrm{t}_0 = 1, \tag{6.3}$$

we can rewrite (6.2) in the following form:

$$\tilde{\mathrm{t}}_s(d) = \sum_{v=0}^{s-d} \sum_{u_1+\cdots+u_{d-2}+v=s-d} \mathrm{t}_{u_1} \cdots \mathrm{t}_{u_{d-2}} \left(\sum_{u_{d-1}+u_d=v} \mathrm{t}_{u_{d-1}}\, \mathrm{t}_{u_d} \right)$$

$$= \sum_{u_1+\cdots+u_{d-2}+u_{d-1}=s-d+1} \mathrm{t}_{u_1} \cdots \mathrm{t}_{u_{d-2}}\, \mathrm{t}_{u_{d-1}} - \sum_{u_1+\cdots+u_{d-2}=s-d+1} \mathrm{t}_{u_1} \cdots \mathrm{t}_{u_{d-2}}. \tag{6.4}$$

Relation (6.4) implies that

$$\tilde{\mathrm{t}}_s(d) = \begin{cases} \tilde{\mathrm{t}}_s(d-1) - \tilde{\mathrm{t}}_{s-1}(d-2), & \text{for all } 3 \le d \le s, \\ \mathrm{t}_{s-1}, & \text{for } d=2 \text{ and } s \ge 2, \\ \mathrm{t}_{s-1}, & \text{for } d=1 \text{ and } s \ge 1, \end{cases} \tag{6.5}$$

where the last two relations are easy to obtain directly. It follows from (6.5) that

$$\tilde{\mathrm{t}}_s(d) \le t_{s-1}, \quad \text{for all } 1 \le d \le s. \tag{6.6}$$

Let us return to (6.4) and rewrite it in the form

$$\tilde{\mathrm{t}}_s(d) = \sum_{v=0}^{s-d} \tilde{\mathrm{t}}_{s-v-1}(d-1) \cdot \mathrm{t}_v, \quad 2 \le d \le s. \tag{6.7}$$

Assuming that $d \geq 3$ and applying (6.6) to the right-hand side of (6.7), we get the inequality

$$\tilde{\mathrm{t}}_s(d) \leq \sum_{v=0}^{s-d} \mathrm{t}_{s-2-v}\,\mathrm{t}_v \leq \mathrm{t}_{s-1} - \mathrm{t}_{s-2}. \tag{6.8}$$

Using the explicit expression for the Catalan numbers (3.2), it is easy to show that

$$2\mathrm{t}_s \leq \mathrm{t}_{s+1} \leq 4\mathrm{t}_s, \quad s \geq 1. \tag{6.9}$$

Then we deduce from (6.8) that

$$\tilde{\mathrm{t}}_s(d) \leq \left(\frac{3}{4}\right) \mathrm{t}_{s-1}, \quad d \geq 3. \tag{6.10}$$

Relation (6.1) holds for $d = 3$, $s \geq 3$. The standard reasoning by recurrence based on (6.7) proves the bound

$$\tilde{\mathrm{t}}_s(d) \leq \left(\frac{3}{4}\right)^{d-2} \mathrm{t}_{s-1}, \quad 3 \leq d \leq s. \tag{6.11}$$

Remembering (6.5), we get that (6.11) is also true in the case of $d = 2, s \geq 2$. Using the first inequality of (6.9), we deduce from (6.11) the upper bound (6.1). Lemma 6.1 is proved. ∎

6.2. Tree-type walks with multiple edges

Given a Catalan tree $\mathcal{T}_s$, let us denote by $\mathrm{N}^{(2)}(\mathcal{T}_s)$ the number of choices of two edges of $\mathcal{T}_s$ that have the same parent vertex. Clearly, the sum $\mathrm{N}_s^{(2)} = \sum_{\mathcal{T}_s \in \mathbb{T}_s} \mathrm{N}^{(2)}(\mathcal{T}_s)$ represents the number of even closed tree-type walks of $2s$ steps whose graphs have exactly one p-edge passed four times and $s-2$ grey edges passed two times.

Lemma 6.2. *For any given $s \geq 2$, the following relations hold:*

$$\mathrm{N}_s^{(2)} = \frac{(2s)!}{(s-2)!\,(s+2)!} = \left(s - \frac{3s}{s+2}\right) \mathrm{t}_s, \tag{6.12}$$

and therefore $\mathrm{N}_s^{(2)} \leq s\mathrm{t}_s$; the lower bound $\mathrm{N}_s^{(2)} \geq (s\mathrm{t}_s)/2$ is true for all $s \geq 4$.

P r o o f. It is easy to see that

$$\mathrm{N}_s^{(2)} = \sum_{u+v_1+v_2+v_3=s-2} (2u+1)\,\mathrm{t}_u \mathrm{t}_{v_1} \mathrm{t}_{v_2} \mathrm{t}_{v_3}, \quad s \geq 2, \tag{6.13}$$

where the sum runs over all integers $u \ge 0$ and $v_i \ge 0$. Then the generating function $\Phi^{(2)}(\varsigma) = \sum_{k\ge 0} \mathrm{N}_k^{(2)} \varsigma^k$ with $\mathrm{N}_0^{(2)} = \mathrm{N}_1^{(2)} = 0$ is given by the relation

$$\Phi^{(2)}(\varsigma) = 2\varsigma^3\, \mathrm{f}'(\varsigma)\, \mathrm{f}^3(\varsigma) + \varsigma^2\, \mathrm{f}^4(\varsigma), \tag{6.14}$$

where $\mathrm{f}(\varsigma) = \sum_{s=0}^{\infty} \mathrm{t}_s\, \varsigma^s$ is the generating function of the Catalan numbers. It follows from (6.3) that $\mathrm{f}(\varsigma)$ verifies the well-known equation

$$\mathrm{f}(\varsigma) = 1 + \varsigma\, \mathrm{f}^2(\varsigma), \tag{6.15}$$

and then

$$\mathrm{f}(\varsigma) = \frac{1 - \sqrt{1 - 4\varsigma}}{2\varsigma}\ . \tag{6.16}$$

It follows from (6.15) that

$$\mathrm{f}'(\varsigma) = \mathrm{f}^2(\varsigma) + 2\varsigma\, \mathrm{f}'(\varsigma)\, \mathrm{f}(\varsigma). \tag{6.17}$$

Using (6.15) and (6.17) and taking into account that

$$\mathrm{f}'(\varsigma) = \sum_{k=0}^{\infty} (k+1)\mathrm{t}_{k+1}\, \varsigma^k, \tag{6.18}$$

one can easily derive from (6.14) relation (6.12). Lemma 6.2 is proved. ■

Let us consider a general case of the tree-type walks such that their graphs have exactly one edge passed $2l$ times and other $s-l$ edges passed two times. The number of these walks $\mathrm{N}_s^{(l)}$ is given by the the total number of possibilities to mark l edges that have the same parent vertex at the Catalan trees. Similarly to (6.13), we can write that

$$\mathrm{N}_s^{(l)} = \sum_{u+v_1+\dots+v_{2l-1}=s-l} (2u+1)\, \mathrm{t}_u \mathrm{t}_{v_1} \mathrm{t}_{v_2} \dots \mathrm{t}_{v_{2l-1}}. \tag{6.19}$$

The corresponding generating function $\Phi^{(l)}(\varsigma)$ is given by the relation

$$\Phi^{(l)}(\varsigma) = 2\varsigma^{l+1}\, \mathrm{f}'(\varsigma)\, \mathrm{f}^{2l-1}(\varsigma) + \varsigma^l\, \mathrm{f}^{2l}(\varsigma), \tag{6.20}$$

and therefore

$$\Phi^{(l)}(\varsigma) = \varsigma^l\, \mathrm{f}^{2l-1}(\varsigma) \left(\frac{2}{\sqrt{1-4\varsigma}} - \mathrm{f}(\varsigma) \right). \tag{6.21}$$

Using (6.15) and (6.18), we can deduce from (6.20) that

$$\mathrm{N}_s^{(l)} \le 2^l s\, \mathrm{t}_s, \quad 2 \le l \le s. \tag{6.22}$$

This inequality means that the constant $\eta = \ln(4/3)$ of the exponential estimate (6.1) can be considerably increased for large values of s and d.

