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# The System of Kaup Equations with a Self-Consistent Source in the Class of Periodic Functions

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In the paper, a method of the inverse spectral problem is used to integrate the system of Kaup equations with a self-consistent source in the class of periodic functions.

*Key words*: quadratic pencil of Sturm–Liouville equations, spectral data, inverse problem, system of Dubrovin equations, system of Kaup equations with a self-consistent source.

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## 1. Introduction

In [1], D.J. Kaup proved that the nonlinear system of equations

$$\begin{cases} \eta_\tau = \Phi_{xx} + \beta^2\Phi_{xxxx} - \varepsilon(\Phi_x\eta)_x \\ \eta = \Phi_\tau + \frac{1}{2}\varepsilon\Phi_x^2, \end{cases}$$

is completely integrable. The system describes the waves propagation in shallow water. In [2], the complex finite-gap multiphase solutions expressed in terms of the Riemann theta-functions are considered, the multi-soliton solutions are found and the asymptotic behavior of these solutions is studied. In [3, 4] and [5, pp. 169–179], the real finite-gap regular solutions of Kaup system were studied.

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It is not difficult (see [2]) to verify that after transformations

$$\eta = \frac{4\beta^2}{\varepsilon}(q+p^2) + \frac{1}{\varepsilon}, \quad \Phi_\tau = \frac{4\beta^2}{\varepsilon}(q+3p^2) + \frac{1}{\varepsilon}, \quad \Phi_x = -\frac{4\beta}{i\varepsilon}p, \quad t = i\beta\tau,$$

the system of Kaup equations takes a simpler form

$$\begin{cases} p_t = -6pp_x - q_x \\ q_t = p_{xxx} - 4qp_x - 2pq_x. \end{cases}$$

This system we will also call the Kaup system.

The Kaup system can be considered as a compatibility condition (see [2])

$$y_{xxt} - y_{txx} \equiv [(q_t - p_{xxx} + 4qp_x + 2pq_x) + 2\lambda(p_t + 6pp_x + q_x)]y = 0$$

for the system of the linear equations

$$\begin{cases} -y_{xx} + qy + 2\lambda py - \lambda^2 y = 0 \\ y_t + 2(p+\lambda)y_x - p_x y = 0. \end{cases}$$

The first of these equations is called the quadratic pencil of Sturm–Liouville equations.

The inverse problem for the quadratic pencil of Sturm–Liouville equations in the class of "rapidly decreasing" coefficients by scattering data on the half line and whole line was solved in the works of M. Jaulent [6], M. Jaulent, I. Miodek [7], M. Jaulent, C. Jean [8, 9], F.G. Maksudov, G.Sh. Guseinov [10], by the Weyl–Titchmarsh function, in the work of V.A. Yurko [11], on the finite interval by spectrum and normalization constants as well as by two spectra was studied by M.G. Gasimov, G.Sh. Guseinov in [12], with periodical potential on the whole line by G.Sh. Guseinov in [13–16], B.A. Babazhanov, A.B. Khasanov, A.B. Yakhshimuratov in [17] and A.B. Yakhshimuratov in [18].

In this paper, the method of the inverse spectral problem for the quadratic pencil of Sturm–Liouville equations with periodic coefficients is used to integrate the system of Kaup equations with a self-consistent source in the class of periodic functions. We note that some nonlinear equations with a self-consistent source in the class of periodic functions were studied in [19–21].

## 2. Problem Statement

We consider the system of Kaup equations with a self-consistent source

$$p_t = -6pp_x - q_x + \int_{-\infty}^{\infty} \beta(\lambda,t)s(\pi,\lambda,t)(\psi_+\psi_-)_x d\lambda, \tag{1}$$

$$q_t = p_{xxx} - 4qp_x - 2pq_x$$
$$+2\int_{-\infty}^{\infty} \beta(\lambda,t)s(\pi,\lambda,t)\{-p_x\psi_+\psi_- + (\lambda - 2p)(\psi_+\psi_-)_x\}d\lambda, \tag{2}$$

in the class of real-valued $\pi$-periodic on the spatial variable $x$ functions $p = p(x,t)$ and $q = q(x,t)$ which satisfy the regularity of assumptions

$$p \in C_x^3(t > 0) \cap C_t^1(t > 0) \cap C(t \geq 0), \quad q \in C^1(t > 0) \cap C(t \geq 0)$$

with the initial conditions

$$p(x,t)|_{t=0} = p_0(x)\,, \qquad q(x,t)|_{t=0} = q_0(x). \tag{3}$$

Here $p_0 \in C^3(R)$, $q_0 \in C^2(R)$ are the given real-valued $\pi$-periodic functions such that for any nontrivial function $y \in W_2^2[0,\pi]$ satisfying the equalities $y'(0)\bar{y}(0) - y'(\pi)\bar{y}(\pi) = 0$ and $(y,y) = 1$, the following inequality holds:

$$(p_0 y, y)^2 + (q_0 y, y) + (y', y') > 0,$$

where $(\cdot\,,\ \cdot)$ is a scalar product of the space $L_2(0,\ \pi)$. The last condition we will call condition (A).

