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Conditions on a Surface $F^2 \subset E^n$ to lie in E^4

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We consider a surface F^2 in E^n with a non-degenerate ellipse of normal curvature whose plane passes through the corresponding surface point. The definition of three types of points is given in dependence of the position of the point relatively to the ellipse. If in the domain $D \subset F^2$ all the points are of the same type, then the domain D is said also to be of this type. This classification of points and domains is linked with the classification of partial differential equations of the second order. The theorems on the surface to lie in E^4 are proved under the fulfilment of certain boundary conditions. Some examples of the surfaces are constructed to show that the boundary conditions of the theorems are essential.

Key words: an ellipse of normal curvature, asymptotic lines, characteristics, boundary conditions.

Mathematics Subject Classification 2000: 53A05.

1. Introduction

Conditions for a two-dimensional surface in E^4 to lie in a hyperplane E^3 are well known [1, p. 146]. In this paper we consider the surface F^2 in E^n with non-degenerate ellipse of normal curvature *such that the plane of this ellipse for each point x of the surface passes through this point x*. This condition means that the *point codimension* of the surface is equal to 2. Notice that the condition on the plane is fulfilled for all surfaces lying in E^4. We say that the point $x \in F^2$ is of

1) hyperbolic type if x lies outside the ellipse of normal curvature,
2) parabolic type if x lies on this ellipse,
3) elliptic type if x lies inside this ellipse.

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If all points of some domain G of a surface are of one of the types, then the domain G is said also to be of this type. If the domain G has the points of different types, then it is said to be of mixed type. Below we will connect this splitting into types with the classification of differential equations in partial derivatives of the second order. We will also give the conditions on the surfaces $F^2 \subset E^n$ of types 1)–3), under which F^2 is also to lie in some E^4.

2. About the Surfaces of Hyperbolic Type in E^n

Let us consider the surface F^2 with the non-degenerate ellipse of normal curvature in the space E^n such that all points $x \in F^2$ are of hyperbolic type.

We recall the definition of the normal curvature ellipse.

Denote by $k_n(\tau)$ the vector of normal curvature of F^2 determined by the tangent vector $\tau \in T_x$ at every point $x \in F^2$. The vector $k_n(\tau)$ is in the normal space N_x. Let the start of this vector lie at the point x. Then the set of end-points of vectors $k_n(\tau)$, when τ rotates in T_x, forms the indicatrix of normal curvature. This set is a closed plane curve, namely an ellipse, perhaps degenerated at a segment or a point.

If τ is defined by the differentials of du^1, du^2, and $L^\alpha_{ij} du^i du^j$ are the second fundamental forms with respect to the unit normals $n_\alpha, \alpha = 1, \ldots, n-2$, then

$$k_n(\tau) = \frac{L^\alpha_{ij} du^i du^j}{ds^2} n_\alpha,$$

where $ds^2 = g_{ij} du^i du^j$ is the metric form.

In a normal space N_x, we introduce the Cartesian coordinates X^α with origin at the point x and basic vectors n_α. Then the coordinates of the point M of the indicatrix have the form

$$X^\alpha = \frac{L^\alpha_{ij} du^i du^j}{g_{ij} du^i du^j}. \tag{1}$$

On the surface F^2 we construct geometrically some net of curves. In this case, through the point x it is possible to draw two straight lines tangent to the ellipse. We denote the points of contacts by P and Q. Each of this points corresponds to the vector of normal curvature k_{nP} or k_{nQ}. In the tangent plane of F^2 for the vector k_{nP} there exists a tangent direction such that the vector of normal curvature for this direction coincides with k_{nP}. We have this direction at each point of the considering domain and hence we have the field of tangent directions. This field generates some family of integral curves which we call *characteristics*. With the help of the vector k_{nQ} we get another family of curves — also characteristics. The curves of these two families are not tangent to each other. On the surface, these families form some net of curves which we will call *specially hyperbolic net*.

Theorem 1. *Let the surface $F^2 \subset E^n$ of the class C^4 with non-degenerate ellipse of normal curvature be of hyperbolic type. And let D be a triangular domain on F^2 bounded by two characteristics — curves η_1 and η_2 from different families of the specially hyperbolic net beginning at the point x, and some curve γ of the class C^2 which crosses η_1, η_2 and is not tangent to the characteristics. Assume that there exists some hyperplane E^4 such that both the curve $\gamma \subset E^4$ and the tangent surface strip along γ lie in E^4.*

Then the whole domain D lies in E^4.

P r o o f. On the surface of the domain D introduce some coordinates u^1, u^2 of the class C^4. Then the fundamental forms of F^2 are the following:

$$ds^2 = g_{ij} du^i du^j,$$
$$II^\alpha = L^\alpha_{ij} du^i du^j.$$

The normals to F^2 will be chosen as follows. Let the vectors n_1, n_2 define the plane of the ellipse. Other normals $n_k, k = 3, 4, \dots, n-2$, are taken to be orthogonal to this plane. Hence $L^k_{ij} = 0, k = 3, 4, \dots, n-2$.

The Gauss decomposition has the form

$$r_{ij} = \Gamma^k_{ij} r_k + L^\alpha_{ij} n_\alpha.$$

Consider this decomposition for $i = 1, j = 2$

$$r_{12} = \Gamma^k_{12} r_k + L^\alpha_{12} n_\alpha.$$

Due to the choice of normals, we have

$$r_{12} = \Gamma^k_{12} r_k + L^1_{12} n_1 + L^2_{12} n_2. \tag{2}$$

Lemma 1. *The coefficients $\bar{L}^1_{12}$, and $\bar{L}^2_{12}$ in the special hyperbolic system of coordinates are equal to zero.*

P r o o f. We construct a new system of the coordinates ξ, η such that the coordinate lines are the characteristics. In this system, the coefficients of the second fundamental forms $\bar{L}^\alpha_{12}$ are

$$\overline{L}^\alpha_{12} = L^\alpha_{ij} \frac{\partial u^i}{\partial \xi} \frac{\partial u^j}{\partial \eta}, \quad \alpha = 1, 2.$$

Take a straight line which does not cross the ellipse when passing through x. Then the normal n_1 is the direction vector of this straight line, and n_2 is a vector orthogonal to n_1. We suppose that n_2 is directed to the half-plane where the ellipse lies.