Similarly to (6.12), it is not hard to show that

$$\mathrm{N}_s^{(3)} = \frac{(2s)!}{(s-3)!\,(s+3)!} \;=\; \mathrm{t}_s \left(s - 8 - \frac{36s+48}{s^2+5s+6} \right), \qquad (6.23)$$

and therefore $\mathrm{N}_s^{(3)} \le s\mathrm{t}_s$. Regarding $\mathrm{N}_s^{(1)}$ as a number of trees with one marked edge, we can see that

$$\mathrm{N}_s^{(1)} = s\mathrm{t}_s = \frac{(2s)!}{(s-1)!\,(s+1)!}. \qquad (6.24)$$

Relation (6.24) indicates a natural connection among expressions (3.2), (6.12) and (6.23). It is natural to assume that the equality

$$\mathrm{N}_s^{(l)} = \frac{(2s)!}{(s-l)!\,(s+l)!} \qquad (6.25)$$

is true for all values of $l \in [1, \dots, s]$. However, relation (6.21) seems to be not so convenient for proving (6.25). It would be useful to find the representations of $\mathrm{N}_s^{(l)}$ different from (6.19) and (6.20). Let us note that (6.25) would imply a useful upper bound $\mathrm{N}_s^{(l)} \le s\mathrm{t}_s$ for all s and $l \le s$ that is stronger than (6.20). Clearly, the last upper bound is in complete accordance with the Galton–Watson view of Catalan trees.

6.3. *D*-lemma

Our aim is to prove the exponential-type estimate of the form (4.18). For simplicity, we consider the case when the non-trivial tree-type sub-clusters with $d_i > 0$ correspond either to the proper or to the imported cells at $\breve{\beta}$. The case of mirror cells will be considered at the end of the present subsection.

Lemma 6.3 *Consider a family of walks* $\mathbb{W}_{2s}^{(u)}(\mathrm{D}, \bar{d}_R; \langle \mathcal{G}_R^\star \rangle_s, \mathcal{H}_R, \Upsilon)$ *(4.16). Then for any integer* $m \ge 0$,

$$\sum_{u=1}^{s} u^m \sum_{\mathcal{H}_R} |\mathbb{W}_{2s}^{(u)}(\mathrm{D}, \bar{d}_R; \langle \mathcal{G}_R^\star \rangle_s, \mathcal{H}_R, \Upsilon)| \le A_s^{(m)} \cdot 4^R \,(D_R+1)\, e^{-\eta D_R}\, \mathrm{t}_s, \quad (6.26)$$

where we denoted

$$A_s^{(m)} = A_s^{(m)}(\mathcal{G}, \mathrm{D}, k_0) \;=\; s^{(m+r)/2}\; B_{m+r}\; 2^r\, \mathrm{D}^p\, k_0^q\, (s(\mathrm{D}+k_0))^{\mu_3'}, \qquad (6.27)$$

with

$$B_{m+r} = \sup_{s\ge 1} B^{(s)}_{m+r}, \quad B^{(s)}_r = \frac{1}{\mathrm{t}_s}\sum_{u=1}^{s}\left(\frac{u}{\sqrt{s}}\right)^r \cdot |\mathbb{T}^{(u)}_s| \tag{6.28}$$

and

$$\mathbb{T}^{(u)}_s = \{\mathcal{T}_s:\ \theta^*(\mathcal{T}_s) = u\}\,. \tag{6.29}$$

We prove Lemma 6.3 by recurrence. Let us introduce the following denotations related with (6.29). Given natural a and u, let us denote by $\dot{\mathbb{T}}^{(u)}_a = \mathbb{T}^{(u)}_a$ a family of Catalan trees $\mathcal{T}_a$ of a edges such that for any such a tree the corresponding Dyck path has the height u, $\theta^*(\mathcal{T}_a) = u$. In this case we simply say that $\mathcal{T}_a$ has the height u. Let $\ddot{\mathbb{T}}^{(u)}_a$ be a family of Catalan trees such that $\theta^*(\mathcal{T}_a) \le u$. We denote the cardinalities of these sets of trees by $\dot{\mathrm{T}}^{(u)}_a$ and $\ddot{\mathrm{T}}^{(u)}_a$, respectively,

$$\dot{\mathrm{T}}^{(u)}_a = |\,\{\mathcal{T}_a \in \mathbb{T}_a:\ \theta^*(\mathcal{T}_a) = u\}\,| \quad \text{and} \quad \ddot{\mathrm{T}}^{(u)}_a = |\,\{\mathcal{T}_a \in \mathbb{T}_a:\ \theta^*(\mathcal{T}_a) \le u\}\,|\,. \tag{6.30}$$

We assume that $\dot{\mathrm{T}}^{(u)}_a = 0$ when $a < u$, $\dot{\mathrm{T}}^{(0)}_a = \ddot{\mathrm{T}}^{(0)}_a = 0$ for any $a \ge 1$ and $\dot{\mathrm{T}}^{(0)}_0 = \ddot{\mathrm{T}}^{(0)}_0 = 1$.

We also denote by $\dot{\mathbb{T}}^{(u,[l])}_a$ the family of trees of the height u that have the descending part of length l. The families $\dot{\mathbb{T}}^{(u,\{l\})}_a$, $\ddot{\mathbb{T}}^{(u,[l])}_a$ and $\ddot{\mathbb{T}}^{(u,\{l\})}_a$ as well as the families of trees that have a sub-cluster at one of the nest cell are determined in obvious manner, (see (5.26)). This will be clarified in the computations that follow.

6.3.1. The initial step of recurrence. In this subsection, we study a family of walks $\mathbb{W}^{(u)}_{2s}(d, \langle \mathcal{G}^\star_{\tau_1}\rangle_s, \phi, \Upsilon)$ given by (4.16) with $R = 1$ such that the first non-trivial sub-cluster of the tree-type part $\tilde{W}$ with d edges is attached to the nest cell $\varpi_1 = (\tau_1, \phi_1)$ of the corresponding tree (see also (5.26)). The variable τ_1 denotes one of the three possible values that are x_1, y_1 or z_1, and $\phi_1 = \phi$ is equal to 0, ℓ_1 or φ_1, respectively. If $\tau_1 = y_1$, then ϕ can also be equal to ψ_1.

According to the definitions of (6.30), the subset of trees (5.26) can be represented as follows:

$$\mathbb{T}^{(u)}_s(d, \tau_1, \phi) =$$

$$\bigsqcup_{l=1}^{\tau_1}\left\{\dot{\mathcal{T}}^{(u,[l])}_{\tau_1} \otimes \ddot{\mathcal{T}}^{(u,\{l\})}_{s-\tau_1}(d, \tau_1, \phi)\right\} \sqcup \bigsqcup_{l=1}^{\tau_1}\left\{\ddot{\mathcal{T}}^{(u-1,[l])}_{\tau_1} \otimes \dot{\mathcal{T}}^{(u,\{l\})}_{s-\tau_1}(d, \tau_1, \phi)\right\}\,. \tag{6.31}$$

The accurate version of (5.27) is given then by the two sums,

$$|\mathbb{W}^{(u)}_{2s}(d, \langle \mathcal{G}^*\rangle_s, \phi, \Upsilon)|$$

$$= \sum_{l=1}^{\tau_1} \sum_{\dot{\mathcal{T}}_{\tau_1}^{(u,[l])}} \sum_{\langle \mathcal{G}_\circ \rangle_{[0,t_1]}} |\Gamma_{t_1}^{(\phi,\mathrm{u})}(\breve{\beta})| \sum_{\ddot{\mathcal{T}}_{s-\tau_1}^{(u,\{l\})}(d,\tau_1,\phi)} \sum_{\langle \mathcal{G}_\circ \rangle_{[t_1+2,2s]}^{(\tau_1,\phi)}} 1$$

$$+ \sum_{l=1}^{\tau_1} \sum_{\ddot{\mathcal{T}}_{\tau_1}^{(u-1,[l])}} \sum_{\langle \mathcal{G}_\circ \rangle_{[0,t_1]}} |\Gamma_{t_1}^{(\phi,\mathrm{u})}(\breve{\beta})| \sum_{\dot{\mathcal{T}}_{s-\tau_1}^{(u,\{l\})}(d,\tau_1,\phi)} \sum_{\langle \mathcal{G}_\circ \rangle_{[t_1+2,2s]}^{(\tau_1,\phi)}} 1, \qquad (6.32)$$

where we denoted $t_1 = \xi_{\tau_1} - 1$. It should be noted that the sum $\sum_{\langle \mathcal{G}_\circ \rangle_{[t_1+2,2s]}^{(\tau_1,\phi)}} 1$ is bounded from above by $|\mathcal{G}_\circ^{(2)}|$ (5.29) uniformly with respect to ϕ.