In the previous expressions, $\beta(\lambda,t)$ is a given real-valued continuous function having a uniform asymptotic behavior $\beta(\lambda,t) = O(\lambda^{-2})$, $\lambda \to \pm\infty$ and $\psi_\pm = \psi_\pm(x,\lambda,t)$ are the Floquet solutions (normalized by the condition $\psi_\pm(0,\lambda,t) = 1$) of the quadratic pencil of Sturm–Liouville equations

$$T(\lambda,t)y \equiv -y'' + qy + 2\lambda py - \lambda^2 y = 0\,, \quad x \in R. \tag{4}$$

We denote by $s(x,\lambda,t)$ the unique solution of equation (4) satisfying the initial conditions $s(0,\lambda,t) = 0$, $s'(0,\lambda,t) = 1$.

The aim of this work is to develop a procedure for constructing the solution $(p(x,t),\, q(x,t),\, \psi_+(x,\lambda,t),\, \psi_-(x,\lambda,t))$ of problem (1)–(4) by means of the inverse spectral problem for the quadratic pencil of Sturm–Liouville equations (4).

We note that the Lax pair for system (1), (2) consists of equation (4) and the equation

$$y_t + 2(p+\lambda)y_x - p_x y + F(x,\lambda,t) = 0,$$

where

$$F(x,\lambda,t) = \int_{-\infty}^{\infty} \frac{\beta(\mu,t)s(\pi,\mu,t)\psi_+(x,\mu,t)W\{\psi_-(x,\mu,t), y(x,\lambda,t)\}}{\lambda - \mu} d\mu.$$

Here $W\{z(x), y(x)\} = z(x)y'(x) - z'(x)y(x)$.

The Floquet solutions for (4) are defined in a similar way as for the Sturm–Liouville equation. Using the expression for the Floquet solutions, it is easy to derive the identity

$$s(\pi,\lambda,t)\psi_+(\tau,\lambda,t)\psi_-(\tau,\lambda,t)=s(\pi,\lambda,t,\tau), \tag{5}$$

where $s(x,\lambda,t,\tau)$ is the solution of the quadratic pencil of Sturm–Liouville equations with coefficients $p(x+\tau,t)$ and $q(x+\tau,t)$ satisfying the initial conditions $s(0,\lambda,t,\tau)=0$, $s'(0,\lambda,t,\tau)=1$. In particular, from equality (5) it follows that the integrals in equations (1) and (2) are $\pi$-periodic on $x$, so we can speak about periodic solutions of system (1), (2).

Equality (5) and asymptotic formulas (see [22])

$$s(\pi,\lambda,t,\tau)=O(\lambda^{-1}),\quad \frac{\partial s(\pi,\lambda,t,\tau)}{\partial\tau}=O(\lambda^{-1}),\quad \lambda\to\pm\infty,$$

provide a uniform convergence of the integrals in equations (1) and (2). Identity (5) will also play the main role in Section 6.

The function $\psi_+(x,\lambda,t)\psi_-(x,\lambda,t)$ has the pole of the first order at the zeros $\lambda=\xi_n(t)$, $n\in Z\backslash\{0\}$ of the function $s(\pi,\lambda,t)$. Thus, we will understand the expression

$$s(\pi,\xi_n(t),t)\psi_+(x,\xi_n(t),t)\psi_-(x,\xi_n(t),t)$$

as

$$s(\pi,\xi_n(t),t)\psi_+(x,\xi_n(t),t)\psi_-(x,\xi_n(t),t)=\lim_{\lambda\to\xi_n(t)}s(\pi,\lambda,t)\psi_+(x,\lambda,t)\psi_-(x,\lambda,t).$$

## 3. Preliminaries

In this section, for the sake of completeness, we will give some information about the theory of the inverse problem for the quadratic pencil of Sturm–Liouville equations (see [13–16]).

We consider the quadratic pencil of Sturm–Liouville equations (4). The spectrum of problem (4) coincides with the set

$$\sigma(T)=\big\{\lambda\in C:\ \operatorname{Im}\Delta(\lambda)=0,\ \Delta^2(\lambda)\le 4\big\},$$

where $\Delta(\lambda)=c(\pi,\lambda,t)+s'(\pi,\lambda,t)$ is called a Lyapunov function or Hill discriminant of the quadratic pencil (4) (see [4], [5, pp. 169–179]). Here $c(x,\lambda,t)$ is the solution of equation (4) which satisfies the initial conditions $c(0,\lambda,t)=1$, $c'(0,\lambda,t)=0$. If $q\in L_2[0,\pi]$ and $p\in W_2^1[0,\pi]$ are the real-valued $\pi$-periodic functions satisfying condition (A), then the spectrum of problem (4) is real and it coincides with the set

$$\sigma(T) = \{\lambda \in R| -2 \leq \Delta(\lambda) \leq 2\} = R \backslash \bigcup_{n=-\infty}^{\infty} (\lambda_{2n-1},\ \lambda_{2n}).$$

The intervals $(\lambda_{2n-1},\ \lambda_{2n})$, $n \in Z$ are called the gaps or lacunas. The numbering is introduced such that $\lambda_{-1} < 0 < \lambda_0$.