Let M be an arbitrary point outside the ellipse. Then its coordinate $Y > 0$. For the non-zero shift du^1, du^2, we have

$$Y = \frac{L^2_{ij} du^i du^j}{ds^2} > 0.$$

If φ is the angle between the strait line xM and n_1, then

$$\mathrm{ctg}\varphi = \frac{X}{Y} = \frac{L^1_{ij} du^i du^j}{L^2_{ij} du^i du^j}.$$

The extremal value of the angle φ will be attained when the straight line xM is the tangent of the ellipse. If the ellipse is non-degenerate, then we have two extremal values of φ and two straight lines xM that are tangents of the ellipse. As in the case with the main directions of the 2-dimensional surface in E^3, for determining the corresponding directions in the tangent plane of F^2, it is necessary to solve the equation

$$|L^1_{ij} - \lambda L^2_{ij}| = 0.$$

Let λ_1 and λ_2 be the roots of this equation, and $\tau = \{\tau^i\}$, $\nu = \{\nu^i\}$ be two directions $\{du^1, du^2\}$ in the tangent plane of F^2 corresponding to λ_1 and λ_2. Then the following system takes place:

$$\begin{aligned}
(L^1_{11} - \lambda_1 L^2_{11})\tau^1 + (L^1_{12} - \lambda_1 L^2_{12})\tau^2 = 0,\\
(L^1_{11} - \lambda_1 L^2_{12})\tau^1 + (L^1_{22} - \lambda_1 L^2_{22})\tau^2 = 0,\\
(L^1_{11} - \lambda_2 L^2_{11})\nu^1 + (L^1_{12} - \lambda_2 L^2_{12})\nu^2 = 0,\\
(L^1_{12} - \lambda_2 L^2_{12})\nu^1 + (L^1_{22} - \lambda_2 L^2_{22})\nu^2 = 0.
\end{aligned}$$

Multiply the first equation on ν^1, the second equation on ν^2, and take their sum to obtain

$$L^1_{ij}\tau^i\nu^j - \lambda_1 L^2_{ij}\tau^i\nu^j = 0.$$

Similarly, for the third and fourth equations we have

$$L^1_{ij}\tau^i\nu^j - \lambda_2 L^2_{ij}\tau^i\nu^j = 0.$$

Therefore,

$$(\lambda_1 - \lambda_2) L^2_{ij}\tau^i\nu^j = 0.$$

Because $(\lambda_1 - \lambda_2) \neq 0$, we have $L^2_{ij}\tau^i\nu^j = 0$ and $L^1_{ij}\tau^i\nu^j = 0$. We take the functions $\eta(u^1, u^2)$ and $\xi(u^1, u^2)$ such that the lines $\eta = \mathrm{const}$ are the integral curves of the vector field τ, and the lines $\xi = \text{const}$ are the integral curves of the field ν. Then,

$$\begin{aligned} \eta_{u^1} &= -q\tau^2, \quad \xi_{u^1} = -p\nu^2, \\ \eta_{u^2} &= q\tau^1, \qquad \xi_{u^2} = p\nu^1, \end{aligned}$$

with some functions $p \neq 0$ and $q \neq 0$.

Let us write the transformation from the coordinates ξ, η to u^1, u^2

$$\begin{aligned} u^1 &= u^1(\xi, \eta), \\ u^2 &= u^2(\xi, \eta). \end{aligned}$$

If $J = J\left(\frac{u^1, u^2}{\xi, \eta}\right)$ is the Jacobian of the transformation, then

$$\xi_{u^1} = \frac{1}{J}\frac{\partial u^2}{\partial \eta}, \quad \eta_{u^1} = -\frac{1}{J}\frac{\partial u^2}{\partial \xi}, \quad \tau^1 = \frac{1}{qJ}\frac{\partial u^1}{\partial \xi}, \quad \nu^1 = -\frac{1}{pJ}\frac{\partial u^1}{\partial \eta},$$

$$\xi_{u^2} = -\frac{1}{J}\frac{\partial u^1}{\partial \eta}, \quad \eta_{u^2} = \frac{1}{J}\frac{\partial u^1}{\partial \xi}, \quad \tau^2 = \frac{1}{qJ}\frac{\partial u^2}{\partial \xi}, \quad \nu^2 = -\frac{1}{pJ}\frac{\partial u^2}{\partial \eta}.$$

Hence,

$$0 = L^\alpha_{ij}\tau^i\nu^j = -\frac{1}{pqJ^2}L^\alpha_{ij}\frac{\partial u^i}{\partial \xi}\frac{\partial u^j}{\partial \eta} = -\frac{1}{pqJ^2}\bar{L}^\alpha_{12}.$$

Therefore, $\bar{L}^\alpha_{12} = 0$.

Lemma 1 is proved.

Lemma 2. *The surface F^2 with respect to the coordinates ξ, η belongs to the regularity class C^2.*

P r o o f. For the proof of this lemma we will need the result of the following lemma.

Lemma 3. *Suppose F^2 has the regularity of the class C^4 with respect to the coordinates (u^1, u^2). Then the coefficients $L^\alpha_{ij} = (r_{ij}, n_\alpha)$ have the regularity of the class C^2.*

P r o o f. First, we will find the regularity class of the normals $n_\alpha, \alpha = 3, \ldots, n-2$. As the plane of the ellipse passes through x, the vectors r_{ij} lie in the 4-dimensional space spanned by $r_{u_1}, r_{u_2}, n_1, n_2$. Therefore, five vectors $r_{u_1}, r_{u_2}, r_{u_1u_1}, r_{u_1u_2}, r_{u_2u_2}$ are linearly dependent. As the ellipse of normal curvature is non-degenerate, there exist four linearly independent vectors between the first and the second derivatives of r, for example, $r_{u_1}, r_{u_2}, r_{u_1u_2}, r_{u_2u_2}$. Let the normal n_α have the coordinates $\xi_j, j = 1, \ldots, n$. To determine the normals, we can write the following system:

$$(r_{u_1}, n_\alpha) = x_{1u_1}\xi_1 + x_{2u_1}\xi_2 + \ldots + x_{nu_1}\xi_n = 0,$$
$$(r_{u_2}, n_\alpha) = x_{1u_2}\xi_1 + x_{2u_2}\xi_2 + \ldots + x_{nu_2}\xi_n = 0,$$
$$(r_{u_1u_2}, n_\alpha) = x_{1u_1u_2}\xi_1 + x_{2u_1u_2}\xi_2 + \ldots + x_{nu_1u_2}\xi_n = 0,$$
$$(r_{u_2u_2}, n_\alpha) = x_{1u_2u_2}\xi_1 + x_{2u_2u_2}\xi_2 + \ldots + x_{nu_2u_2}\xi_n = 0.$$

The equations of this system are linearly independent. Let, for example, the determinant be not equal to zero

$$\Delta = \begin{vmatrix} x_{1u_1} & \ldots & x_{4u_1} \\ x_{1u_2} & \ldots & x_{4u_2} \\ x_{1u_1u_2} & \ldots & x_{4u_1u_2} \\ x_{1u_2u_2} & \ldots & x_{4u_1u_2} \end{vmatrix} \neq 0.$$

By solving the system of the linear equations, we get

$$\xi_1 = \Delta^{-1}(\xi_5 A_1 + \xi_6 A_2 + \ldots + \xi_n A_{n-4}),$$
$$\xi_2 = \Delta^{-1}(\xi_5 B_1 + \xi_6 B_2 + \ldots + \xi_n B_{n-4}),$$
$$\xi_3 = \Delta^{-1}(\xi_5 C_1 + \xi_6 C_2 + \ldots + \xi_n C_{n-4}),$$
$$\xi_4 = \Delta^{-1}(\xi_5 D_1 + \xi_6 D_2 + \ldots + \xi_n D_{n-4}),$$

where $\Delta, A_i, B_i, C_i, D_i$ are some minors of the regularity of the class C^2. Setting the values $\xi_5, \ldots, \xi_n$, we get the $n-4$ normals. Hence, the normals n_α, $\alpha = 3, \ldots, n-2$ have the regularity of the class C^2. To obtain the regular fields of the normals n_1, n_2, we write the system

$$(r_{u_k}, n_i) = 0, \quad i = 1, 2,$$
$$(n_\alpha, n_i) = 0, \quad \alpha = 3, \ldots, n-2.$$

As in the above, we get that $n_i, i = 1, 2$ belong to the class C^2. Hence, $L^\alpha_{ij} \in C^2$.