Let us estimate the cardinality of the family $\ddot{\mathbb{T}}_{s-\tau_1}^{(u,\{l\})}(d,\tau_1,\phi)$,

$$|\ddot{\mathbb{T}}_{s-\tau_1}^{(u,\{l\})}(d,\tau_1,\phi)| = \sum_{\ddot{\mathcal{T}}_{s-\tau_1}^{(u,\{l\})}(d,\tau_1,\phi)} 1 .$$

Using (6.30), we can write that

$$|\ddot{\mathbb{T}}_{s-\tau_1}^{(u,\{l\})}(d,\tau_1,\phi)| = \sum_{\substack{\bar{b}_d,\bar{c}_l,\\ |\bar{b}_d|+|\bar{c}_l|=s-\tau_1}} \ddot{\mathcal{T}}_{\{d,\bar{b}_d\}}^{(u-l+\phi-1)} \, \ddot{\mathcal{T}}_{[l+1,\bar{c}_l]}^{(u)} , \qquad (6.33)$$

where $\bar{b}_d = (b_1, \dots, b_d)$, $|\bar{b}_d| = b_1 + \dots + b_d$, $\bar{c}_l = (c_0, c_1, \dots, c_l)$, $|\bar{c}_l| = c_0 + c_1 + \dots + c_l$,

$$\ddot{\mathcal{T}}_{\{d,\bar{\beta}_d\}}^{(u-l+\phi-1)} = \prod_{k=1}^{d} \ddot{\mathrm{T}}_{b_k}^{(u-l+\phi-1)}$$

and

$$\ddot{\mathcal{T}}_{[l+1,\bar{c}_l]}^{(u)} = \prod_{i=0}^{l} \ddot{\mathrm{T}}_{c_i}^{(u-l+i)}.$$

Regarding (6.34), we can use the obvious inequalities

$$\ddot{\mathrm{T}}_{b_k}^{(u-l+\phi-1)} \le \ddot{\mathrm{T}}_{b_k}^{(u-l+\phi)} \le \mathrm{t}_{b_k} \qquad (6.34)$$

and write that for a given $b \ge d$, due to Lemma 6.1,

$$\sum_{\bar{b}_d \colon |\bar{b}_d| = b-d} \ddot{\mathcal{T}}_{\{d,\bar{b}_d\}}^{(u-l+\phi-1)} \le \sum_{\bar{b}_d \colon |\bar{b}_d| = b-d} \prod_{k=1}^{d} \mathrm{t}_{b_k} = \tau_b(d) \le e^{-\eta d}\, \mathrm{t}_b. \qquad (6.35)$$

Then

$$|\ddot{\mathbb{T}}_{s-\tau_1}^{(u,\{l\})}(d,\tau_1,\phi)| \le \sum_{b=d}^{s-\max\{\tau_1,1\}} e^{-\eta d}\, \mathrm{t}_b \sum_{\substack{\bar{c}_l,\\ |\bar{c}_l|=s-\tau_1-b}} \ddot{\mathcal{T}}_{[l+1,\bar{c}_l]}^{(u)} . \qquad (6.36)$$

Now we can perform the sum over ϕ in the right-hand side of (6.32) and get with the help of (5.19), the upper bound (cf. (5.5))

$$\sum_{\phi=1}^{u} \sum_{\langle \mathcal{G}_\circ \rangle_{[0,t_1]}} |\Gamma_{t_1}^{(\phi,\mathrm{u})}(\breve{\beta})| \cdot |\mathcal{G}_\circ^{(2)}| \le (2u)^r\, \mathrm{D}^p\, k_0^q\, (s(\mathrm{D}+k_0))^{\mu_3'} = |\mathcal{G}_\circ|.$$

The sum over ϕ being performed, the last expression of (6.36) can be inserted into the first term of the right-hand side of (6.32). This gives the sum

$$\sum_{l=1}^{\tau_1} \sum_{\dot{\mathcal{T}}_{\tau_1}^{(u,[l])}} \sum_{\bar{c}_l, |\bar{c}_l| = s-\tau_1-b} \ddot{T}_{[l+1,\bar{c}_l]}^{(u)} = |\grave{\mathbb{T}}_{s-b}^{(u,\tau_1)}|, \tag{6.37}$$

where we have denoted by $\grave{\mathbb{T}}_{s-b}^{(u,\tau_1)}$ the family of all trees $\mathcal{T}_{s-b}$ such that the height $\theta^*(\mathcal{T}_{s-b}) = u$ is attained for the first time during the time interval $[0, t_1] = [0, \xi_{\tau_1} - 1]$.

Let us consider the second term of the right-hand side of (6.32) and estimate the cardinality

$$|\dot{\mathbb{T}}_{s-\tau_1}^{(u,\{l\})}(d, \tau_1, \phi)| = \sum_{\dot{\mathcal{T}}_{s-\tau_1}^{(u,\{l\})}(d,\tau_1,\phi)} 1.$$

Using the first inequality of (6.35), we can write that

$$|\dot{\mathbb{T}}_{s-\tau_1}^{(u,\{l\})}(d, \tau_1, \phi)| \le \sum_{\substack{\bar{b}_d, \bar{c}_l, \\ |\bar{b}_d| + |\bar{c}_l| = s-\tau_1}} \left(\ddot{T}_{\{d,\bar{b}_d\}}^{(u-l+\phi)}\, \dot{T}_{[l+1,\bar{c}_l]}^{(u)} + \dot{T}_{\{d,\bar{b}_d\}}^{(u-l+\phi)}\, \ddot{T}_{[l+1,\bar{c}_l]}^{(u)} \right), \tag{6.38}$$

where

$$\dot{T}_{\{d,\bar{b}_d\}}^{(u-l+\phi)} = \sum_{k=1}^{d} \ddot{\mathrm{T}}_{b_1}^{(u-l+\phi-2)} \cdots \ddot{\mathrm{T}}_{b_{k-1}}^{(u-l+\phi-2)}\, \dot{\mathrm{T}}_{b_k}^{(u-l+\phi-1)}\, \ddot{\mathrm{T}}_{b_{k+1}}^{(u-l+\phi-1)} \cdots \ddot{\mathrm{T}}_{b_d}^{(u-l+\phi-1)}. \tag{6.39}$$

Let us consider the first term of the right-hand side of (6.38). The same computation as before shows that for any given $b \ge d$,

$$\sum_{\bar{b}_d:\, |\bar{b}_d| = b-d} \ddot{T}_{\{d,\bar{b}_d\}}^{(u-l+\phi)} \le e^{-\eta d}\, \mathrm{t}_b. \tag{6.39}$$

Similarly to (6.37), we observe that

$$\sum_{l=1}^{\tau_1} \sum_{\ddot{\mathcal{T}}_{\tau_1}^{(u,[l])}} \sum_{\bar{c}_l, |\bar{c}_l| = s-\tau_1-b} \dot{T}_{[l+1,\bar{c}_l]}^{(u)} = |\acute{\mathbb{T}}_{s-b}^{(u,\tau_1)}|, \tag{6.40}$$

where we $\acute{\mathbb{T}}^{(u,\tau_1)}_{s-b}$ denotes the family of trees $\mathcal{T}_{s-b}$ such that the height $\theta^*(\mathcal{T}_{s-b}) = u$ is attained for the first time during the time interval $[\xi_{\tau_1}, 2s]$.