The numbers $\xi_n = \xi_n(t)$ with the signs $\sigma_n = \sigma_n(t) = \operatorname{sign}\{s'(\pi, \xi_n, t) - c(\pi, \xi_n, t)\}$, $n \in Z\backslash\{0\}$ are called the spectral parameters of problem (4). Notice that $\xi_n$ coincides with the eigenvalues of the Dirichlet problem for equation (4). Moreover, the inclusions $\xi_n \in [\lambda_{2n-1}, \lambda_{2n}]$ and the equality

$$s(\pi, \lambda, t) = \pi \prod_{0 \neq k=-\infty}^{\infty} \frac{\xi_k - \lambda}{k} \tag{6}$$

are fulfilled.

R e m a r k 1. If $\xi_n = \lambda_{2n-1}$ or $\xi_n = \lambda_{2n}$, then $s'(\pi, \xi_n, t) - c(\pi, \xi_n, t) = 0$. For determinacy, in this case we will assume $\sigma_n = 1$.

The boundaries $\lambda_n$ of the spectrum and the spectral parameters $\xi_n$, $\sigma_n$ are called the spectral data of problem (4). The determination of spectral data of problem (4) is called a direct problem and conversely, the restoration of the coefficients $p$ and $q$ of problem (4) by spectral data is called an inverse problem.

The spectrum of the quadratic pencil of Sturm–Liouville equations with the coefficients $p(x+\tau, t)$ and $q(x+\tau, t)$ does not depend on the real parameter $\tau$, but the spectral parameters do. It is not difficult to prove that the spectral parameters are $\pi$-periodic on $\tau$. The spectral parameters satisfy the system of Dubrovin differential equations

$$\frac{\partial \xi_n}{\partial \tau} = 2(-1)^{n-1} \operatorname{sign}(n) \sigma_n \sqrt{(\xi_n - \lambda_{2n-1})(\lambda_{2n} - \xi_n)}\, h_n(\xi), \quad n \in Z\backslash\{0\}, \tag{7}$$

where

$$h_n(\xi) = h_n(\dots, \xi_{-1},\ \xi_1,\ \dots) = \sqrt{(\xi_n - \lambda_{-1})(\xi_n - \lambda_0) \prod_{k \neq n, 0} \frac{(\xi_n - \lambda_{2k-1})(\xi_n - \lambda_{2k})}{(\xi_n - \xi_k)^2}}\,.$$

The system of Dubrovin equations and the following first and second trace formulas:

$$p(\tau, t) = \frac{\lambda_{-1} + \lambda_0}{2} + \sum_{0 \neq k=-\infty}^{\infty} \left( \frac{\lambda_{2k-1} + \lambda_{2k}}{2} - \xi_k \right),$$

$$q(\tau,t)+2p^2(\tau,t)=\frac{(\lambda_{-1})^2+(\lambda_0)^2}{2}+\sum_{0\neq k=-\infty}^{\infty}\left(\frac{(\lambda_{2k-1})^2+(\lambda_{2k})^2}{2}-\xi_k^2\right)$$

provide the method for solving the inverse problem.

## 4. Main Result

The main result of the paper is included in the theorem below.

**Theorem.** *Let $(p(x,t),\ q(x,t),\ \psi_+(x,\lambda,t),\ \psi_-(x,\lambda,t))$ be the solution of problem* (1)–(4). *Then the spectrum of problem* (4) *does not depend on $t$, and the spectral parameters $\xi_n$ satisfy the analogue of the system of Dubrovin equations*

$$\frac{\partial \xi_n}{\partial t}=2(-1)^n\sigma_n \mathrm{sign}(n)\sqrt{(\xi_n-\lambda_{2n-1})(\lambda_{2n}-\xi_n)}\,g_n(\xi)h_n(\xi),\quad n\in Z\backslash\{0\},\tag{8}$$

*where*

$$g_n(\xi)=g_n(\ldots,\xi_{-1},\ \xi_1,\ \ldots)=\lambda_{-1}+\lambda_0+\sum_{0\neq k=-\infty}^{\infty}(\lambda_{2k-1}+\lambda_{2k}-2\xi_k)+2\xi_n$$

$$+\int_{-\infty}^{\infty}\frac{\beta(\lambda,t)s(\pi,\lambda,t)}{\xi_n-\lambda}d\lambda.$$

*The sign $\sigma_n=\pm1$ changes at each collision of the point $\xi_n$ with the boundaries of its gap $[\lambda_{2n-1},\ \lambda_{2n}]$. Moreover, the following initial conditions are fulfilled*:

$$\xi_n|_{t=0}=\xi_n^0,\quad \sigma_n|_{t=0}=\sigma_n^0,\quad n\in Z\backslash\{0\},\tag{9}$$

