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An Application of Kadets–Pełczyński Sets to Narrow Operators

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A known analogue of the Pitt compactness theorem for function spaces asserts that if $1 \le p < 2$ and $p < r < \infty$, then every operator $T : L_p \to L_r$ is narrow. Using a technique developed by M.I. Kadets and A. Pełczyński, we prove a similar result. More precisely, if $1 \le p \le 2$ and F is a Köthe–Banach space on $[0,1]$ with an absolutely continuous norm containing no isomorph of L_p such that $F \subset L_p$, then every regular operator $T : L_p \to F$ is narrow.

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To the memory of M.I. Kadets

1. Introduction

Narrow operators were introduced and studied by A.M. Plichko and the second named author in [11]. Let us recall the definition for function spaces on the Lebesgue measure space $([0,1], \Sigma, \mu)$. Let L_0 denote the linear space of all equivalence classes of Σ-measurable functions $x : [0,1] \to \mathbb{R}$, and $L_p = L_p[0,1]$ for $1 \le p \le \infty$. By $\mathbf{1}_A$ we denote the characteristic function of a set $A \in \Sigma$. We set $\Sigma(A) = \{B \in \Sigma : B \subseteq A\}$, $\Sigma^+(A) = \{B \in \Sigma(A) : \mu(B) > 0\}$ and, as a partial case, $\Sigma^+ = \Sigma^+([0,1])$. The notation $A = B \sqcup C$ means that $A = B \cup C$ and $B \cap C = 0$. By a *sign* we mean any $\{-1,0,1\}$-valued element $x \in L_0$. More precisely, a sign x is called a sign on a set $A \in \Sigma$ provided that $\operatorname{supp} x = A$.

A sign x is said to be *of mean zero* if $\int_{[0,1]} x\, d\mu = 0$. Observe that $x \in L_0$ is a sign on $A \in \Sigma$ if and only if $x = \mathbf{1}_B - \mathbf{1}_C$ for some $B, C \in \Sigma$ with $A = B \sqcup C$, and, in addition, $\mu(B) = \mu(C)$ means that x is of mean zero. A Banach space $E \subset L_1$ is called a *Köthe–Banach space on* $[0,1]$ if the following conditions hold:

(1) $\mathbf{1}_{[0,1]} \in E$;

(2) for each $x \in L_0$ and $y \in E$ the condition $|x| \le |y|$ implies $x \in E$ and $\|x\| \le \|y\|$.

Note that, in the terminology of Lindenstrauss–Tzafriri [10, p. 28], a *Köthe function space* is a somewhat general notion which concerns the linear subspaces E of L_0, because we additionally assume the inclusion $E \subseteq L_1$. Using this inclusion and the closed graph theorem, one can show that the inclusion embedding of E to L_1 is continuous. A further convenience of the integrability assumption $E \subseteq L_1$ is shown in the following useful observation. Let E and F be Köthe–Banach spaces on $[0,1]$ with $E \subseteq F$. Then the inclusion embedding $J : E \to F$, $Jx = x$ for all $x \in E$, is continuous. Indeed, given any Köthe–Banach space G on $[0,1]$, by continuity of the inclusions $G \subseteq L_1 \subseteq L_0$ where the convergence in L_0 is equivalent to the convergence in measure, we have that every convergent sequence in G converges in measure. Using this fact and the closed graph theorem, one can easily prove that any inclusion of Köthe–Banach spaces is continuous.

A Köthe–Banach space E on $[0,1]$ is said to have an *absolutely continuous norm* if $\lim_{\mu(A)\to 0} \|x \cdot \mathbf{1}_A\| = 0$ for every $x \in E$.

By $\mathcal{L}(X,Y)$ we denote the set of all linear bounded operators from a Banach space X to a Banach space Y, and set $\mathcal{L}(X) = \mathcal{L}(X,X)$. Let E be a Köthe-Banach space on $[0,1]$ and let X be a Banach space. An operator $T \in \mathcal{L}(E,X)$ is called *narrow* if for every $A \in \Sigma$ and every $\varepsilon > 0$ there is a mean zero sign x on A with $\|Tx\| < \varepsilon$. It is not very hard to show that if E has an absolutely continuous norm, then every compact operator $T \in \mathcal{L}(E,X)$ is narrow [11]. Thus, narrow operators generalize compact operators (as well as some other natural classes of "small" operators). Some properties of compact operators inherit by narrow operators, but not all of them (see [11], a recent survey [12] and a forthcoming book [13]).