Let us consider the second term of the right-hand side of (6.38). Applying (6.34) to all the factors $\ddot{\mathrm{T}}$ of (6.39), we can write that

$$\dot{T}^{(u-l+\phi)}_{\{d,\bar{b}_d\}} \ \le \ \sum_{k=1}^{d} \mathrm{t}_{b_1}\cdots \mathrm{t}_{b_{k-1}}\, \dot{\mathrm{T}}^{(u-l+\phi-1)}_{b_k}\, \mathrm{t}_{b_{k+1}}\cdots \mathrm{t}_{b_d}.$$

Then for any given $b \ge d$,

$$\sum_{\bar{b}_d:\, |\bar{b}_d| = b-d} \dot{T}^{(u-l+\phi)}_{\{d,\bar{b}_d\}} \ \le \ d\, e^{-\eta(d-1)} \sum_{b_1 = u-l+\phi-1}^{b-d} \dot{\mathrm{T}}^{(u-l+\phi-1)}_{b_1}\ \mathrm{t}_{b-b_1-1}. \tag{6.41}$$

It is useful to note that the factor $\dot{\mathrm{T}}^{(u-l+\phi-1)}_{b_1}\, \ddot{T}^{(u)}_{[l+1,\bar{c}_l]}$ being substituted into the second term of (6.32) produces an expression

$$\sum_{l=1}^{\tau_1} \sum_{\ddot{\mathcal{T}}^{(u-1,[l])}_{\tau_1}} \dot{\mathrm{T}}^{(u-l+\phi-1)}_{b_1} \sum_{\substack{\bar{c}_l,\\ |\bar{c}_l| = s-\tau_1-(b-b_1-1)}} \ddot{T}^{(u)}_{[l+1,\bar{c}_l]} \ = \ |\check{\mathbb{T}}^{(u,\tau_1,\phi,1)}_{s-(b-b_1-1)}|, \tag{6.42}$$

where we denote by $\check{\mathbb{T}}^{(u,\tau_1,\phi,1)}_{s-b'}$, $b' = b - b_1 - 1$ the family of trees $\mathcal{T}_{s-b'}$ such that the height $\theta^*(\mathcal{T}_{s-b'}) = u$ is attained for the first time during the chronological run over a sub-tree attached by exactly one edge to the nest cell (τ_1, ϕ) of $\mathcal{T}_{s-b'}$. This definition is self-explained by the left-hand side of (6.42). It is clear that

$$|\check{\mathbb{T}}^{(u,\tau_1,\phi,1)}_{s-b'}| \le |\dot{\mathbb{T}}^{(u)}_{s-b'}| \quad \text{and} \quad |\grave{\mathbb{T}}^{(u,\tau_1)}_{s-b}| + |\acute{\mathbb{T}}^{(u,\tau_1)}_{s-b}| = |\dot{\mathbb{T}}^{(u)}_{s-b}|. \tag{6.43}$$

Remembering that all of the upper bounds (6.36), (6.39) and (6.41) are valid for any given values of ϕ, we turn back to (6.32) and get the upper bound

$$|\mathbb{W}^{(u)}_{2s}(d, \langle \mathcal{G}^*\rangle_s, \phi, \Upsilon)| \ \le \ |\mathcal{G}_\circ|\ e^{-\eta d} \left(\sum_{b=d}^{s-1} \mathrm{t}_b\ |\dot{\mathbb{T}}^{(u)}_{s-b}| + \frac{4d}{3} \sum_{b'=d-1}^{s-1} \mathrm{t}_{b'}\ |\dot{\mathbb{T}}^{(u)}_{s-b'}| \right). \tag{6.44}$$

Extracting the factor u^r from $|\mathcal{G}_\circ|$ (5.5), we can write that

$$\sum_{u=1}^{s-b} u^{m+r}\ |\dot{\mathbb{T}}^{(u)}_{s-b}| \ = (\sqrt{s})^{m+r} \sum_{u=1}^{s-b} \left(\frac{u}{\sqrt{s}}\right)^{m+r} |\Theta^{(u)}_{2s-2b}| \le \ s^{(m+r)/2}\, \mathrm{t}_{s-b}\, B_{m+r}\,. \tag{6.45}$$

Taking into account that $d \ge 1$ and using the elementary relations based on (3.2) and (6.3),

$$\sum_{b=1}^{s-1} \mathrm{t}_b\, \mathrm{t}_{s-b} = \mathrm{t}_{s+1} - 2\mathrm{t}_s \le 2\mathrm{t}_s$$

and

$$\frac{4}{3}\sum_{b'=0}^{s-1} \mathrm{t}_{b'}\,\mathrm{t}_{s-b'} = \frac{4}{3}(\mathrm{t}_{s+1}-\mathrm{t}_s) \le 4\mathrm{t}_s,$$

we deduce from (6.44) and (6.45) that

$$\sum_{u=1}^{s} u^m \sum_{\phi=1}^{u} |\mathbb{W}_{2s}^{(u)}(d, \langle \mathcal{G}^* \rangle_s, \phi, \Upsilon)| \le (2+4d)\, e^{-\eta d}\, \mathrm{t}_s$$

$$\times\, s^{(m+r)/2}\, B_{m+r}\, 2^r\, D^p\, k_0^q\, (s(D+k_0))^{\mu_3'}. \qquad (6.46)$$

This proves relations (6.26), (6.27) with $R = 1$.

6.3.2. Recurrent estimates. Let us denote by τ_{R+1} the marked instant that corresponds to the imported cell (τ_{R+1}, ϕ_{R+1}) and write down the expression (cf. (6.32))

$$|\mathbb{W}_{2s}^{(u)}(\mathrm{D}, \bar{d}_{R+1}; \langle \mathcal{G}^\star_{R+1} \rangle_s, (\mathcal{H}_R, \phi_{R+1}), \Upsilon)| = \Sigma^{(1)}(\phi_{R+1}) + \Sigma^{(2)}(\phi_{R+1}), \qquad (6.47)$$

where

$$\Sigma^{(1)}(\phi_{R+1}) = \sum_{l=1}^{\tau_{R+1}} \sum_{\dot{\mathcal{T}}_{\tau_{R+1}}^{(u,[l])}} \sum_{\langle \mathcal{G}_\circ^{(1)} \rangle} |\Gamma_{t_{R+1}}^{(\phi_{R+1},\mathrm{u})}(\breve{\beta})| \sum_{\ddot{\mathcal{T}}_{s-\tau_{R+1}}^{(u,\{l\})}(d_{R+1},\tau_{R+1},\phi_{R+1})} \sum_{\langle \mathcal{G}_\circ^{(2)} \rangle} 1$$

and

$$\Sigma^{(2)}(\phi_{R+1}) = \sum_{l=1}^{\tau_{R+1}} \sum_{\ddot{\mathcal{T}}_{\tau_{R+1}}^{(u,[l])}} \sum_{\langle \mathcal{G}_\circ^{(1)} \rangle} |\Gamma_{t_{R+1}}^{(\phi_{R+1},\mathrm{u})}(\breve{\beta})| \sum_{\dot{\mathcal{T}}_{s-\tau_{R+1}}^{(u,\{l\})}(d_{R+1},\tau_{R+1},\phi_{R+1})} \sum_{\langle \mathcal{G}_\circ^{(2)} \rangle} 1.$$

In these relations, by $\langle \mathcal{G}_\circ^{(1)} \rangle$ are denoted the realizations of values of the red edge-windows of $\langle \mathcal{G}_{\mathcal{Q}_R} \rangle_s$ such that the marked instants of the corresponding blue edge-windows are strictly less than τ_{R+1} and by $\langle \mathcal{G}_\circ^{(2)} \rangle$ those that have marked instants greater or equal to τ_{R+1}.