*where $\xi_n^0$, $\sigma_n^0$ are the spectral parameters of the quadratic pencil of Sturm–Liouville equations corresponding to the coefficients $p_0(x)$ and $q_0(x)$.*

P r o o f. Denoting the sum of the last two addends in equations (1) and (2) by $G_1(x,t)$ and $G_2(x,t)$, we rewrite the system of equations (1), (2) in the form

$$\begin{cases} p_t=-6pp_x-q_x+G_1\\ q_t=p_{xxx}-4qp_x-2pq_x+G_2.\end{cases}\tag{10}$$

Let $y_n(x,t)$ be the normalized eigenfunction of the Dirichlet problem for equation (4) corresponding to the eigenvalue $\xi_n$. It is easy to see that

$$y_n(x,t)=\frac{1}{c_n(t)}s(x,\xi_n(t),t),\tag{11}$$

where

$$c_n^2(t)=\int_0^{\pi}s^2(x,\xi_n(t),t)dx.$$

Differentiating the identity

$$-(y_n'', y_n) + (qy_n, y_n) + 2\xi_n(py_n, y_n) - \xi_n^2 = 0$$

with respect to $t$, we get

$$\begin{gathered}-(\dot{y}_n'', y_n) - (y_n'', \dot{y}_n) + (q_t y_n + q\dot{y}_n, y_n) + (qy_n, \dot{y}_n)\\ +2\dot{\xi}_n(py_n, y_n) + 2\xi_n(p_t y_n + p\dot{y}_n, y_n) + 2\xi_n(py_n, \dot{y}_n) - 2\xi_n\dot{\xi}_n = 0\,.\end{gathered}$$

From the last equality we obtain

$$\begin{gathered}\left(-\dot{y}_n'' + q\dot{y}_n + 2\xi_n p\dot{y}_n, y_n\right) + \left(-y_n'' + qy_n + 2\xi_n py_n, \dot{y}_n\right)\\ + (q_t y_n + 2\xi_n p_t y_n, y_n) + 2\dot{\xi}_n(py_n, y_n) - 2\xi_n\dot{\xi}_n = 0,\\ 2\dot{\xi}_n[\xi_n - (py_n, y_n)] = (q_t y_n + 2\xi_n p_t y_n, y_n)\,,\end{gathered}$$

that is,

$$2\dot{\xi}_n\left(\xi_n - \int_0^\pi py_n^2 dx\right) = \int_0^\pi (q_t + 2\xi_n p_t)y_n^2 dx. \tag{12}$$

Now, substituting expressions (10) into formula (12), we deduce the equality

$$\begin{gathered}2\dot{\xi}_n\left(\xi_n - \int_0^\pi py_n^2 dx\right) = \int_0^\pi p_{xxx}y_n^2 dx - \int_0^\pi (2p + 2\xi_n)y_n^2 q_x dx\\ -\int_0^\pi (4q + 12\xi_n p)y_n^2 p_x dx + \int_0^\pi (G_2 + 2\xi_n G_1)y_n^2 dx.\end{gathered} \tag{13}$$

Integrating the first two integrals in the right-hand side of (13) by parts and using the Dirichlet conditions, we obtain the identities

$$\int_0^\pi p_{xxx}y_n^2 dx = \int_0^\pi (2(y_n')^2 + 2y_n y_n'')p_x dx, \tag{14}$$

$$\int_0^\pi (2py_n^2 + 2\xi_n y_n^2)q_x dx = -\int_0^\pi (2p_x y_n^2 + 4py_n y_n' + 4\xi_n y_n y_n')q dx. \tag{15}$$

Substituting (14) and (15) into (13), we deduce the equality

$$2\dot{\xi}_n\left(\xi_n - \int_0^\pi py_n^2 dx\right)$$

$$= \int_0^\pi [2p_x(y_n')^2 + 2p_x y_n y_n'' - 12\xi_n p p_x y_n^2 + (-2p_x y_n + 4py_n' + 4\xi_n y_n')qy_n]dx$$

$$+ \int_0^\pi (G_2 + 2\xi_n G_1)y_n^2 dx. \tag{16}$$

Using the identity

$$qy_n = \xi_n^2 y_n + y_n'' - 2\xi_n p y_n,$$

we can rewrite the first integral in the right-hand side of (16) in the form

$$I_1 \equiv \int_0^\pi \{(2p_x(y_n')^2 + 4py_n'y_n'') - (2\xi_n^2 p_x y_n^2 + 4\xi_n^2 p y_n y_n')$$

$$-(8\xi_n p p_x y_n^2 + 8\xi_n p^2 y_n y_n')\}dx + \int_0^\pi [4\xi_n^3 y_n y_n' + 4\xi_n y_n' y_n'']dx.$$