The classical Pitt theorem [9, p. 76] asserts that for any $1 \le p < r < \infty$ every operator $T \in \mathcal{L}(\ell_r, \ell_p)$ is compact. Using the notion of infratype for Banach spaces, the following result was obtained in [8].

Theorem 1.1. *If $1 \le p < 2$ and $p < r < \infty$, then every operator $T \in \mathcal{L}(L_p, L_r)$ is narrow.*

Theorem 1.1 can be considered as an analogue of the Pitt compactness theorem in the setting of function spaces. We remark that Theorem 1.1 is false for any other values of p and r. If $p \geq 2$, then the composition $J_r \circ I_p$ of the identity embedding $I_p : L_p \to L_2$ and the isomorphic embedding $J_r : L_2 \to L_r$ is evidently not narrow. And if $1 \leq p < 2$ and $1 \leq r \leq p$, then the identity embedding of L_p into L_r is not narrow.

Recall that a linear operator $T : E \to F$ between Köthe–Banach spaces (more general, between vector lattices) E and F is called *positive* if $Tx \geq 0$ for every $x \in E$ with $x \geq 0$. Here and in sequel $x \leq y$ for elements of L_1 means that $x(t) \leq y(t)$ holds a.e. on $[0, 1]$. A linear operator $T : E \to F$ is called *regular* if it is a difference of two positive linear operators from E to F.

The main result of the paper is the following theorem.

Theorem 1.2. *Let $1 \leq p \leq 2$ and let F be a Köthe–Banach space on $[0, 1]$ with an absolutely continuous norm containing no subspace isomorphic to L_p such that $F \subset L_p$. Then every regular operator $T \in \mathcal{L}(L_p, F)$ is narrow.*

Theorems 1.2 and 1.1 are incomparable: Theorem 1.2 covers much more range spaces, however it is restricted to regular operators.

2. Kadets–Pełczyński Sets

In seminal paper [7] (1962), which became one of the most cited classical papers on the geometric theory of Banach spaces, M.I. Kadets and A. Pełczyński introduced special sets M_ε^p in the space L_p, $1 \leq p < \infty$ depending on a positive parameter $\varepsilon > 0$ and consisting of all elements $x \in L_p$ such that the subgraph of the decreasing rearrangement of $|x|$ contains a square with sides ε. Let us give a precise definition for the general setting of the Köthe–Banach spaces on $[0, 1]$.

Definition 2.1. *Let E be a Köthe–Banach space on $[0, 1]$ and $\varepsilon > 0$. Set*

$$M_\varepsilon^E = \Big\{ x \in E : \ \mu\big\{ t \in [0, 1] : |x(t)| \geq \varepsilon \|x\|_E \big\} \geq \varepsilon \Big\}.$$

Obviously, $M_{\varepsilon'}^E \subseteq M_{\varepsilon''}^E$ whenever $\varepsilon' \geq \varepsilon''$ and $\bigcup_{\varepsilon > 0} M_\varepsilon^E = E$.

Remark that the sets M_ε^E for the setting of the Köthe–Banach spaces were used by various authors, see, e.g., [10, Proposition 1, p. 8], [4, 5]. The idea of using these sets can be explained as follows. Given a normalized sequence (x_n) in E, either it is contained in some universal set M_ε^E, or for every $\varepsilon > 0$ there is n such that $x_n \notin M_\varepsilon^E$. In the first case, the norm of E and the L_1-norm are equivalent on (x_n), and in the second case, (x_n) contains subsequences with arbitrarily "narrow" elements. This leads to different interesting alternatives for the sequences and subspaces of E. One of the alternatives which we will need later is obtained in the following lemma (see [13, Lemma 10.63]; we provide its proof below for the sake of completeness).