Taking into account the uniform with respect to $\phi_{\tau_{R+1}}$ estimate of $\langle \mathcal{G}_\circ^{(2)} \rangle$ (5.29) and denoting $\tau' = \tau_{R+1}$, we can write with the help of (6.37) that the sum $\Sigma^{(1)} = \sum_{\phi_{R+1}=1}^{u} \Sigma^{(1)}(\phi_{R+1})$ is bounded from above as follows:

$$\Sigma^{(1)} \le\; e^{-\eta d_{R+1}} \sum_{b=d_{R+1}}^{s-1-\tau'} \mathrm{t}_b \left(\sum_{\mathcal{T} \in \dot{\mathbb{T}}_{s-b}^{(u,\tau')}} \sum_{\langle \mathcal{G}_\circ^{(1)} \rangle} 1 \right) \cdot |\mathcal{G}_\circ^{(2)}| \cdot |\Gamma^{(\mathrm{u}_{R+1})}|. \qquad (6.48)$$

It should be noted that the expression standing in the parenthesis of (6.48) represents the family of walks

$$\grave{\mathbb{W}}^{(u,\tau')}_{2s-2b}(\mathrm{D}, \bar{d}_R; \langle \mathcal{G}^{(\star,1)}_R\rangle_{s-b}, \mathcal{H}_R, \Upsilon) \ = \sum_{\mathcal{T}\in \grave{\mathbb{T}}^{(u,\tau')}_{s-b}} \ \sum_{\langle \mathcal{G}^{(1)}_\circ\rangle} 1 \tag{6.49}$$

such that the conditions of Lemma 6.3 are verified. Here we have denoted by $\langle \mathcal{G}^{(\star,1)}_R\rangle_{s-b}$ the part of the realization of the diagram $\langle \mathcal{G}^{\star}_R\rangle_s$ that takes into account the instants of self-intersections that are strictly less than $\tau' = \tau_{R+1}$.

Regarding the sum $\Sigma^{(2)} = \sum_{\phi_{R+1}=1}^{u} \Sigma^{(2)}(\phi_{R+1})$, we use representation (6.38) and write that

$$\Sigma^{(2)} \le \acute{\Sigma}^{(2)} + \dot{\Sigma}^{(2)}, \tag{6.50}$$

where

$$\acute{\Sigma}^{(2)} \le \ e^{-\eta d_{R+1}} \sum_{b=d_{R+1}}^{s-1-\tau'} \mathrm{t}_b \left(\sum_{\mathcal{T}\in \acute{\mathbb{T}}^{(u,\tau')}_{s-b}} \ \sum_{\langle \mathcal{G}^{(1)}_\circ\rangle} 1 \right) |\mathcal{G}^{(2)}_\circ| \ |\Gamma^{(\mathrm{u}_{R+1})}| \tag{6.51}$$

and

$$\dot{\Sigma}^{(2)}(\phi') \le d_{R+1}\, e^{-\eta(d_{R+1}-1)} \sum_{b=d_{R+1}-1}^{s-1} \mathrm{t}_b \ |\mathcal{G}^{(2)}_\circ|$$

$$\times \left(\sum_{l=1}^{\tau'} \ \sum_{\ddot{\mathcal{T}}^{(u,[l])}_{\tau'}} \ \sum_{\langle \mathcal{G}^{(1)}_\circ\rangle} \ \sum_{|\bar{c}_l| = s-\tau'-b} \ddot{\mathrm{T}}^{(u-1)}_{[l+1,\bar{c}_l]} \ \dot{\mathrm{T}}^{(u-l+\phi'-1)}_{b-1} \right) |\Gamma^{(\mathrm{u}_{R+1})}|. \tag{6.52}$$

It is easy to see that

$$\Sigma^{(1)} + \acute{\Sigma}^{(2)} \le \mathrm{D}^{p_2}\, k_0^{q_2} \ (2s(\mathrm{D}+k_0))^{\mu'_3(2)} \ |\Gamma^{(\mathrm{u}_{R+1})}|$$

$$\times e^{-\eta d_{R+1}} \sum_{b=1}^{s-1} \mathrm{t}_b \ (2u)^{r_2} \ \cdot |\grave{\mathbb{W}}^{(u,\tau')}_{2s-2b}(\mathrm{D}, \bar{d}_R; \langle \mathcal{G}^{(\star,1)}_R\rangle_{s-b}, \mathcal{H}_R, \Upsilon)|. \tag{6.53}$$

Regarding the sum

$$\Sigma'_{R+1} = \sum_{u=1}^{s} u^m \sum_{\mathcal{H}_R} \left(\Sigma^{(1)} + \acute{\Sigma}^{(2)} \right),$$

we apply to the right-hand side of (6.53) the main proposition of the lemma (6.27) and get the inequality

$$\Sigma'_{R+1} \le 2\cdot 4^R\, D_R \ e^{-\eta D_{R+1}}\, \mathrm{t}_s \cdot B_{m+r}\ 2^r\, \mathrm{D}^p\, k_0^q \ (s(\mathrm{D}+k_0))^{2\mu'_3}. \tag{6.54}$$

Let us consider the right-hand side of (6.52). The expression standing in the parenthesis of (6.52) is bounded from above by the number of walks $\mathcal{W}_{2s-2b}$ that on the time interval $[0, \xi_{\tau_{R+1}} - 1]$ verify the conditions of Lemma 6.3 and such that the height u of the corresponding Dyck path $\theta(\mathcal{W}_{2s-2b})$ is attained for the first time above the edge attached to the nest cell (τ', ϕ'). Let us denote this set of walks by $\dot{\mathbb{W}}^{(u,\tau',\phi')}_{2s-2b}(\mathrm{D}, \bar{d}_R; \langle \mathcal{G}^{(\star,1)}_R \rangle_{s-b}, \mathcal{H}_R, \Upsilon)$ (cf. (6.49)). To simplify the proof, it is useful to observe that

$$|\dot{\mathbb{W}}^{(u,\tau',\phi')}_{2s-2b}(\mathrm{D}, \bar{d}_R; \langle \mathcal{G}^{(\star,1)}_{\mathcal{Q}_R} \rangle_{s-b}, \mathcal{H}_R, \Upsilon)| \le |\,\dddot{\mathbb{W}}^{(u,\Delta_i^c)}_{2s-2b}\,(\mathrm{D}, \bar{d}_R; \langle \mathcal{G}^{(\star,1)}_R \rangle_{s-b}, \mathcal{H}_R, \Upsilon)|,$$

where $\dddot{\mathbb{W}}^{(u,\Delta_i^c)}_{2s-2b}\,(\mathrm{D}, \bar{d}_R; \langle \mathcal{G}^{(\star,1)}_R \rangle_{s-b}, \mathcal{H}_R, \Upsilon)$ is the family of walks with the same properties as before and the only difference that the height $\theta^*(\mathcal{W}_{2s}) = u$ is attained somewhere excepting those parts of θ that lie over the exit sub-clusters Δ_i, $1 \le i \le d_R$.

We are going to show that

$$\sum_{u=1}^{s} u^m \sum_{\mathcal{H}_R} |\,\dddot{\mathbb{W}}^{(u,\Delta_i^c)}_{2s-2b}\,(\mathrm{D}, \bar{d}_R; \langle \mathcal{G}^{(\star,1)}_R \rangle_{s-b}, \mathcal{H}_R, \Upsilon)|$$

$$\le\; 2^R\, e^{-\eta D_R}\, \mathrm{t}_{s-b}\; B_{m+r_1}\, 2^{r_1}\, \mathrm{D}^{p_1}\, k_0^{q_1}\; (s(\mathrm{D}+k_0))^{\mu_3'(1)}\,. \tag{6.55}$$

One can prove (6.55) by recurrence. We consider here the initial case of $R = 1$ only. It is easy to see that

$$|\,\dddot{\mathbb{W}}^{(u,\Delta_i^c)}_{2s}\,(\mathrm{D}, d; \langle \mathcal{G}^{\star}_{\tau_1} \rangle_s, \phi_1, \Upsilon)| = \sum_{l=1}^{\tau_1} \left\{ \sum_{\ddot{\mathcal{T}}^{(u-1,[l])}_{\tau_1}} \sum_{\dot{\mathcal{T}}^{(u,\{l\})}_{s-\tau_1-b}} + \sum_{\dot{\mathcal{T}}^{(u,[l])}_{\tau_1}} \sum_{\ddot{\mathcal{T}}^{(u,\{l\})}_{s-\tau_1-b}} \right\} \sum_{\langle \mathcal{G}_\circ \rangle_{[0,t_1]}} 1$$

$$\times\; |\Gamma^{(\phi_1,\mathrm{u}_1)}_{t_1}(\breve{\beta})| \sum_{\ddot{\mathcal{T}}^{(u-l+\phi_1-1)}_b(d)} \;\sum_{\langle \mathcal{G}_\circ \rangle_{[t_1+2,2s]}} 1, \tag{6.56}$$

where $t_1 = \xi_{\tau_1} - 1$.