Hence, calculating the last integrals, we find that

$$I_1 = 2(p(0,t) + \xi_n)[(y_n'(\pi,t))^2 - (y_n'(0,t))^2]. \tag{17}$$

Now we calculate the last integral in (16)

$$I_2 \equiv \int_0^\pi (G_2 + 2\xi_n G_1)y_n^2 dx = \int_{-\infty}^{\infty} \beta(\lambda,t)s(\pi,\lambda,t)J(\lambda,t)d\lambda, \tag{18}$$

where

$$J(\lambda,t) \equiv -2\int_0^\pi p_x y_n^2 \psi_+\psi_- dx + 2\int_0^\pi (\xi_n + \lambda - 2p)y_n^2(\psi_+\psi_-)_x dx.$$

It is not difficult to verify that

$$J(\lambda,t) = -2\int_0^\pi p_x y_n^2\, \psi_+\psi_-\, dx + \int_0^\pi (\xi_n + \lambda - 2p)y_n^2\,(\psi_+'\psi_- + \psi_+\psi_-')dx$$

$$-\int_0^\pi [-2p_x y_n^2 + 2(\xi_n + \lambda - 2p)y_n y_n']\, \psi_+\psi_-\, dx$$

$$= \int_0^\pi (\xi_n + \lambda - 2p) y_n \psi_- (y_n \psi_+' - y_n' \psi_+) dx$$

$$+ \int_0^\pi (\xi_n + \lambda - 2p) y_n \psi_+ (y_n \psi_-' - y_n' \psi_-) dx. \tag{19}$$

Using the identity

$$(\xi_n + \lambda - 2p) y_n \psi_\pm = \frac{(y_n \psi_\pm' - y_n' \psi_\pm)'}{\xi_n - \lambda},$$

from (19) we can deduce that

$$J(\lambda, t) = \frac{1}{\xi_n - \lambda} [(y_n'(\pi, t))^2 - (y_n'(0, t))^2]. \tag{20}$$

Substituting (20) into (18), we obtain

$$I_2 = \left\{ \int_{-\infty}^{\infty} \frac{\beta(\lambda, t) s(\pi, \lambda, t)}{\xi_n - \lambda} d\lambda \right\} [(y_n'(\pi, t))^2 - (y_n'(0, t))^2]. \tag{21}$$

Hence, by means of expressions (16), (17) and (21), we may conclude that

$$2\dot{\xi}_n \left( \xi_n - \int_0^\pi p y_n^2 dx \right) = [2p(0, t) + 2\xi_n][(y_n'(\pi, t))^2 - (y_n'(0, t))^2]$$

$$+ \left\{ \int_{-\infty}^{\infty} \frac{\beta(\lambda, t) s(\pi, \lambda, t)}{\xi_n - \lambda} d\lambda \right\} [(y_n'(\pi, t))^2 - (y_n'(0, t))^2]. \tag{22}$$

From the identity (see [16], p. 56)

$$2 \int_0^\pi [\lambda - p(x, t)] s^2(x, \lambda, t) dx = s'(\pi, \lambda, t) \frac{\partial s(\pi, \lambda, t)}{\partial \lambda} - s(\pi, \lambda, t) \frac{\partial s'(\pi, \lambda, t)}{\partial \lambda}$$

we find

$$2\xi_n(t) c_n^2(t) - 2 \int_0^\pi p(x, t) s^2(x, \xi_n(t), t) dx = s'(\pi, \xi_n(t), t) \frac{\partial s(\pi, \xi_n(t), t)}{\partial \lambda}. \tag{23}$$

Substituting expression (11) into equality (22), we have

$$2\dot{\xi}_n(t)\left(\xi_n(t)c_n^2(t)-\int\limits_0^\pi ps^2(x,\xi_n(t),t)dx\right)$$

$$=2[p(0,t)+\xi_n(t)][s'^2(\pi,\xi_n(t),t)-1]$$

$$+\left\{\int\limits_{-\infty}^{\infty}\frac{\beta(\lambda,t)s(\pi,\lambda,t)}{\xi_n-\lambda}\,d\lambda\right\}[s'^2(\pi,\xi_n(t),t)-1],$$

and using identity (23,) we get

$$\dot{\xi}_n(t)\frac{\partial s(x,\xi_n(t),t)}{\partial\lambda}=2[p(0,t)+\xi_n(t)]\left(s'(\pi,\xi_n(t),t)-\frac{1}{s'(x,\xi_n(t),t)}\right)$$

$$+\left\{\int\limits_{-\infty}^{\infty}\frac{\beta(\lambda,t)s(\pi,\lambda,t)}{\xi_n-\lambda}d\lambda\right\}\left(s'(\pi,\xi_n(t),t)-\frac{1}{s'(x,\xi_n(t),t)}\right). \tag{24}$$