Remark that in the given construction the vectors n_j may not be orthogonal to each other. But it is not difficult to check that the process of orthogonality gives new normals of the same class of regularity. Lemma 3 is proved.

Continue the proof of Lemma 2.

From the equation $|L^1_{ij} - \lambda L^2_{ij}| = 0$ and Lemma 3 it follows that $\lambda_i \in C^2$. Therefore, the fields τ and ν are of the class C^2. We have the system of differential equations of the first order

$$\xi_{u_1}\nu^1 + \xi_{u_2}\nu^2 = 0,$$
$$\eta_{u_1}\tau^1 + \eta_{u_2}\tau^2 = 0.$$

The coefficients of this system are of the class C^2. Then, by the theorem on the regular dependence of solutions of an ordinary differential equation on initial data (see, for example, [10, p. 92 § 2] and for the existence of the solution see [10, p. 255], the functions ξ, η also belong to the class C^2. Lemma 2 is proved.

Equation (2) can be written in the canonical form

$$r_{\xi\eta} = \Gamma^1_{12} r_\xi + \Gamma^2_{12} r_\eta. \tag{3}$$

Both the considering curve γ and the surface strip lie in some space E^4. Let n_0 be a constant vector from a normal space orthogonal to E^4. Denote $U = (r, n_0)$. From (3) it follows that

$$U_{\xi\eta} = \Gamma^1_{12} U_\xi + \Gamma^2_{12} U_\eta. \tag{4}$$

We suppose that the origin of the Cartesian coordinate in E^n lies in E^4. The conditions of the theorem imply that

$$U|_\gamma = 0,$$

$$U_\nu|_\gamma = 0,$$

where U_ν is the derivative on F^2 at the direction orthogonal to γ.

According to the theorem on the uniqueness of solutions in the theory of differential equations of the second order of hyperbolic type (see [2, p. 439], [3, p. 65]), we may conclude that $U(\xi, \eta) \equiv 0$ in the domain D. It means that D lies in the hyperplane orthogonal to n_0. If we take n_0 as an arbitrary constant vector in the space orthogonal to E^4, then the domain D lies in E^4.

3. About the Surfaces of Parabolic Type in E^n

Consider the case when every point $x \in F^2$ lies on the ellipse of normal curvature corresponding to this point. In the plane T_x, which is tangent to the surface F^2, there exists the direction τ for which $k_n(\tau) = 0$. The integral curves of the field of the vectors τ are asymptotic curves in the usual sense of differential geometry.

Introduce a system of the coordinates (ξ, η) on the surface with a family of asymptotic lines as ξ-lines. The second family of the coordinate η-lines can be taken arbitrarily as a regular family of curves crossing asymptotic lines transversally. For example, we can take the family of orthogonal trajectories of the family of asymptotic lines. Then the vector position r of F^2 is

$$r = r(\xi, \eta).$$

Write the Gauss decomposition with $i = j = 1$,

$$r_{11} = \Gamma_{11}^k r_k + L_{11}^\alpha n_\alpha.$$

As the ξ-lines are asymptotic, $L_{11}^\alpha = 0, \alpha = 1, \ldots, n-2$. For $r(\xi, \eta)$, we have the parabolic equation

$$r_{11} = \Gamma_{11}^k r_k.$$

Theorem 2. *Let the analytical surface $F^2 \subset E^n$ with the ellipse of normal curvature of non-degenerate type be of parabolic type. Let the domain $D \subset F^2$ be some strip bounded by two asymptotic curves. Suppose that some curve γ, crossing transversally all asymptotic lines in the strip and the tangent surface strip along γ lie in the some subspace E^4.*

Then D also lies in this subspace E^4.

P r o o f. For proving, we introduce the function $U = (r, n_0)$, where n_0 is the normal vector from the space orthogonal to E^4. We get the equation

$$U_{\xi\xi} = \Gamma_{11}^1 U_\xi + \Gamma_{11}^2 U_\eta. \tag{5}$$

Suppose that γ is given by the equation $\xi = 0$. Conditions on the curve γ give zero conditions for the Cauchy problem

$$U(0, \eta) = 0,$$

$$U_\xi(0, \eta) = 0.$$

By the Cauchy–Kovalevskaya theorem [11, p. 22], in the neighborhood of γ (that is, $\xi = 0$) there exists only one solution $U \equiv 0$. Hence, by the analyticity of F^2, the domain D lies in the subspace orthogonal to n_0. By the arbitrariness of n_0, D lies in E^4.

In the theory of parabolic differential equations there are other uniqueness theorems. We can formulate and prove the theorem not supposing analyticity of the surface, but imposing additional conditions.

Theorem 3. *Let the surface $F^2 \subset E^n$ of the regularity class C^5 with non-degenerate ellipse of normal curvature be of parabolic type. Let the domain $D \subset F^2$ be bounded by two asymptotic lines and by two curves γ and γ_1 crossing the asymptotic lines transversally. Suppose that the geodesic curvature of asymptotic lines in D is nonnegative. Let the curve γ lie in some E^4 together with the tangent surface strip along γ. Then the whole domain D lies in E^4.*

P r o o f. For the proof of this theorem, we use the theorem on the uniqueness of solutions for differential equations of parabolic type proved by E.M. Landis [4], where the equation

$$\frac{\partial^2 U}{\partial x^2} = a(t,x)\frac{\partial U}{\partial t} + b(t,x)\frac{\partial U}{\partial x} + c(t,x)U \tag{6}$$

is considered in some domain G. On the coefficients of this equation the following conditions are imposed:

1. The modules of the coefficients are limited by 1,

2. The coefficient $a(t,x)$ has a derivative with respect to t, the coefficient $b(t,x)$ has a derivative with respect to x, and

$$\left|\frac{\partial a}{\partial t}\right| < 1, \quad \left|\frac{\partial b}{\partial x}\right| < 1.$$

3. In the domain G $a \geq 0$, $c \geq 0$.

4. The solution $U(x,t)$ of (6) in G is of the class C^2.

In [4], the following uniqueness theorem is proved.

Theorem 4. *Let G be a part of the strip $\Pi = t_1 < t < t_2$ situated between two non-crossing curves Γ_1 and Γ_2 with one-to-one projections on the t-axis and connecting the opposite sides of Π. Let $U(x,t)$ be a solution of (6) belonging to C^1 in $\bar{D}$. Suppose that Γ_2 is a smooth curve, and*

$$U\mid_{\Gamma_2} = \frac{\partial U}{\partial n}\mid_{\Gamma_2} = 0.$$

Then in G

$$U \equiv 0.$$

Here $\frac{\partial U}{\partial n}$ is a derivative in the direction orthogonal to Γ_2.

Give an explanation of the nonnegative geodesic curvature of asymptotic line. If the coordinates ξ, η are introduced in D, then on the basic curve $\eta = 0$ the vector of the curvature is directed inside D. The curvature vector of asymptotic lines inside D builds an acute angle or $\frac{\pi}{2}$ with positive direction of the η-line.