Using the upper bound (5.5) and inequality (6.35), we deduce from (6.56) the following inequality:

$$\sum_{u=1}^{s} u^m \sum_{\phi_1=1}^{u} |\,\dddot{\mathbb{W}}^{(u,\Delta_i^c)}_{2s}\,(\mathrm{D}, d; \langle \mathcal{G}^{*}_{\tau_1} \rangle_s, \phi_1, \Upsilon)|$$

$$\le \sum_{b=d_1}^{s-1} \sum_{u=1}^{s} u^{m+r}\, |\dot{\mathbb{T}}^{(u)}_{s-b}| \cdot e^{-\eta d_1}\, \mathrm{t}_b\, 2^r\, \mathrm{D}^p\, k_0^q\, (2s(\mathrm{D}+k_0))^{\mu_3'}. \tag{6.57}$$

Standard computations show that (6.57) proves (6.55) in the case of $R = 1$.

With the help of (6.55), we can deduce from (6.52) the following inequality:

$$\Sigma''_{R+1} = \sum_{u=1}^{s} u^m \sum_{\mathcal{H}_R} \sum_{\phi'=1}^{u} \dot{\Sigma}^{(2)}(\phi') \leq \frac{4d_{R+1}}{3}\, 2^{R+1}\, e^{-\eta D_{R+1}}\, \mathrm{t}_s$$

$$B_{m+r}\, 2^r\, \mathrm{D}^p\, k_0^q\, (s(\mathrm{D}+k_0))^{2\mu'_3}.$$

Remembering (6.54), we obtain that

$$\Sigma'_{R+1} + \Sigma''_{R+1} \leq \left(2 \cdot 4^R\, D_R + \frac{4d_{R+1}}{3}\, 2^{R+1}\right)\, e^{-\eta D_{R+1}}\, \mathrm{t}_s$$

$$\times B_{m+r}\, 2^r\, \mathrm{D}^p\, k_0^q\, (s(\mathrm{D}+k_0))^{2\mu'_3}.$$

It is clear that the last inequality implies (6.27) with R replaced by $R+1$. Lemma 6.3 is proved. ■

6.3.3. Walks with mirror cells at $\breve{\beta}$. Let us consider a family of walks $\mathbb{W}_{2s}^{(u)}(\mathrm{D}, \bar{d}_R; \langle \mathcal{G}_R^\star \rangle_s, \mathcal{H}_R, \Upsilon)$ (4.16) with the given set $(\bar{x}, \bar{m})_I$ and assume that x_1 is attributed by a number $m_1 > 0$. This means that the walk arrives at $\breve{\beta}$ at the marked instant x_1 and then performs a tree-type sub-walk $\breve{\mathcal{W}} = \sqcup_{i=1}^{m_1} \breve{\mathcal{W}}^{(i)}$ such that during this sub-walk it arrives m_1 times by non-marked steps at $\breve{\beta}$. Moreover, each of the m_1 tree-type sub-walks $\breve{\mathcal{W}}^{(i)}$ has a number of marked instants $x_j^{(i)}$ such that $\breve{\mathcal{W}}^{(i)}(\xi_{x_j^{(i)}}) = \breve{\beta}$. These marked instants being determined, let us denote the maximal one by $x'_1 = \max\{x_j^{(i)}\}$.

The construction of the sub-trees and the nest cell is as follows: we consider a tree $\mathcal{T}_{x_1}^{[l_1]}$ such that the vertex υ_1 is on the distance of l_1 from the tree root ϱ. Then we add l_2 edges to the vertex υ_1 and get the vertex υ_2. On l_2 roots obtained, we construct sub-trees with the help of $x'_1 - x_1$ edges and get the tree $\mathcal{T}_{x'_1}^{[l_1+l_2]}$. The mirror cell we consider will correspond to the arrival at υ_2 by λ_2 steps from υ_2. Then the ordinary procedure of constructing the trees with u-condition like (5.26) can be used.

In the proof of Lemma 6.3 by recurrence, we needed the construction of the last proper imported cell only. Therefore it is clear that the construction of the mirror cell presented above fits completely this scheme. We do not present the detailed arguments here.

7. Estimates from Below

Let us consider the random matrices $H^{(n,\rho)}$ (2.1) with random variables a_{ij} that have all moments finite. Then the following statement for the moments $\mathrm{M}_n^{(n,\rho)}$ (2.5) is true.

Theorem 7.1. *Let $s_n = \chi n^{2/3}$ and $\rho_n = \zeta n^{2/3}$ with given $\chi, \zeta > 0$. Then*

$$\liminf_{n\to\infty} \mathrm{M}_{2s_n}^{(n,\rho_n)} \ \geq\ \frac{4V_4}{\zeta(\pi\chi)^{1/2}} e^{-e\chi^3}(1+o(1)) \ , \tag{7.1}$$

where $V_4 = \mathbf{E}|a_{ij}|^4$ and $\mathbf{E}|a_{ij}|^2 = v^2 = 1/4$.

P r o o f. Let us consider a Dyck path θ_{2s} and the corresponding Catalan tree $\mathcal{T}_s$. Regarding the chronological run $\mathfrak{R}_\mathcal{T}$, let us determine the marked instants τ_1 and τ_2 such that the corresponding vertices of $\mathcal{T}_s$ have the same parent. We denote such a pair by $(\tau_1, \tau_2)_p$. Regarding the example tree $\mathcal{T}_8$ from Figure 1, we can take $(\tau_1, \tau_2)_p = (3, 4)$ or, say, $(\tau_1', \tau_2')_p = (6, 8)$.

It is clear that the tree-type walk $\mathcal{W}_{2s} = \mathcal{W}_{2s}^{[\theta]}(\tau_1, \tau_2)$ that has the Dyck structure $\theta_s = \theta(\mathcal{T}_s)$ and one simple self-intersection $(\tau_1, \tau_2)_p$ is a walk with one p-edge. It is also clear that the family of all possible walks

$$\mathbb{W}_{2s}^{[1,p]} = \bigsqcup_{\mathcal{T}_s} \ \bigsqcup_{(\tau_1,\tau_2)_p} \mathcal{W}_{2s}^{[\theta]}(\tau_1, \tau_2)$$

has the cardinality $|\mathbb{W}_{2s}^{[1,p]}| = \mathrm{N}_s^{(2)}$ (6.12).

Let us denote the elements of $\mathbb{W}_{2s}^{[1,p]}$ by $w = w_{2s}$. We are going to construct a family of walks $\mathbb{W}_{2s}(w; \mu_2)$ by introducing μ_2 additional simple self-intersections in it, $0 \leq \mu_2 \leq M$. The graph of the walk w_{2s} has s vertices and therefore the cardinality of this family is bounded from below as follows:

$$|\mathbb{W}_{2s}(w; \mu_2)| = \frac{s!}{2^{\mu_2}\, \mu_2!(s-2\mu_2)!} \geq \frac{1}{\mu_2!} \left(\frac{(s-2M)^2}{2} \right)^{\mu_2} . \tag{7.2}$$

It is clear that the weight of any trajectory $\mathcal{I}_{2s}$ (2.8) such that $\mathcal{W}(\mathcal{I}_{2s}) \in \mathbb{W}(w; \mu_2)$ admits the lower bound

$$\Pi_a(\mathcal{I}_{2s})\, \Pi_b(\mathcal{I}_{2s}) \ \geq\ \frac{V_2^{s-1}\, V_4}{n^{s-2}\, \rho}. \tag{7.3}$$

Remembering (3.1), we conclude that the number of trajectories in the class $\mathcal{C}(\mathcal{W}_{2s})$, $\mathcal{W}_{2s} \in \mathbb{W}_{2s}(w; \mu_2)$ is bounded as follows:

$$|\mathcal{C}(\mathcal{W}_{2s})| = \prod_{i=0}^{s-\mu_2-1} (n-i) = n^{s-\mu_2} \prod_{k=1}^{s-\mu_2-1} \left(1 - \frac{k}{n}\right) \geq n^{s-\mu_2} \exp\left\{ -\frac{s^2}{2n} \right\}. \tag{7.4}$$

The last inequality is obtained by the same argument as the upper bound (5.15).