Lemma 2.2. *Let E be a Köthe–Banach space on $[0,1]$ with an absolutely continuous norm. Let (x_n) be an order bounded sequence from E such that for every $\varepsilon > 0$ there exist $n \in \mathbb{N}$ such that $x_n \notin M_\varepsilon^E$. Then there exists a subsequence (y_n) of (x_n) and a disjoint sequence (z_n) in E such that $|z_n| \le |y_n|$ for all n, and $\|y_n - z_n\| \to 0$.*

Before the proof, we recall some lattice definitions. A subset X of a Köthe–Banach space E is called *order bounded* provided there exists $y \in E$ such that $|x| \le y$ for each $x \in X$. A linear operator $T : E \to F$ between Köthe–Banach spaces E and F is called *order bounded* if T sends order bounded sets from E to order bounded sets in F. Evidently, any positive operator (hence, any regular operator) is order bounded. By E^+ we denote the positive cone of E, that is, $E^+ = \{x \in E : \ x \ge 0\}$.

P r o o f. Let $e \in E^+$ be such that $|x_n| \le e$ for all $n \in \mathbb{N}$. Choose a subsequence (x_n') of (x_n) so that $x_n' \notin M_{2^{-n}}^E$ for all n. For every $n \in \mathbb{N}$, let $A_n = \{t \in [0,1] : \ |x_n'(t)| \ge 2^{-n}\|x_n'\|\}$ and $B_n = \bigcup_{k=n}^\infty A_k$.

Note that $\mu(A_n) < 2^{-n}$, $B_{n+1} \subseteq B_n$, and $\mu(B_n) \le 2^{-n+1}$ for each n. Choose a strictly increasing sequence of the integers $(n_i)_i$ such that $\|e \cdot \mathbf{1}_{B_{n_{i+1}}}\| \le 1/i$ (this is possible because of the absolute continuity of the norm).

Observe that the sets $C_i = A_{n_i} \setminus B_{n_{i+1}}$ are disjoint. Let $y_i = x_{n_i}'$ and $z_i = y_i \cdot \mathbf{1}_{C_i}$ for $i = 1, 2, \ldots$. Then (z_i) is a disjoint sequence, $|z_i| \le |y_i|$, and

$$\begin{aligned}\|y_i - z_i\| &= \|x_{n_i}' \cdot \mathbf{1}_{[0,1]\setminus C_i}\| \le \|x_{n_i}' \cdot \mathbf{1}_{[0,1]\setminus A_{n_i}}\| + \|x_{n_i}' \cdot \mathbf{1}_{B_{n_{i+1}}}\| \\ &\le \|2^{-n_i}\|x_{n_i}'\| \cdot \mathbf{1}_{[0,1]\setminus A_{n_i}}\| + \|e \cdot \mathbf{1}_{B_{n_{i+1}}}\| \\ &\le 2^{-n_i}\|e\|\|\mathbf{1}_{[0,1]}\| + 1/i \to 0 \ \text{ as } \ i \to \infty. \qquad \blacksquare\end{aligned}$$

We need the following lemma which in a certain degree develops the previous one.

Lemma 2.3. *Let E be a Köthe–Banach space on $[0,1]$ with an absolutely continuous norm. Let (x_n) be an order bounded sequence from E such that $\|x_n\| \ge \delta$ for some $\delta > 0$ and all $n \in \mathbb{N}$. Then there exists $\varepsilon > 0$ such that $x_n \in M_\varepsilon^E$ for all n.*

P r o o f. Let $y \in E$ be such that $|x_n| \le y$ for all $n \in \mathbb{N}$. Supposing the lemma is false, choose by Lemma 2.2 a subsequence (y_n) of (x_n) and a disjoint sequence (z_n) in E such that $|z_n| \le |y_n|$ for all n, and $\|y_n - z_n\| \to 0$. Set $A_n = \operatorname{supp} z_n$ for each $n \in \mathbb{N}$. Then $|z_n| \le |y_n| \cdot \mathbf{1}_{A_n} \le z \cdot \mathbf{1}_{A_n}$ and hence

$$\delta \le \|y_n\| \le \|z_n\| + \|y_n - z_n\| \le \left\|z \cdot \mathbf{1}_{A_n}\right\| + \|y_n - z_n\|$$

for all n. This is impossible, because $\|y_n - z_n\| \to 0$ and $\left\|z \cdot \mathbf{1}_{A_n}\right\| \to 0$ by the absolute continuity of the norm in E. $\blacksquare$