To apply the Landis theorem, we have to verify the conditions on the coefficients. We put $x = \xi$ and $t = \eta$. Then the coefficients from Condition 3 in Theorem 4 are $a(t,x) = \Gamma_{11}^2$, $b(t,x) = \Gamma_{11}^1$, and $c(t,x) = 0$ in our case. Remark that $\Gamma_{ij}^k \in C^2$ and, hence, in the domain D both the functions and the derivatives are limited. The boundedness condition of modules by 1 can be satisfied by a new parametrization of the coordinate lines.

Remark that Γ_{11}^2 is connected with the geodesic curvature of asymptotic line. Indeed,

$$\Gamma_{11}^2 = \frac{1}{2W^2}(2g_{11}\frac{\partial g_{12}}{\partial \xi} - g_{12}\frac{\partial g_{11}}{\partial \xi} - g_{11}\frac{\partial g_{11}}{\partial \eta}),$$

where g_{ij} are the coefficients of the metric, and $W = \sqrt{g_{11}g_{22} - g_{12}^2}$. The geodesic curvature $\frac{1}{\rho_g}$ can be written as (see [5, § 83, 127]

$$\frac{1}{\rho_g} = \frac{1}{2Wg_{11}^{\frac{3}{2}}}(2g_{11}\frac{\partial g_{12}}{\partial \xi} - g_{12}\frac{\partial g_{11}}{\partial \xi} - g_{11}\frac{\partial g_{11}}{\partial \eta}).$$

Comparing these two expressions, we have

$$\Gamma_{11}^2 = \frac{g_{11}^{\frac{3}{2}}}{W}\frac{1}{\rho_g}. \tag{7}$$

Hence, the sign Γ_{11}^2 depends on the sign of $\frac{1}{\rho_g}$. By Condition 3 in Theorem 3, the coefficient $a(x,t) \geq 0$ is satisfied if $\frac{1}{\rho_g} \geq 0$. The boundary conditions on γ will coincide with the boundary conditions of Landis' theorem [4] if we take γ instead of Γ_2.

Thus, in D we have $U(\xi,\eta) \equiv 0$. Therefore, $D \subset E^4$.

In the theory of parabolic equations, differential equations are usually written in the form of the generalized heat equations, namely with the derivative chosen with respect to t (see [6]). This theorem will be used for proving the theorem bellow.

Determine a non-closed contour Ω. Suppose the point C has the coordinates (a,T); the point A has the coordinates $(a,0)$; the point B has the coordinates $(b,0)$ and the point K has the coordinates (b,T) (here we suppose that $a < b, T > 0$). Then the contour Ω consists of the intercepts of coordinate lines. Consider the domain D with the boundary $CABKC$.

Theorem 5. *Suppose that $F^2 \subset E^n$ of the class C^5 with non-degenerate ellipse of normal curvature is of parabolic type. Let the contour Ω be situated in some space of E^4. Let the geodesic curvature of asymptotic lines in D be positive. Then the whole domain D lies in E^4.*

P r o o f. Begin with considering the contour Ω.

The intercept AB is an asymptotic line of F^2 and, consequently, the tangent surface strip along AB lies in E^4 automatically. Although, on CA and BK the condition on the tangent surface strip is absent.

The contour Ω consists of an intercept of the coordinate line $\eta = 0$ and two intercepts of the coordinate η-lines. Rewrite Eq. (5) in the form of the generalized heat equation

$$U_\eta = \frac{1}{\Gamma_{11}^2}U_{\xi\xi} + \frac{\Gamma_{11}^1}{\Gamma_{11}^2}U_\xi. \tag{8}$$

By the condition $\frac{1}{\rho_g} > 0$ and (7), we have $\Gamma_{11}^2 \neq 0$. Hence the coefficients of this equation have the regularity of the class C^2. For (8), we apply the uniqueness theorem for parabolic equations (see [6]) and obtain that $U \equiv 0$ in D. The Theorem 5 is proved.

Consider now an infinitely long strip bounded by two complete asymptotic lines. We suppose that a system of the coordinates (ξ, η) in this strip can be introduced by using the family of asymptotic lines and the second family of the lines crossing the curves of the first family transversally. For example, it is possible to construct an orthogonal coordinates system where the strip is bounded by the basic curve $\Gamma : \eta = 0$ and by the curve Γ_1: $\eta = T$ which is said to be free. Let the parameter ξ be the arc length on Γ. Define the width of the strip. Every point $P \subset \Gamma_1$ has some coordinates (ξ, T). Let $l(\xi)$ be the shortest distance between P and Γ along F^2. The number $l(\xi)$ is called the variable width of the strip.

Theorem 6. *Suppose that $F^2 \subset E^n$ of the class C^5 with non-degenerate ellipse of normal curvature is of parabolic type. Suppose that D is an infinite strip between two complete asymptotic lines Γ and Γ_1. Suppose that the asymptotic lines in D have a positive geodesic curvature, and the following conditions in D are fulfilled:*

$$\left|\frac{1}{\Gamma^2_{11}}\right| \leq M,$$

$$\left|\frac{\Gamma^1_{11}}{\Gamma^2_{11}}\right| \leq M(|\xi| + 1),$$

$$|l(\xi)| \leq Be^{\beta\xi^2}, \quad B, \beta > 0,$$

where M, B, and β are some positive numbers. If the basic curve Γ lies in some E^4, then the whole domain D lies in E^4.

P r o o f. We apply Theorem 7 from [8, p. 63], which is a generalization of A.N. Tikhonov's theorem from [9] proved for the classical heat equation. Let $x = (x_1, \ldots, x_n)$ be a varying point in E^n. Suppose that in the domain D, the function $U(x, t)$ has continuous second derivatives with respect to x and continuous first derivatives with respect to the parameter t. The theorem is formulated as follows.

Theorem 7. *Let L be a parabolic operator*

$$L(U) = \sum_{i,j=1}^{n} a_{ij}(x, t) \frac{\partial^2 U}{\partial u^i \partial u^j} + \sum_{i,j=1}^{n} b_i(t, x) \frac{\partial U}{\partial x_i} + c(x, t)U - \frac{\partial U}{\partial t},$$

where

$$\sum a_{ij}(x, t)\xi_i \xi_j > 0$$

for all vectors $\xi = \{\xi_i\} \neq 0$ *with continuous coefficients in* $R^n \times (0, T]$, *and the following conditions be satisfied:*

$$|a_{ij}(x,t)| \leq M,$$
$$|b_i(x,t)| \leq M(|x|+1),$$
$$|c(x,t)| \leq M(|x|^2+1).$$

Then there exists not more than one solution of the equation

$$L(U) = f(x,t)$$

in $R^n \times (0,T]$ *such that* $U(x,0) = \varphi(x)$ *in* R^n *and whenever*

$$|U(x,t)| \leq Bexp\beta|x|^2, \quad B, \beta > 0, \tag{9}$$

where M, B *and* β *are some positive constants.*

Remark that for the heat equation (9), the condition from [9] is essential. Namely, if $|U(x,t)| \leq Be^{\beta(x^{2+\varepsilon})}$ and $\varepsilon > 0$, then the uniqueness theorem is not true.