Now, substituting the values $x=\pi$ and $\lambda=\xi_n(t)$ into the identity

$$c(x,\lambda,t)s'(x,\lambda,t)-c'(x,\lambda,t)s(x,\lambda,t)=1,$$

we find that

$$c(\pi,\xi_n(t),t)=\frac{1}{s'(\pi,\xi_n(t),t)}. \tag{25}$$

According to (25) and the identity

$$[c(\pi,\lambda,t)-s'(\pi,\lambda,t)]^2=(\Delta^2(\lambda)-4)-4c'(\pi,\lambda,t)s(\pi,\lambda,t),$$

we obtain the equality

$$s'(\pi,\xi_n(t),t)-\frac{1}{s'(\pi,\xi_n(t),t)}=\sigma_n\sqrt{\Delta^2(\xi_n(t))-4}. \tag{26}$$

Using (6), (26) and the expansion

$$\Delta^2(\lambda)-4=-4\pi^2(\lambda-\lambda_{-1})(\lambda-\lambda_0)\prod_{0\neq k=-\infty}^{\infty}\frac{(\lambda-\lambda_{2k-1})(\lambda-\lambda_{2k})}{k^2},$$

we deduce

$$\frac{s'(\pi,\xi_n(t),t)-\frac{1}{s'(x,\xi_n(t),t)}}{\frac{\partial s(\pi,\xi_n(t),t)}{\partial\lambda}}$$

$$= 2(-1)^n \sigma_n \mathrm{sign}(n) \sqrt{(\xi_n - \lambda_{2n-1})(\lambda_{2n} - \xi_n)}\, h_n(\xi). \tag{27}$$

Here we also used the equality

$$\mathrm{sign}\left\{ -\frac{\pi}{n} \prod_{k \neq n,0} \frac{\xi_k - \xi_n}{k} \right\} = (-1)^n \mathrm{sign}(n).$$

From (24), (27) and the first trace formula we conclude (8).

We notice that if instead of Dirichlet boundary conditions we consider periodic or anti-periodic boundary value conditions, then equation (22) remains $\dot{\lambda}_n(t) = 0$, $n \in Z$. Hence, the spectrum of problem (4) does not depend on the parameter $t$, and the theorem is proved.

## 5. Solvability of the Cauchy Problem (8), (9)

In the case when $\beta(\lambda, t)$ does not depend on $t$, we study the existence and uniqueness of the solution of the Cauchy problem (8), (9). By following [23, Chapter 9], we do the substitution

$$\xi_n = \lambda_{2n-1} + (\lambda_{2n} - \lambda_{2n-1}) \sin^2 x_n(t), \quad n \in Z \backslash \{0\}. \tag{28}$$

We note that when the variable $\xi_n$ passes through the endpoint of a band gap, either $\sigma_n$ or the product $\sin x_n(t) \cos x_n(t)$ changes the sign. If we choose the initial conditions

$$x_n(0) = x_n^0 = \arcsin \sqrt{\frac{\xi_n^0 - \lambda_{2n-1}}{\lambda_{2n} - \lambda_{2n-1}}}, \quad n \in Z \backslash \{0\}, \tag{29}$$

then $\sigma_n(t)\mathrm{sign}\{\sin x_n(t) \cos x_n(t)\} = \sigma_n(0)$. After substituting (28), the system (8) takes the form

$$\frac{dx_n}{dt} = H_n(\ldots,\, x_{-1}, x_1,\, \ldots), \quad n \in Z \backslash \{0\}, \tag{30}$$

where $H_n(\ldots,\, x_{-1}, x_1,\, \ldots) = (-1)^n \sigma_n(0) \mathrm{sign}(n) g_n(\xi) h_n(\xi)$. To study the solvability of the Cauchy problem (30), (29), we consider the Banach space K of the sequences $\{x \in \mathrm{K} :\ x = (\ldots,\, x_{-1}, x_1,\, \ldots),\ x_n \in R\}$ with the norm

$$\|x\| = \sum_{0 \neq n = -\infty}^{\infty} (\lambda_{2n} - \lambda_{2n-1}) |x_n|.$$

We put $H(x) = (\ldots,\ H_{-1}(x),\ H_1(x),\ \ldots)$. Then the system of equations (30) can be rewritten as one equation in the Banach space K

$$\frac{dx}{dt} = H(x), \tag{31}$$

and initial conditions (29) can be rewritten in the form

$$x(t)\big|_{t=0} = x^0\,, \quad x^0 \in \mathrm{K}. \tag{32}$$

From the conditions $p_0(x) \in C^3(R)$ and $q_0(x) \in C^2(R)$ there follow the asymptotics (see [16])

$$\lambda_{2n-1} = n + c_0 + \frac{c_1}{n} + \frac{c_2}{n^2} + \frac{c_3}{n^3} + \frac{\varepsilon_n^-}{n^3}, \quad \lambda_{2n} = n + c_0 + \frac{c_1}{n} + \frac{c_2}{n^2} + \frac{c_3}{n^3} + \frac{\varepsilon_n^+}{n^3}, \tag{33}$$

where $c_k,\ k = 0,\ 1,\ 2,\ 3$ are constants, and $\{\varepsilon_n^\pm\} \in l_2$. Consequently, taking into consideration that $\xi_n \in [\lambda_{2n-1},\ \lambda_{2n}]$, we get that $\inf\limits_{k \neq n,0} |\xi_n - \xi_k| \geq a > 0$. Using these facts, we deduce the estimates

$$|g_n(\xi)| \leq C_1|n|, \quad \left|\frac{\partial g_n(\xi)}{\partial \xi_m}\right| \leq C_2,$$

$$C_3|n| \leq |h_n(\xi)| \leq C_4|n|, \quad \left|\frac{\partial h_n(\xi)}{\partial \xi_m}\right| \leq C_5|n|,$$

where the constants $C_k,\ k = 1,\ 2,\ 3,\ 4,\ 5$ are positive and do not depend on $n$ and $m$.