3. Enflo Operators and Proof of the Main Result

Let X be a Banach space. An operator $T \in \mathcal{L}(X)$ is called an *Enflo operator* if there is a subspace Y of X isomorphic to X such that the restriction $T|_Y$ of T to Y is an isomorphic embedding. The name "Enflo operator" is due to the following famous Enflo theorem on primarity of L_p: if the space L_p, $1 \le p < \infty$, is decomposed into a direct sum of closed subspaces $L_p = X \oplus Y$, then at least one of X, Y is isomorphic to L_p (see [10, p. 179]).

One of the peculiarities of the spaces L_p with $1 \le p < 2$, which will be used later, is described in the following deep theorem due to varios authors.

Theorem 3.1. *Let $1 \le p \le 2$. Then any non-Enflo operator $T \in \mathcal{L}(L_p)$ is narrow.*

Theorem 3.1 for $p = 1$ can be deduced from the results of [3]. Moreover, the following remarkable result of Rosenthal (the equivalence of (c) and (d) in Theorem 1.5 of [14]) gives much more — necessary and sufficient conditions for an operator $T \in \mathcal{L}(L_1)$ to be narrow.

Theorem 3.2. *An operator $T \in \mathcal{L}(L_1)$ is narrow if and only if for each $A \in \Sigma$ the restriction $T\big|_{L_1(A)}$ is not an isomorphic embedding, where $L_1(A) = \{x \in L_1 : \operatorname{supp} x \subseteq A\}$.*

Theorem 3.1 in the case $1 < p < 2$ was announced by J. Bourgain [1, Theorem 4.12, item 2, p. 54] without a proof, accompanied with a citation to [6]. Formally, Theorem 3.1 cannot be deduced from [6], however an involved proof can be written by using the ideas and methods of [6] (such a proof is to be found in [13, Section 7.3]). Another proof of Theorem 3.1 in the case $1 < p < 2$ is given in [2]. The last our comment is that for $p = 2$ Theorem 3.1 holds trivially, because in this case a non-Enflo operator must be compact and hence narrow.

Now we are ready to prove the main result.

P r o o f of Theorem 1.2. Denote by $J \in \mathcal{L}(F, L_p)$ the inclusion embedding, $Jx = x$ for all $x \in F$. Consider any $T \in \mathcal{L}(L_p, F)$ and assume, on the contrary, that T is not narrow. Choose $A \in \Sigma^+$ and $\delta > 0$ such that $\|Tx\| \ge \delta$ for each mean zero sign x on A.

Set $S = J \circ T \in \mathcal{L}(L_p)$ and show that S is non-narrow as well. Assuming, on the contrary, that S is narrow, we find a sequence (x_n) of mean zero signs on A such that $\|Sx_n\| \to 0$. Since (x_n) is order bounded by $\mathbf{1}_{[0,1]}$ and T is regular, (Tx_n) is an order bounded sequence. And since $\|Tx_n\| \ge \delta$ for all $n \in \mathbb{N}$, by Lemma 2.3, there exists $\varepsilon > 0$ such that $Tx_n \in M_\varepsilon^F$ for all n. Thus,

$$\|Sx_n\|_p^p = \int\limits_{[0,1]} |Tx_n|^p d\mu \ge \varepsilon^{p+1},$$

which contradicts the condition $\|Sx_n\| \to 0$. Thus, S is non-narrow. By Theorem 1.2, S is an Enflo operator. Let E be a subspace of L_p isomorphic to L_p such that $\|Sx\| \geq \alpha\|x\|$ for some $\alpha > 0$ and all $x \in E$. Then

$$\|Tx\| \geq \|J\|^{-1}\|Sx\| \geq \alpha\|J\|^{-1}\|x\|$$

for all $x \in E$ which contradicts the assumption that F contains no subspace isomorphic to L_p. ■

We do not know whether the assumption of the regularity of T is essential in Theorem 1.2.

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