Apply this theorem to Eq. (8) under the conditions $f(x,t) = 0$, $\phi(x) = 0$. Remark that $|U(\xi,\eta)|$ is equal to a distance from the corresponding point of the surface to E^4. By the condition of Theorem 6, we have

$$|U(\xi,\eta)| \leq l(\xi) \leq Be^{\beta\xi^2}.$$

Consequently, all conditions of Theorem 7 are fulfilled. Hence, under the given boundary conditions and limitations imposed on the coefficients, Eq. (8) has the unique solution $U(\xi,\eta) \equiv 0$. Thus, D lies in E^4. Finally, Theorem 6 is proved.

Give a few examples of the surface strips whose metrics satisfy the conditions of Theorem 6:

1) A universal covering of a ring on a plane between two concentric circles. The exterior contour is taken as a basic curve. The metric of the plane can be written in the form $ds^2 = \eta^2 d\xi^2 + d\eta^2$. Then $\Gamma^1_{11} = 0$, $\Gamma^2_{11} = -\eta$, and Eq. (8) takes the form $u_\eta = -\frac{1}{\eta}u_{\xi\xi}$.

2) A ring on a half-sphere.

3) A strip on the Lobachevsky plane to be considered below in Sec. 5.

4) A local convex infinitely long curve γ with the radius position $\rho(\xi)$ on the plane. The curvature k of γ satisfies the restrictions $0 < k_1 \leq k \leq k_2$, where k_1, k_2 are constants. The strip consists of the family of parallel curves. The vector position of a point in the plane has the form $r(\xi,\eta) = \rho(\xi) + \eta\nu(\xi)$, where $\nu(\xi)$

is a unit normal to γ. The metric form of the plane is $ds^2 = (1-\eta k)^2 d\xi^2 + d\eta^2$. Introduce the restrictions on η: $0 < \eta < \frac{1-\varepsilon}{k_2}$, where $0 < \varepsilon < 1$ and $|k_\xi| \le M =$ const. Then,

$$\left|\frac{1}{\Gamma_{11}^2}\right| \le \frac{1}{\varepsilon k_1}, \left|\frac{\Gamma_{11}^1}{\Gamma_{11}^2}\right| \le \left|\frac{(1-\varepsilon)k_\xi}{k_2 \varepsilon^2 k_1}\right| \le M_1 = \text{const.}$$

Thus all conditions of Theorem 6 for this strip are fulfilled.

4. On the Surfaces of Elliptic Type in E^n

Theorem 8. *Let the domain D be homeomorphic to a disk and be of elliptic type. Assume that the surface has the regularity of the class $C^{4,\alpha}$. Suppose the boundary of D is a curve $\gamma \in C^1$ which lies in some E^4. Then the whole domain D lies in E^4.*

P r o o f. In contrast to the theorems from Sections 1–3, the condition that the tangent surface strip along γ lies in E^4 is not necessary.

Theorem 8 can be proved by using the theory of elliptic equations or some geometrical considerations.

First we are to obtain an elliptic equation for the vector position $r(x^1, x^2)$ of F^2 with the coordinates x^1, x^2. Suppose that the coordinates x^1, x^2 are introduced in the whole simply connected domain D. Write the Gauss expansions by using the covariant derivatives

$$r_{,ij} = L_{ij}^\alpha n_\alpha.$$

Recall that $L_{ij}^\alpha \equiv 0, \alpha = 3, \dots, n-2$ because the plane of the ellipse of normal curvature passes through $x \in F^2$. Multiply the right- and the left-sides of the Gauss equations by some numbers Ω^{ij}

$$r_{,ij}\, \Omega^{ji} = L_{ij}^\alpha \Omega^{ji} n_\alpha.$$

Assume that Ω^{ji} have the following properties: $L_{ij}^\alpha \Omega^{ji} = 0, \alpha = 1, 2$ and $\Omega^{ij} = \Omega^{ji}$. Then we obtain the equation for the vector position $r(x^1, x^2)$

$$r_{,11}\, \Omega^{11} + 2r_{,12}\, \Omega^{12} + r_{,22}\, \Omega^{22} = 0. \tag{10}$$

Write Ω^{ij} in terms of the coefficients L_{ij}^α

$$\Omega^{11} = \begin{vmatrix} L_{12}^1 & L_{22}^1 \\ L_{12}^2 & L_{22}^2 \end{vmatrix}, \quad \Omega^{12} = -\frac{1}{2}\begin{vmatrix} L_{11}^1 & L_{22}^1 \\ L_{11}^2 & L_{22}^2 \end{vmatrix}, \quad \Omega^{22} = \begin{vmatrix} L_{11}^1 & L_{12}^1 \\ L_{11}^2 & L_{12}^2 \end{vmatrix}.$$

For Eq. (10) to be elliptic, the inequality

$$(\Omega^{12})^2 - \Omega^{11}\Omega^{22} < 0,$$

should be fulfilled.

Determine the sign of $(\Omega^{12})^2 - \Omega^{11}\Omega^{22}$ which depends on disposition of the point x relatively to the ellipse. We use a special system of coordinates on F^2 and obtain the law of transformation for $(\Omega^{12})^2 - \Omega^{11}\Omega^{22}$ under the transition from x^1, x^2 to the coordinates u^1, u^2. Then, by the law of transformation of Ω^{ji} to $\overline{\Omega}^{ij}$, determine

$$\overline{\Omega}^{ij} = J^3 \Omega^{\alpha\beta} \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta},$$

where $J(\frac{x^1,x^2}{u^1,u^2})$ is a Jacobian of transformation. The obtained $\overline{\Omega}^{ij}$ are said to be relative tensors with weight 3 (see [13, p. 237]).

Consider, for example, the expressions

$$\overline{\Omega}^{11} = \begin{vmatrix} \overline{L}^1_{12} & \overline{L}^1_{22} \\ \overline{L}^2_{12} & \overline{L}^2_{22} \end{vmatrix},$$

where $\overline{L}^\sigma_{ij}$ are the coefficients of the second quadratic form in a new coordinate system. The coordinates transformation influences on the coefficients of the second quadratic form in the following way:

$$\overline{L}^\sigma_{ij} = L^\sigma_{\alpha\beta} \frac{\partial x^\alpha}{\partial u^i} \frac{\partial x^\beta}{\partial u^j}.$$

Consequently,

$$\overline{\Omega}^{11} = \begin{vmatrix} L^1_{\alpha\beta} \frac{\partial x^\alpha}{\partial u^1} \frac{\partial x^\beta}{\partial u^2} & L^1_{\gamma\delta} \frac{\partial x^\gamma}{\partial u^2} \frac{\partial x^\delta}{\partial u^2} \\ L^2_{\alpha\beta} \frac{\partial x^\alpha}{\partial u^1} \frac{\partial x^\beta}{\partial u^2} & L^2_{\gamma\delta} \frac{\partial x^\gamma}{\partial u^2} \frac{\partial x^\delta}{\partial u^2} \end{vmatrix}$$

$$= J \left\{ \begin{vmatrix} L^1_{11} & L^1_{21} \\ L^2_{11} & L^2_{21} \end{vmatrix} \left(\frac{\partial x^1}{\partial u^2} \right)^2 + \begin{vmatrix} L^1_{11} & L^1_{22} \\ L^2_{11} & L^2_{22} \end{vmatrix} \frac{\partial x^1}{\partial u^2} \frac{\partial x^2}{\partial u^2} + \begin{vmatrix} L^1_{12} & L^1_{22} \\ L^2_{12} & L^2_{22} \end{vmatrix} \left(\frac{\partial x^2}{\partial u^2} \right)^2 \right\}.$$