Taking $M = \lfloor cn^{1/3} \rfloor + 1$ with $c > 0$, we deduce from relations (7.2), (7.3) and (7.4) that

$$\mathrm{M}_{2s}^{(n,\rho)} \ \geq\ \sum_{w\in\mathbb{W}_{2s}^{[1,p]}} \ \sum_{\mu_2=0}^{M} \ \sum_{\mathcal{I}_{2s}\in\mathcal{C}(\mathcal{W})} \ \sum_{\mathcal{W}\in\mathbb{W}_{2s}(w;\mu_2)} \frac{V_2^{s-1}\, V_4}{n^{s-1}\, \rho}$$

$$\geq n\,\mathrm{N}_s^{(2)} \exp\left\{-\frac{s^2}{2n}\right\} \sum_{\mu_2=0}^{M} \frac{1}{\mu_2!}\left(\frac{(s-M)^2}{2n}\right)^{\mu_2} \frac{V_2^{s-2}V_4}{\rho}. \qquad (7.5)$$

Elementary use of the Stirling formula shows that

$$\sum_{\mu_2=M+1}^{\infty} \frac{1}{\mu_2!}\left(\frac{(s-M)^2}{2n}\right)^{\mu_2} \geq \frac{1}{\sqrt{2\pi c n^{1/3}}} \sum_{\mu_2=M+1}^{\infty} \left(\frac{e\chi^2}{2c}\right)^{\mu_2} = o(1)$$

for $c = e\chi^2$, and therefore with this choice of c we have

$$\exp\left\{-\frac{s^2}{2n}\right\} \sum_{\mu_2=0}^{M} \frac{1}{\mu_2!}\left(\frac{(s-M)^2}{2n}\right)^{\mu_2} \geq \frac{1}{2}\exp\{-e\chi^3/2\}.$$

Remembering the lower bound $\mathrm{N}_s^{(2)} \geq (s\mathrm{t}_s)/2$ (see Lemma 6.2), we deduce from (7.5) the inequality

$$\mathrm{M}_{2s}^{(n,\rho)} \geq n\mathrm{t}_s\, V_2^s\, \frac{s\,V_4}{4V_2^2\rho}\exp\{-e\chi^3\}.$$

Then (7.1) follows. Theorem 7.1 is proved. ■

8. Discussion

We have studied the asymptotic properties of the probability distribution of the spectral norm of large dilute random matrices. We have shown that the probability distribution of the maximal eigenvalue of dilute Wigner random matrices $H^{(n,\rho_n)}$, when regarded at the scale $n^{-2/3}$, admits a universal upper bound in the limit of infinite n, ρ_n such that $n^{2/3(1+\varepsilon)} \leq \rho_n \leq n$, $\varepsilon > 0$ and $s_n = \chi n^{2/3}$, $\chi > 0$. This result is a consequence of the existence of a universal upper bound $\mathfrak{L}(\chi)$ of the moments $\tilde{\mathrm{M}}_{2s_n}^{(n,\rho_n)}$, $s_n = \chi n^{2/3}$ of $\tilde{H}^{(n,\rho_n)}$ (2.6) and, in more general situation, of the moments $\hat{\mathrm{M}}_{2s_n}^{(n,\rho_n)}$ of corresponding random matrices with truncated elements.

According to the general scheme developed in papers [23, 25], in the case of Wigner ensemble of random matrices, this kind of asymptotic behavior of the moments $\mathrm{M}_{2s_n} = \mathbf{E}\mathrm{L}_{2s_n}$ can be regarded as the strong evidence of the universality of the probability distribution of one maximal eigenvalue of $H^{(n,\rho_n)}$, or its several consecutive neighbors (see also [3, 24]). Indeed, as it is described in [25], the study of the correlation functions of $\mathrm{L}_{2s_n'}$ and $\mathrm{L}_{2s_n''}$ can be reduced, in a major part, to the study of the related moment $\mathrm{M}_{2s_n'+2s_n''-2}$ whose behavior can be shown to be universal (see, however, [9]).

Therefore one can expect that the dilute Wigner random matrices in the limit of the *weak dilution*, i.e., for the values of ρ_n such that $n^{2/3} \ll \rho_n \leq n$, $n \to$

∞, belong to the class of universality determined by the Gaussian Orthogonal Ensemble of random matrices (GOE) in the case of real symmetric $H^{(n,\rho_n)}$, or to the class of GUE in the hermitian case [16].

Theorem 7.1 shows that in the asymptotic regime $n, \rho_n \to \infty$ such that $\rho_n = \zeta n^{2/3}$, the estimate from below of $\mathrm{M}_{2s_n}^{(n,\rho_n)}$ involves the factor V_4. Therefore the upper bound $\limsup_{n\to\infty} \mathrm{M}_{2s_n}^{(n,\rho_n)} \le \mathfrak{M}(\chi)$ (2.6) cannot be true in this asymptotic regime. This means that in the asymptotic regimes of the *moderate and strong* dilutions, when $\rho_n = \zeta^{2/3}$ or $\rho_n = o(n^{2/3})$, respectively, the limiting probability distribution of the maximal eigenvalue cannot be universal in the sense that the limiting expressions should depend on the moments higher than V_2 of the random variables a_{ij}.

Moreover, inequality (7.1) shows that in the asymptotic regime $n, \rho_n \to \infty$ when $\rho_n = \sigma n^{\epsilon}$ with $0 < \epsilon < 2/3$, to get the finite upper bound for the moment $\mathrm{M}_{2s_n}^{(n,\rho_n)}$ of the order s_n, one should restrict the growth of s_n and consider the case when s_n is proportional to ρ_n but not to $n^{2/3}$ as before. According to the general considerations based on the inequalities of the form (4.26), one can conclude that the scale at the border of the limiting spectrum of $H^{(n,\rho_n)}$ should also be changed to be proportional to ρ_n and not to $n^{2/3}$ as it is in the case of $n^{2/3(1+\varepsilon)} \le \rho_n$.

Therefore we can put forward a conjecture that the rate $\rho_n = n^{2/3}$ represents the critical point where the eigenvalue distribution at the edge of the spectrum changes its properties, such as the scale and the dependence on the probability distribution of the matrix elements a_{ij}.

Another important observation concerns the subsequent terms of the estimate from below given by (7.1). Repeating the computations of the proof of Theorem 7.1 and using (6.23), we observe that the moments $\mathrm{M}_{2s}^{(n,\rho)}$ (2.5) admit the following asymptotic expansion:

$$\mathrm{M}_{2s}^{(n,\rho)} \simeq \frac{n\mathrm{t}_s}{4^s} \left(c^{(1)} + \sum_{k\ge 1} c_k^{(2)} \left(\frac{s\,V_4}{\rho}\right)^k + \sum_{l\ge 1} c_l^{(3)} \left(\frac{s\,V_6}{\rho^2}\right)^l + o(s/\rho^2) \right), \tag{8.1}$$

where $s = \chi n^{2/3}$, $\rho = \zeta n^{2/3}$ and $c^{(i)} > 0$ depend on χ and ζ but do not depend on n. In this case, the terms with V_4 are present in the asymptotic development of $\mathrm{M}_{2s}^{(n,\rho)}$, but the higher moments $V_6, V_8, \dots$ disappear from it. Therefore we can formulate a conjecture that the regimes of moderate and strong dilutions exhibit a new kind of universality, say, V_4-*universality* at the border of the limiting spectra.