Next, for the derivatives of the functions $f_n(\xi) = g_n(\xi)h_n(\xi)$ we obtain the estimate

$$\left|\frac{\partial f_n(\xi)}{\partial \xi_m}\right| \leq Cn^2,$$

where the constant $C > 0$ does not depend on $n$ and $m$. By using this estimate and asymptotics (33), the Lipchitz condition can be easily proved

$$\|H(x) - H(y)\| \leq L\|x - y\|, \quad \forall\ x,\ y \in \mathrm{K},$$

where the constant $L > 0$ does not depend on $x$ and $y$. Thus the solution of the Cauchy problem (31), (32), and hence of the Cauchy problem (30), (29), exists and it is unique for all $t > 0$.

## 6. Corollaries and Remarks

In this section we give some conclusions concerning the main result proved in the previous section.

**Corollary 1.** *If instead of $p(x,t)$ and $q(x,t)$ we consider the functions $p(x+\tau,t)$ and $q(x+\tau,t)$, then, as seen from the previous section, the eigenvalues of the periodic and antiperiodic problems do not depend on the parameters $\tau$ and $t$. However, the eigenvalues $\xi_n$ of the Dirichlet problem and the signs $\sigma_n$ depend on $\tau$ and $t$: $\xi_n = \xi_n(\tau,t)$, $\sigma_n = \sigma_n(\tau,t) = \pm 1$.*

**Corollary 2.** *The theorem gives a method for solving problem* (1)–(4)*. First we find the spectral data* $\lambda_n$, $n \in Z$, $\xi_n^0(\tau)$, $\sigma_n^0(\tau)$, $n \in Z\backslash\{0\}$ *of the quadratic pencil of Sturm–Liouvelle equations corresponding to the coefficients* $p_0(x+\tau)$ *and* $q_0(x+\tau)$*. Then we solve the Cauchy problem* $\xi_n(\tau,t)|_{t=0} = \xi_n^0(\tau)$, $\sigma_n(\tau,t)|_{t=0} = \sigma_n^0(\tau)$, $n \in Z\backslash\{0\}$ *for Dubrovin system* (8)*. Finally, by using the formulas of the first and second traces, we get the expressions of* $p(\tau,t)$ *and* $q(\tau,t)$*. After that the Floquet solutions* $\psi_\pm(x,\lambda,t)$ *of equation* (4) *can be found easily.*

R e m a r k 2. We show that the constructed functions $p(\tau,t)$, $q(\tau,t)$ satisfy system (1), (2). For this we use the system of Dubrovin equations (7) and the following trace formula (see [16], pp. 96–97):

$$-\frac{3}{4}p_{\tau\tau}(\tau,t) + 4p^3(\tau,t) + 3p(\tau,t)q(\tau,t)$$

$$= \frac{(\lambda_{-1})^3 + (\lambda_0)^3}{2} + \sum_{0\neq k=-\infty}^{\infty}\left(\frac{(\lambda_{2k-1})^3 + (\lambda_{2k})^3}{2} - \xi_k^3\right). \tag{34}$$

From the system of Dubrovin equations (7) and (8) we have

$$\frac{\partial \xi_k}{\partial t} = -\left\{2p(\tau,t) + 2\xi_k + \int_{-\infty}^{\infty}\frac{\beta(\lambda,t)s(\pi,\lambda,t,\tau)}{\xi_k - \lambda}d\lambda\right\}\frac{\partial \xi_k}{\partial \tau}, \quad k \in Z\backslash\{0\}. \tag{35}$$

From the first trace formula and equalities (35) we can find

$$p_t = -\sum_{0\neq k=-\infty}^{\infty}\frac{\partial \xi_k}{\partial t} = 2p\sum_{0\neq k=-\infty}^{\infty}\frac{\partial \xi_k}{\partial \tau} + 2\sum_{0\neq k=-\infty}^{\infty}\xi_k\frac{\partial \xi_k}{\partial \tau}$$

$$+\int_{-\infty}^{\infty}\beta(\lambda,t)\left\{\sum_{0\neq k=-\infty}^{\infty}\frac{s(\pi,\lambda,t,\tau)}{\xi_k - \lambda}\frac{\partial \xi_k}{\partial \tau}\right\}d\lambda. \tag{36}$$