Substitute the determinants consisting of L^σ_{ij} by Ω^{ij} and use the derivatives $\frac{\partial u^\beta}{\partial x^j}$ instead of $\frac{\partial x^i}{\partial u^\alpha}$. We obtain

$$\overline{\Omega}^{11} = J^3 \left(\Omega^{22} \left(\frac{\partial u^1}{\partial x^2} \right)^2 + 2\Omega^{12} \frac{\partial u^1}{\partial x^2} \frac{\partial u^1}{\partial x^1} + \Omega^{11} \left(\frac{\partial u^1}{\partial x^1} \right)^2 \right) = J^3 \Omega^{\alpha\beta} \frac{\partial u^1}{\partial x^\alpha} \frac{\partial u^1}{\partial x^\beta}.$$

For the ellipticity to be preserved, the expression $(\overline{\Omega}^{12})^2 - \overline{\Omega}^{11}\overline{\Omega}^{22}$ should not change its sign when transforming from one coordinate system to another

$$\begin{vmatrix} \overline{\Omega}^{11} & \overline{\Omega}^{12} \\ \overline{\Omega}^{12} & \overline{\Omega}^{22} \end{vmatrix} = J^6 \begin{vmatrix} \Omega^{\alpha\beta} \frac{\partial u^1}{\partial x^\alpha} \frac{\partial u^1}{\partial x^\beta} & \Omega^{\gamma\delta} \frac{\partial u^1}{\partial x^\gamma} \frac{\partial u^2}{\partial x^\delta} \\ \Omega^{\alpha\beta} \frac{\partial u^2}{\partial x^\alpha} \frac{\partial u^1}{\partial x^\beta} & \Omega^{\gamma\delta} \frac{\partial u^2}{\partial x^\gamma} \frac{\partial u^2}{\partial x^\delta} \end{vmatrix}$$

$$= J^5 \frac{\partial u^1}{\partial x^\beta} \frac{\partial u^2}{\partial x^\delta} \left(\Omega^{1\beta}\Omega^{2\delta} - \Omega^{2\beta}\Omega^{1\delta} \right)$$
$$= J^4 \left(\Omega^{11}\Omega^{22} - (\Omega^{12})^2 \right). \tag{11}$$

At it is seen from the last relation, the sign of (11) is not changed.

It is easy to ascertain that under the rotation of normal basis n_1, n_2, the sign of the numbers Ω^{ij} is not changed.

On the surface, take some point x_0. In the neighborhood of x_0, construct special orthogonal coordinate system in the following way. If at the point x_0 the ellipse of normal curvature is not a circle, then take a vector of normal curvature whose end-point lies at one of summits of the ellipse (for example, at the largest one). Let the vector τ in the tangent plane be corresponding to the vector of normal curvature. Draw the coordinate u^1-curve on F^2 tangential to τ at x_0 and thus get the family of the u^1-lines, namely the first family. If the ellipse of normal curvature is a circle, then τ can be taken arbitrarily. Taking the orthogonal trajectories of the first family, we get the second family of the coordinate lines. Chose the normals n_1, n_2 at x_0 to be parallel to the axes of the ellipse (if it is not a circle), and take them arbitrarily in opposite case. Additionally, put $g_{11} = g_{22} = 1$ at x_0. Under this choice of the coordinates and normals, at x_0 we have

$$\begin{aligned} L^1_{11} &= \alpha + a, & L^2_{11} &= \beta, \\ L^1_{12} &= 0, & L^2_{12} &= b, \\ L^1_{22} &= \alpha - a, & L^2_{22} &= \beta, \end{aligned}$$

where α and β are the coordinates of the origin of the ellipse, and a, b are its half-axes.

Consequently, at x_0 we get

$$(\Omega^{12})^2 - \Omega^{11}\Omega^{22} = \left(\begin{vmatrix} \alpha + a & \alpha - a \\ \beta & \beta \end{vmatrix}^2 - 4 \begin{vmatrix} 0 & \alpha - a \\ b & \beta \end{vmatrix} \begin{vmatrix} \alpha + a & 0 \\ \beta & b \end{vmatrix} \right) =$$

$$4(\beta^2 a^2 - b^2 a^2 + \alpha^2 b^2) < 0.$$

Geometrically, this inequality means that x_0 lies inside the ellipse. If we introduce the function

$$F(X^1, X^2) = \frac{(X^1 - \alpha)^2}{a^2} + \frac{(X^2 - \beta)^2}{b^2} - 1$$

on the plane of the ellipse, then $F = 0$ for the points of this ellipse and $F < 0$ for the point x_0 which has the coordinates $X^1 = 0, X^2 = 0$. Hence

$$\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} - 1 < 0.$$

From Eq. (11), we obtain the elliptic equation for $U = (r, n_0)$,

$$U_{11}\Omega^{11} + 2U_{12}\Omega^{12} + U_{22}\Omega^{22} = 0.$$

Subsequently, from the uniqueness theorem for elliptic differential equations (see [2, p. 334], [12, p. 109]) there follows Theorem 8.

By using another way of proving Theorem 8 proposed by A.A. Borisenko, an $(n-1)$-dimensional sphere S^{n-1} with the center in E^4 containing the domain D is considered. This is a generalization of the method first suggested by A.V. Pogorelov.

Introduce the Cartesian coordinates in E^n $y_1, \dots, y_n$ such that the axes $y_1, \dots, y_4$ lie in E^4, and $y_5, \dots, y_n$ are orthogonal to E^4. Consider the family of ellipsoids, obtained from S^{n-1} by being compressed along the axes $y_5, \dots, y_n$ to E^4. If the domain D does not lie in E^4, then at some moment of compression one ellipsoid touches D at some inner point x. The vectors of normal curvature at this point for every tangent vector are directed inside the ellipsoid. Consequently, the point x lies outside the ellipse of normal curvature. It means that x is of hyperbolic type which contradicts the theorem conditions.

5. Examples of Surfaces in E^5 with the Plane of Normal Curvature Ellipse Passing through a Point of a Surface

We show the way of constructing the surfaces in E^5 with non-degenerate ellipse of normal curvature whose plane passes through the point x. The surfaces are constructed with a sufficiently large arbitrariness.

The following theorem shows that the boundary conditions imposed on the previous theorems are essential.

Theorem 9. *Let $\Gamma \subset E^5$ be a curve of the regularity class C^5 with the curvatures $k_i \neq 0, i = 1, \dots, 4$. Then through Γ it is possible to draw a surface of the regularity class C^2 whose ellipse of normal curvature is non-degenerate and the plane passes through the point x of this surface.*

P r o o f. If $k_4 \neq 0$, then neither the curve Γ nor the surface containing Γ lies in E^4. The plane of the ellipse of normal curvature of every ruled surface in E^5 passes through the point x. Moreover, every ruled surface with non-degenerate ellipse of normal curvature is of parabolic type. Our construction provides non-degeneracy of this ellipse.