The next observation is that the asymptotic expansion of the form

$$\frac{4^s}{n\mathrm{t}_s}\mathrm{M}_{2s}^{(n,\rho)} \simeq c^{(1)} + \sum_{k\ge 1} c_k^{(2)} \left(\frac{s\,V_4}{\rho}\right)^k, \quad s = \chi n^{2/3}, \quad \rho = \zeta n^{2/3} \tag{8.2}$$

is related with the corresponding limit of the moments of sparse random matrices studied in [13]. The fact that the right-hand side of (8.2) depends on the terms with V_4 only could essentially simplify the recurrent relations obtained in [13]. Moreover, it is natural to assume that the coefficients $c_k^{(2)}$ will be related with the corresponding terms of $\lim_{s/\rho=\chi/\zeta,\ s\to\infty} m_s^{(\rho)}$, where $m_s^{(\rho)}$ are determined by the following recurrent relation with obvious initial conditions:

$$m_s^{(\rho)} = v^2 \sum_{a_1+a_2=s-1} m_{a_1}^{(\rho)} m_{a_2}^{(\rho)} + \frac{V_4}{\rho} \sum_{b_1+\cdots+b_4=s-2} m_{b_1}^{(\rho)} m_{b_2}^{(\rho)} m_{b_3}^{(\rho)} m_{b_4}^{(\rho)}. \tag{8.3}$$

One can show that the asymptotic expression of the numbers $m_s^{(\rho)}$ (8.3) should be related with the generating functions of the numbers of ternary trees. We postpone the study of (8.3) to subsequent publications.

Our last remark is related with the difference between the ensembles of real symmetric and hermitian matrices. As it is mentioned in Section 2, the upper bounds $\mathfrak{M}_{\mathrm{GOE}}(\chi)$ and $\mathfrak{M}_{\mathrm{GUE}}(\chi)$ are slightly different (see also relations (4.35) and (4.36)). This difference is due to the contribution of walks with simple open self-intersections and the breaks of the tree structure performed by the walk at them. The contribution of these walks should vanish in the asymptotic regime of the strong dilution, when $\rho_n \le n^{2/3-\varepsilon}$ and $s_n \le n^{2/3-\varepsilon}$, and only the tree-type walks give the non-vanishing contribution. This means that the difference between the spectral properties of real symmetric random matrices and their hermitian analogs could disappear in the asymptotic regime of the strong dilution. It would be interesting to study this phenomenon in more details. It should be noted that the moments $\mathrm{M}_{2s}^{(n,\rho)}$ of random matrices $H^{(n,\rho)}$ (2.1) with $\rho = O(1)$ as $n \to \infty$ were studied in [13] in the case of $s = O(1)$. The explicit expressions obtained there as well as the technique developed in [7] could be useful in the studies of more complicated asymptotic regime described above, when $s_n \to \infty$ as $n \to \infty$.

Acknowledgement. The author is grateful to the anonymous referee for the careful reading of the manuscript.

References

[1] *G.W. Anderson, A. Guionnet, and O. Zeitouni*, An Introduction to Random Matrices. Cambridge Studies in Advanced Mathematics, 118. Cambridge University Press, Cambridge, 2010.

[2] *Z.D. Bai and Y.Q. Yin*, Necessary and Sufficient Conditions for the Almost Sure Convergence of the Largest Eigenvalue of Wigner Matrices. — *Ann. Probab.* **16** (1988), 1729–1741.

[3] *O.N. Feldheim and S. Sodin*, A Universality Result for the Smallest Eigenvalues of Certain Sample Covariance Matrices. — *Geom. Funct. Anal.* **20** (2010), 88–123.

[4] *Z. Füredi and J. Komlós*, The Eigenvalues of Random Symmetric Matrices. — *Combinatorica* **1** (1981), 233–241.

[5] *S. Geman*, A Limit Theorem for the Norm of Random Matrices. — *Ann. Probab.* **8** (1980), 252–261.

[6] *V.L. Girko*, Spectral Properties of Random Matrices. Nauka, Moscow, 1988. (Russian)

[7] *A. Khorunzhy*, Sparse Random Matrices: Spectral Edge and Statistics of Rooted Trees. — *Adv. Appl. Probab.* **33** (2001), 124–140.

[8] *O. Khorunzhy*, High Moments of Large Wigner Random Matrices and Asymptotic Properties of the Spectral Norm. — *Random Oper. Stoch. Eq.* **20** (2012), 25–68.

[9] *O. Khorunzhiy*, On Correlation Function of High Moments of Wigner Random Matrices. Preprint *arXiv:1011.3965*

[10] *O. Khorunzhiy, B. Khoruzhenko, L. Pastur, and M. Shcherbina*, The large-n Limit in Statistical Mechanics and the Spectral Theory of Disordered Systems. In: *Phase Transitions and Critical Phenomena* **15**, 74–239, Academic Press, London, 1992.

[11] *O. Khorunzhy, W. Kirsch, and P. Müller*, Lifshitz Tails for Spectra of Erdős-Rényi Random Graphs. — *Ann. Appl. Probab.* **16** (2006), 295–309.

[12] *O. Khorunzhy and J.-F. Marckert*, Uniform Bounds for Exponential Moment of Maximum of a Dyck Path. — *Electr. Commun. Probab.* **14** (2009), 327–333.

[13] *O. Khorunzhy, M. Shcherbina, and V. Vengerovsky*, Eigenvalue Distribution of Large Weighted Random Graphs. — *J. Math. Phys.* **45** (2004), 1648–1672.

[14] *O. Khorunzhiy and V. Vengerovsky*, Even Walks and Estimates of High Moments of Large Wigner Random Matrices. Preprint *arXiv:0806.0157*

[15] *V.A. Marchenko and L.A. Pastur*, Distribution of Eigenvalues for Some Sets of Random Matrices. — *Math. USSR Sb.* **1** (1967), No. 4, 457–483.

[16] *M.L. Mehta*, Random Matrices. Amsterdam: Elsevier/Academic Press, 2004.

[17] *A.D. Mirlin and Ya.V. Fyodorov*, Universality of Level Correlation Function of Sparse Random Matrices. — *J. Phys. A* **24** (1991), 2273–2286.

[18] *L. Pastur*, On the Spectrum of Random Matrices. — *Theor. Math. Phys.* **10** (1972), 102–111.

[19] *C. Porter (Ed.)*, Statistical Theories of Spectra: Fluctuations. Acad. Press, New-York, 1965.

[20] *G.J. Rodgers and A.J. Bray*, Density of States of a Sparse Random Matrix. — *Phys. Rev. B* **37** (1988), 3557–3562.

[21] *A. Ruzmaikina*, Universality of the Edge Distribution of the Eigenvalues of Wigner Random Matrices with Polynomially Decaying Distributions of Entries. — *Commun. Math. Phys.* **261** (2006), 277–296.

[22] *Ya. Sinai and A. Soshnikov*, Central LimitTtheorem for Traces of Large Symmetric Matrices with Independent Matrix Elements. — *Bol. Soc. Brazil. Mat.* **29** (1998), 1–24.

[23] *Ya. Sinai and A. Soshnikov*, A Refinement of Wigner's Semicircle Law in a Neighborhood of the Spectrum Edge for Random Symmetric Matrices. — *Func. Anal. Appl.* **32** (1998), 114–131.

[24] *S. Sodin*, The Tracy-Widom Law for Some Sparse Random Matrices. — *J. Stat. Phys.* **136** (2009), 834–841.

[25] *A. Soshnikov*, Universality at the Edge of the Spectrum in Wigner Random Matrices. — *Commun. Math. Phys.* **207** (1999), 697–733.

[26] *C.A. Tracy and H. Widom*, Level Spacing Distribution and the Airy Kernel. — *Commun. Math. Phys.* **159** (1994), 151–174.

[27] *E. Wigner*, Characteristic Vectors of Bordered Matrices with Infinite Dimensions. — *Ann. Math.* **62** (1955), 548–564.