Differentiating the first and the second trace formulas with respect to $\tau$, we obtain

$$\sum_{0\neq k=-\infty}^{\infty}\frac{\partial \xi_k}{\partial \tau} = -p_\tau, \quad 2\sum_{0\neq k=-\infty}^{\infty}\xi_k\frac{\partial \xi_k}{\partial \tau} = -4pp_\tau - q_\tau. \tag{37}$$

Using these equalities and identity (6), from (36) we deduce

$$p_t = -6pp_\tau - q_\tau + \int_{-\infty}^{\infty}\beta(\lambda,t)\frac{\partial s(\pi,\lambda,t,\tau)}{\partial \tau}d\lambda. \tag{38}$$

Taking into account equality (5), from (38) we obtain equation (1). Differentiating trace formula (34) with respect to $t$, we get

$$q_t = -4pp_t - 2\sum_{0\neq k=-\infty}^{\infty} \xi_k \frac{\partial \xi_k}{\partial t}$$

$$= -4pp_t + 4p\sum_{0\neq k=-\infty}^{\infty} \xi_k \frac{\partial \xi_k}{\partial \tau} + 4\sum_{0\neq k=-\infty}^{\infty} \xi_k^2 \frac{\partial \xi_k}{\partial \tau}$$

$$+2\int_{-\infty}^{\infty} \beta(\lambda, t)\left\{\sum_{0\neq k=-\infty}^{\infty} \frac{\xi_k s(\pi, \lambda, t, \tau)}{\xi_k - \lambda}\frac{\partial \xi_k}{\partial \tau}\right\}d\lambda.$$

From (34), (37), (6) and the last formula, we find

$$q_t = -4pp_t + 2p(-4pp_\tau - q_\tau) + (p_{\tau\tau\tau} - 16p^2 p_\tau - 4p_\tau q - 4pq_\tau)$$

$$+2\int_{-\infty}^{\infty} \beta(\lambda, t)\left\{s(\pi, \lambda, t, \tau)(-p_\tau) + \lambda\frac{\partial s(\pi, \lambda, t, \tau)}{\partial \tau}\right\}d\lambda.$$

Substituting expression (38) into the last formula and taking into account equality (5), we obtain equation (2).

**Corollary 3.** *If the number of zones is finite, that is, there are two nonnegative integer numbers $N$ and $M$ such that $\lambda_{2k-1} = \lambda_{2k} = \xi_k$ for all $k > N$ and $k < -M$, then system (8) has the form*

$$\frac{\partial \xi_n}{\partial t} = 2(-1)^n \sigma_n(\tau, t)\mathrm{sign}(n)\sqrt{(\xi_n - \lambda_{2n-1})(\lambda_{2n} - \xi_n)}$$

$$\times\sqrt{(\xi_n - \lambda_{-1})(\xi_n - \lambda_0)\prod_{\substack{k=-M,\\ k\neq n,0}}^{N} \frac{(\xi_n - \lambda_{2k-1})(\xi_n - \lambda_{2k})}{(\xi_n - \xi_k)^2}}$$

$$\times\left\{\lambda_{-1} + \lambda_0 + \sum_{0\neq k=-M}^{N} (\lambda_{2k-1} + \lambda_{2k} - 2\xi_k) + 2\xi_n\right.$$

$$\left.+\int_{-\infty}^{\infty} \frac{\beta(\lambda, t)s(\pi, \lambda, t, \tau)}{\xi_n - \lambda}d\lambda\right\}, \quad n = -M, \ldots, -1, 1, \ldots, N.$$

**Corollary 4.** *In* [17]*, there was proved the theorem which states that the lengths of the gaps of the quadratic pencil of Sturm–Liouvelle equations with $\pi$-periodic real-valued coefficients decrease exponentially if and only if the coefficients are analytic. From this theorem we conclude that if $p_0(x)$ and $q_0(x)$ are real analytical functions, then the lengths of the gaps corresponding to these coefficients decrease exponentially. For the coefficients $p(x,t)$ and $q(x,t)$ there correspond the same gaps. Thus the solutions of problem* (1)–(4)*, $p(x,t)$ and $q(x,t)$, are real analytical functions on $x$.*

**Corollary 5.** *In* [18]*, an analogue of Borg's inverse theorem was proved: the number $\frac{\pi}{2}$ is a period of the coefficients of the quadratic pencil of Sturm–Liouvelle equations with $\pi$-periodic real-valued coefficients if and only if all eigenvalues of antiperiodic problem are double. From this theorem we conclude that if the functions $p_0(x)$ and $q_0(x)$ have the period $\frac{\pi}{2}$, then all eigenvalues of antiperiodic problem corresponding to these coefficients are double. For the coefficients $p(x,t)$ and $q(x,t)$ there correspond the same eigenvalues with the same multiplicities. Thus the solutions $p(x,t)$ and $q(x,t)$ of problem* (1)–(4) *are the $\frac{\pi}{2}$-periodic functions on $x$.*

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