If the vector position of the curve Γ is $\rho(s)$, where s is the arc length, then in E^5 we take the vector position of the ruled surface

$$r(s,t) = \rho(s) + t\xi_3(s).$$

Here ξ_i are the vectors of natural basis of Γ.

Calculate the first and second fundamental forms for this surface. We have

$$r_s = \xi_1 + t(-k_2\xi_2 + \xi_4 k_3),$$
$$r_t = \xi_3.$$

From here it is seen that $g_{11} = 1 + t^2(k_2^2 + k_3^2)$, $g_{12} = 0$, $g_{22} = 1$.

Further,

$$r_{ss} = \xi_1 t k_1 k_2 + \xi_2(k_1 - tk_2') + \xi_3(-tk_2^2 - tk_3^2) + \xi_4 t k_3' + \xi_5 k_3 k_4 t,$$

$$r_{st} = -\xi_2 k_2 + \xi_4 k_3,$$

$$r_{tt} = 0.$$

Write the normals to the surface

$$n_1 = \lambda[\frac{-k_2\xi_2 + k_3\xi_4}{k_2^2 + k_3^2} - t\xi_1], \quad \lambda = \sqrt{\frac{k_2^2 + k_3^2}{1 + t^2(k_2^2 + k_3^2)}}\,,$$

$$n_2 = \frac{k_3\xi_2 + k_2\xi_4}{\sqrt{k_3^2 + k_2^2}}\,,$$

$$n_3 = \xi_5.$$

Calculate the coefficients of the second quadratic forms of the surface

$$L_{11}^1 = (n_1, r_{ss}) = \lambda(-t^2 k_1 k_2 - \frac{k_1 k_2 - tk_2' k_2 - tk_3 k_3'}{k_3^2 + k_2^2}),$$

$$L_{12}^1 = \lambda,$$

$$L_{22}^\alpha = 0, \quad \alpha = 1, 2, 3,$$

$$L_{11}^2 = \frac{(k_1 - k_2' t)k_3 + k_2 k_3' t}{\sqrt{k_3^2 + k_2^2}}\,,$$

$$L_{12}^2 = 0, \quad L_{11}^3 = k_4 k_3 t, \quad L_{12}^3 = 0.$$

Check whether the normal curvature ellipse at the points of the curve Γ is non-degenerate.

The coordinates of the ellipse are given by equation (1). On the surface F^2, consider the first fundamental form dl^2. At $t = 0$, we have $dl^2 = (ds)^2 + (dt)^2$. Then for the ellipse of normal curvature we get

$$\cos\varphi = \frac{ds}{\sqrt{(ds)^2 + (dt)^2}}\,, \quad \sin\varphi = \frac{dt}{\sqrt{(ds)^2 + (dt)^2}}\,.$$

Here the angle φ is formed by the direction τ and the coordinate line s. For the coordinates of indicatrix we can write

$$X^\alpha(\varphi) = L^\alpha_{11}\cos^2\varphi + 2L^\alpha_{12}\sin\varphi\cos\varphi + L^\alpha_{22}\sin^2\varphi.$$

After transformation of this expression, we have

$$X^\alpha(\varphi) = \frac{L^\alpha_{11} + L^\alpha_{22}}{2} + \frac{L^\alpha_{11} - L^\alpha_{22}}{2}\cos 2\varphi + L^\alpha_{12}\sin 2\varphi.$$

Recall that $L^\alpha_{22} = 0$. Let us introduce the vectors

$$M = \begin{pmatrix} L^1_{11} \\ L^2_{11} \\ L^3_{11} \end{pmatrix}, \quad N = \begin{pmatrix} L^1_{12} \\ L^2_{12} \\ L^3_{12} \end{pmatrix}.$$

For the normal curvature ellipse at the points of the curve Γ not to degenerate into a segment of a straight line, it is necessary and sufficient for the vectors M and N be non-collinear. Hence, the minor, composed of the components of these vectors, is not equal to zero

$$\begin{vmatrix} L^1_{11} & L^1_{12} \\ L^2_{11} & L^2_{12} \end{vmatrix} = \begin{vmatrix} L^1_{11} & L^1_{12} \\ L^2_{11} & 0 \end{vmatrix} = k_1 k_3 \neq 0.$$

On a continuity, the ellipse of normal curvature will also be non-degenerate in some neighborhood of the curve Γ.

Theorem 9 is proved.

Give another example of the surface of parabolic type. Suppose that the metric of surface is the metric of the Lobachevsky plane in the Poincaré interpretation

$$ds^2 = \frac{d\xi^2 + d\eta^2}{\eta^2}.$$

Suppose also that asymptotic lines are horocycles $\eta = \text{const}$. For this metrics we have $\Gamma^1_{11} = 0$, $\Gamma^2_{11} = \frac{1}{\eta}$. Then Eq. (5) has the form

$$r_\eta = \eta r_{\xi\xi}.$$

If replacing $t = \frac{\eta^2}{2}$, $\xi = x$, then this equation can be written in a classical form of the classical heat conduction equation

$$r_{xx} = r_t. \tag{12}$$

Thus, *the classical object, namely the Lobachevsky plane, leads to the classical heat conduction equation.*

Let some continuous limited curve in E^5 with the vector position $\rho = \rho(x)$, $-\infty < x < +\infty$ be given. Applying the Poisson formula (see [6, p. 225], or [7, p. 481]), we can define the surface $F^2 \subset E^5$

$$r(x,t) = \frac{1}{2\sqrt{\pi}} \int\limits_{-\infty}^{+\infty} \frac{1}{\sqrt{t}} e^{-\frac{(x-\xi)^2}{4t}} \rho(\xi) d\xi,$$

where $r(x,0) = \rho(x)$.

Thus, we can state that except the ruled surfaces there are also other surfaces of parabolic type whose vector positions satisfy heat equation (12). Therefore, $L_{11}^{\alpha} = 0$. Hence, a curve $t = \text{const}$ is an asymptotic line on the surface. It means that the surface is of parabolic type.

6. About the Surfaces in E^5 in the Explicit Form

In this section, we consider the surface in E^5 given in the explicit form. The plane of the ellipse of normal curvature for the point x is to pass through this point. Let $e_1, \dots, e_5$ be the fixed basis in E^5 with the Cartesian coordinates x_1, x_2, u, v, w.

Then the surface $F^2 \subset E^5$ is given as follow:

$$u = u(x_1, x_2),$$

$$v = v(x_1, x_2),$$

$$w = w(x_1, x_2).$$

Write the vector position and its derivatives in the form

$$r = \begin{pmatrix} x_1 \\ x_2 \\ u(x_1,x_2) \\ v(x_1,x_2) \\ w(x_1,x_2) \end{pmatrix}, \quad r_{x_1} = \begin{pmatrix} 1 \\ 0 \\ u_1 \\ v_1 \\ w_1 \end{pmatrix}, \quad r_{x_2} = \begin{pmatrix} 0 \\ 1 \\ u_2 \\ v_2 \\ w_2 \end{pmatrix}, \quad r_{x_i x_j} = \begin{pmatrix} 0 \\ 0 \\ u_{ij} \\ v_{ij} \\ w_{ij} \end{pmatrix}.$$

Write the normals n_1, n_2, n_3 of F^2 in terms of the coordinates

$$n_1 = \{\xi_i\}, \quad n_2 = \{\eta_i\}, \quad n_3 = \{\zeta_i\}, \quad i = 1, \dots, 5.$$

The system of equations for the normal n_1 is the following:

$$\xi_\alpha + u_\alpha \xi_3 + v_\alpha \xi_4 + w_\alpha \xi_5 = 0, \tag{13}$$

where $\alpha = 1, 2$. The similar systems also exist for n_2 and n_3.

It is obvious that

$$\begin{aligned} L^1_{ij} &= \xi_3 u_{ij} + \xi_4 v_{ij} + \xi_5 w_{ij}, \\ L^2_{ij} &= \eta_3 u_{ij} + \eta_4 v_{ij} + \eta_5 w_{ij}, \\ L^3_{ij} &= \zeta_3 u_{ij} + \zeta_4 v_{ij} + \zeta_5 w_{ij} \end{aligned} \tag{14}$$

are the coefficients of the second fundamental forms.

Lemma 4. *If the plane of the ellipse of normal curvature is defined and it passes through the point x of the surface, then*

$$\Delta = \begin{vmatrix} L^1_{11} & L^2_{11} & L^3_{11} \\ L^1_{12} & L^2_{12} & L^3_{12} \\ L^1_{22} & L^2_{22} & L^3_{22} \end{vmatrix} = 0.$$

On contrary, if $\Delta = 0$, then the plane of this ellipse passes through the point x.

P r o o f. The vector of normal curvature can be rewritten as

$$k_n = (L^1_{11} n_1 + L^2_{11} n_2 + L^3_{11} n_3) \frac{(du^1)^2}{ds^2} + 2(L^1_{12} n_1 + L^2_{12} n_2 + L^3_{12} n_3) \frac{du^1 du^2}{ds^2}$$

$$+ (L^1_{22} n_1 + L^2_{22} n_2 + L^3_{22} n_3) \frac{(du^2)^2}{ds^2}.$$

The expressions in brackets, i.e., $N_{kl} = \sum\limits_{\alpha} L^{\alpha}_{kl} n_{\alpha}$ $k, l = 1, 2$, are vectors. It is known that the end of the vector k_n describes a flat curve, namely, an ellipse. As the plane of normal curvature ellipse passes through the point x of the surface, the vectors N_{kl} are coplanar. Hence we have

$$\Delta = \begin{vmatrix} L^1_{11} & L^2_{11} & L^3_{11} \\ L^1_{12} & L^2_{12} & L^3_{12} \\ L^1_{22} & L^2_{22} & L^3_{22} \end{vmatrix} = 0. \tag{15}$$

On contrary, if $\Delta = 0$, then three vectors N_{kl} are linearly dependent. Therefore, k_n for all du^1, du^2 lies in one plane passing through the point x.

The lemma is proved.

Lemma 5. *For a surface in the explicit form, the equation $\Delta = 0$ gives*

$$\begin{vmatrix} u_{11} & v_{11} & w_{11} \\ u_{12} & v_{12} & w_{12} \\ u_{22} & v_{22} & w_{22} \end{vmatrix} = 0. \tag{16}$$

P r o o f. In the determinant Δ, substitute the expressions for the coefficients of the second quadratic forms (14) to get

$$\Delta = \begin{vmatrix} \xi_3 u_{11} + \xi_4 v_{11} + \xi_5 w_{11} & \eta_3 u_{11} + \eta_4 v_{11} + \eta_5 w_{11} & \zeta_3 u_{11} + \zeta_4 v_{11} + \zeta_5 w_{11} \\ \xi_3 u_{12} + \xi_4 v_{12} + \xi_5 w_{12} & \eta_3 u_{12} + \eta_4 v_{12} + \eta_5 w_{12} & \zeta_3 u_{12} + \zeta_4 v_{12} + \zeta_5 w_{12} \\ \xi_3 u_{22} + \xi_4 v_{22} + \xi_5 w_{22} & \eta_3 u_{22} + \eta_4 v_{22} + \eta_5 w_{22} & \zeta_3 u_{22} + \zeta_4 v_{22} + \zeta_5 w_{22} \end{vmatrix} = 0.$$

This determinant can be written in the form of the product of two determinants

$$\begin{vmatrix} u_{11} & v_{11} & w_{11} \\ u_{12} & v_{12} & w_{12} \\ u_{22} & v_{22} & w_{22} \end{vmatrix} \begin{vmatrix} \xi_3 & \eta_3 & \zeta_3 \\ \xi_4 & \eta_4 & \zeta_4 \\ \xi_5 & \eta_5 & \zeta_5 \end{vmatrix} = 0.$$

The second determinant can not be equal to zero. If the determinant

$$\begin{vmatrix} \xi_3 & \eta_3 & \zeta_3 \\ \xi_4 & \eta_4 & \zeta_4 \\ \xi_5 & \eta_5 & \zeta_5 \end{vmatrix}$$

is equal to zero, then from Eqs. (13) it follows that the normals n_1, n_2, n_3 are linearly dependent. Therefore, Eq. (16) is proved.

Equations (15) and (16) allow to construct a number of simple examples of the surfaces in E^5, where the plane of the ellipse of normal curvature passes through the corresponding point of the surface.

Let the surface be of the form

$$r = \begin{pmatrix} x_1 \\ x_2 \\ u \\ v \\ w \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ u(x_1, x_2) \\ v = \frac{1}{2}(c_1 x_1^2 + c_2 x_1 x_2 + c_3 x_2^2) \\ w = \frac{1}{2}(d_1 x_1^2 + d_2 x_1 x_2 + d_3 x_2^2) \end{pmatrix}.$$

We find the derivatives

$$r_{x_1 x_1} = \begin{pmatrix} 0 \\ 0 \\ u_{11} \\ c_1 \\ d_1 \end{pmatrix}, \quad r_{x_2 x_2} = \begin{pmatrix} 0 \\ 0 \\ u_{22} \\ c_3 \\ d_3 \end{pmatrix}.$$

Construct two classes of surfaces. For the relations $d_3 = -d_1$, $c_3 = -c_1$ and $u_{11} + u_{22}$=0, we get the Laplace equation

$$r_{x_1 x_1} + r_{x_2 x_2} = 0. \tag{17}$$

At $d_3 = d_1$, $c_3 = c_1$, $u_{11} - u_{22} = 0$, we have the wave equation

$$r_{x_1x_1} - r_{x_2x_2} = 0. \tag{18}$$

Write the Gauss expansions

$$r_{x_1x_1} = \Gamma^i_{11} r_i + L^\alpha_{11} n_\alpha,$$
$$r_{x_2x_2} = \Gamma^i_{22} r_i + L^\alpha_{22} n_\alpha.$$

From the Gauss equations, the linear independence of the normals n_α, and the Laplace equation (17), we get the conditions on the coefficients of the second fundamental forms

$$L^\alpha_{11} = -L^\alpha_{22},$$

and from the wave equation (18), we obtain the relations

$$L^\alpha_{11} = L^\alpha_{22}, \quad \alpha = 1, 2, 3.$$

From the last two equations and (15) it follows that for both classes of surfaces, the plane of the ellipse of normal curvature passes through a point of the surface.

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